Nonlinearities in Quantum Mechanics

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Many of the paradoxes encountered in the Copenhagen interpretation of quantum mechanics can be shown to have plausible, more logical parallels in terms of nonlinear dynamics and chaos. These include the statistical exponential decay laws, interpretations of Bell's inequalities, spontaneous symmetry breaking, and perhaps diffractive behavior and even quantization itself. Many of the so-called alternative explanations of quantum mechanics have toyed with ideas that approach chaotic behavior, but as they were formulated before the advent of modern chaos theory, they remained within linear systems or at most nonlinear perturbations to linear systems; however, only strongly nonlinear systems can provide the proper parallels to the Copenhagen paradoxes. Several examples of these will be covered qualitatively. Strongly nonlinear behavior related to quantum mechanics does not involve "hidden variables," but chaos provides a bridge between the statistical behavior of quantum mechanics and deterministic behavior of classical mechanics. Perhaps both Einstein and Bohr were correct in their debates—chaos fundamentally provides the determinism so dear to Einstein, but in practice it must be interpreted statistically in the manner of Bohr.

1 Introduction

Quantum mechanics has always been regarded as the epitome of a linear science. This is the essence of the Copenhagen interpretation proffered by Bohr, Heisenberg, and their school. Yet there have always been objections to this orthodox interpretation, the most famous being those of Einstein in the Einstein-Bohr debates [1, 2], the *reductio ad absurdum* of Schrödinger's cat [3], and the hidden-variables extensions of de Broglie [4] and Bohm [5]. Still, the Copenhagen interpretation ruled supreme for most of the twentieth century, with relatively few people questioning or investigating the foundations of quantum mechanics.

The rise of quantum information science, with its possible implications for quantum computing, has changed that somewhat—for example, during the last decade(s) there have been numerous investigations, both theoretical and experimental, into Bell's theorem and inequalities, and at least two important ongoing series of international workshops [6, 7] have been established specifically to reinvestigate the foundations of quantum mechanics, together with its apparent paradoxes. In addition, nonlinear extensions to quantum mechanics have been proposed by a number of authors [8, 9, 10, 11], albeit with limited success, for they encounter severe difficulties such as the introduction of superluminal signals. Part of the difficulty stems from the fact that all of them essentially introduce nonlinear perturbations onto basically linear systems, and only in "strongly nonlinear" systems can chaotic behavior ensue, which we shall find necessary to account for most of the quantum mechanical paradoxes. Mielnik recognizes the problems and succinctly sums up the difficulties:

"I cannot help concluding that we do not know truly whether or not nonlinear QM generates superluminal signals—or perhaps, it resists embedding into too narrow a scheme of tensor products. After all, if the scalar potentials were an obligatory tool to describe the vector fields, some surprising predictions could as well arise! ...the nonlinear theory would be in a peculiar situation of an Orwellian 'thought-crime' confined to a language in which it cannot even be expressed. ...A way out, perhaps, could be a careful revision of all traditional concepts... "

As opposed to all of the above "from the top down" approaches, I have adopted a "from the bottom up" approach, using the "quasi-experimental" technique of gathering specific examples where strongly nonlinear, chaotic techniques can mimic the so-called imponderables, i.e., paradoxes and inconsistencies, at the heart (generated by the Copenhagen interpretation) of quantum mechanics. Thus far, the relevant "specimens" are (in descending order of comprehension—and increasing speculativeness):

- Exponential decay produced by unimodal maps.
- Interpreting Bell's inequalities via nonlinear dynamics and nonextensive thermodynamics.
- Attractors, basins of attraction, and implications for quantization.
- Spontaneous symmetry breaking—parity nonconservation.
- Decoherence and the transition from Hamiltonian to dispersive systems.
- Diffraction—order in chaos.

- Barrier penetration as a nonlinear phenomenon.

Because of space limitations, here I limit myself to the first two—indeed, they have received by far the most attention—but additional information can be found in a series of papers, which outline my progress (including occasional revised thinking) over the last several years [12, 13, 14, 15].

2 Statistical Exponential Decay Laws and Chaotic Escape from Unimodal Mappings

One of the minor imponderables related to quantum mechanics is how exponential, first-order laws arise in the breakup of quantum states—from nuclear and particle decay to transitions in atoms and molecules. A fundamental tenet of quantum mechanics is that identical particles or bodies are indeed indistinguishable. Given, say, a collection of radioactive nuclei, it is impossible to predict when a particular nucleus will disintegrate; yet, if the collection contains enough nuclei to be statistically significant, then one can be assured that the decay of the collection will follow a precise exponential decay law. At times, especially in more elementary texts or courses, this is likened to the statistical nature of actuarial tables in the insurance industry, but such an analogy is not at all apropos, since the actuarial behavior arises from complex systems and causes, which are the very antitheses of identical quantum states.

A satisfactory parallel can be found, however, in the extreme sensitivity of chaotic systems to initial conditions, viz., the "butterfly effect." In the spirit of Ockham's razor, we invoke the simplest possible construct that can produce such behavior, which is the iteration of the quadratic (or logistic) map-it turns out that any unimodal map will produce similar behavior. This is treated in some detail in a previous paper [13], where both the quadratic and sine maps are considered, so the results will only be summarized here.

The quadratic map is generated by iterating the equation,

$$x_{n+1} = x_n^2 + c,$$

or its more familiar equivalent form, the logistic equation,

$$x_{n+1} = Ax_n(1 - x_n).$$

In the logistic equation *x* represents a population ranging from 0 (extinction) to 1 (maximum), and the control parameter *A* represents a birth rate. For $A < 1$, iteration of the map invariably leads to extinction; for $1 < A < 3$, iteration leads to a single final value of *x*; for $A > 3$, a series of successive bifurcations sets in, with the final value of *x* alternating between two, then four, eight, ... final values; and for $A > 3.44948...$, chaos sets in, with the final value of *x* most often (there being windows of order, predictability in chaos) being unpredictable and sensitive to *infinitesimal* difference in the initial values chosen for *x*. (The greatest value that *A* can take on without divergence setting in is 4. Bifurcation diagrams for such iterations can be found in [13] or in almost any introductory book on chaos theory.)

An analogy can be made between these iterations of the map in a chaotic region and radioactive decay. Let a very narrow interval of initial values of x represent the initial radioactive state(s). This is consistent with (but not dependent on) the Uncertainty Principle. (It should be noted also that de Broglie and Bohm almost toyed with ideas such as this when they considered an "unobservable, thermodynamic, statistical sub-background" that interacted with quantum systems to produce stochastic behavior!) Similarly, a narrow interval in the final values can represent the final state. Then the process of iteration represents the decay process itself, say, the number of attempts of an α particle to penetrate the Coulomb barrier to leave its nucleus or the number of oscillations of a nuclear multipole to produce a γ ray. (The transition probability for such a process is quadratic, which is an additional justification for the argument.) One then takes randomly-generated numbers within the initial interval, carries out the iterations, and keeps track of how many iterations it takes for each initial value to "escape" into the final interval.

We have run computer experiments using many different maps, typically starting with tens of thousand of randomly-generated initial values in intervals having widths of 10^{-11} escaping into slightly larger final intervals. And plots of the numbers of remaining active trajectories (paths from each initial value) versus the number of iterations (time) invariably produced statistical exponential decay curves for an assortment of unimodal (smooth one-humped functions without inflection points). This, of course, is far from any sort of proof that the model is apt, but to the best of my knowledge, it by far the simplest, least convoluted explanation that has been touted so far.

3 Classical Vulnerability in Bell-Type Inequalities

Ever since publication of the EPR paradox [1], there have been debates both about the (in)completeness of quantum mechanics and about the breakdown of "local reality" (leading to Einstein's "spooky action at a distance"). The EPR paradox involves correlations between conjugate variables such as position and momentum of widely-separated, incommunicado particles, and it was designed to demonstrate that quantum mechanics was incomplete, but it was cast in the form of an abstract *Gedankenexperiment* with little possible connection to experimental reality. Bohm [16] made EPR more realistic by considering the specific case of two spin-1/2 fermions, but it was Bell [17] who transformed it into something approaching practicability. What Bell did was to treat correlations between widely-separated particles statistically and to derive an inequality that places an upper bound on possible specific classical statistical correlations but which allows quantum statistical correlations to exceed this upper bound under certain conditions. There have been numerous refinements to and variants on Bell's original inequality, but perhaps the most straightforward is what is known as the CHSH inequality [18], which was specifically constructed with experimental testing in mind. It will be used for the following brief discussion (for more details, see [15]).

Alice and Bob, the standard information theory cartoon characters, are separated by an effectively infinite distance (i.e., are incommunicado). Pairs of correlated particles, say, an electron pair in a singlet state or a photon pair having opposite polarizations, are prepared, and one particle from each pair is sent to Alice, the other to Bob, who then proceed to make independent binary measurements on their particles.

Alice can make measurement *Q* or *R* on each particle received, each outcome being either +1 or -1; e.g., +1 could be spin-up and -1 spin-down, or perhaps +1 could be vertical polarization and -1 horizontal polarization. And *Q* could be measurements with respect to a vertical axis and *R* with respect to a skewed axis. Bob's corresponding measurements on each of his particles are termed *S* and *T*. Now, Alice chooses randomly each time whether to make measurement *Q* or *R*, and similarly Bob chooses randomly *S* or *T*. Since there is no communication between them, at times they will measure with respect to the same set of axes and at other times with respect to different sets—and the random choice can be made after the particles have started on the flight path from their mutual source.

After making many, many measurements in order to achieve statistical significance, Alice and Bob get together to compare statistical notes. The quantity of interest for them to compare is

$$QS + RS + RT - QT = (Q + R)S + (R - Q)T.$$

Note the single minus sign on each side—because *Q* and *R* independently take on the values +1 or -1, one or the other of the terms on the right side of the equation must equal 0. Either way,

$$QS + RS + RT - QT = \pm 2,$$

or in terms of probabilities, where *E(QS)*, e.g., is the mean value of the measurements for the combination *QS*, we arrive at the so-called "classical" version of the CHSH inequality,

$$E(QS) + E(RS) + E(RT) - E(QT) \leq 2.$$

In other words, classical mechanics through this inequality places an upper bound on the possible statistical correlations (for specific combinations of results) obtained by presumably independent and randomly chosen measurements made on widely-separated particles.

The parallel derivation for quantum mechanical qubits is similar, but the pairs are assumed to be produced in the *entangled* Bell singlet state,

$$\mid \Psi\rangle = (\mid 01\rangle - \mid 10\rangle)/\sqrt{2}.$$

The first qubit from each ket is sent to Alice and the second to Bob, who proceed to make measurements as before, but on the following combinations of observables:

$$Q = Z_1$$

$$R = X_1$$

$$S = \frac{(-Z_2 - X_2)}{\sqrt{2}}$$

$$T = \frac{(Z_2 - X_2)}{\sqrt{2}}$$

Here *X* and *Z* are the "bit flip" and "phase flip" quantum information matrices, corresponding to the Pauli σ_1 and σ_3 spin matrices, respectively. It can be shown that the expectation values of the pairs *QS*, *RS*, and *RT* are all $+1/\sqrt{2}$, while that of *QT* is $-1/\sqrt{2}$. This leads to the CHSH quantum mechanical combination,

$$\langle QS\rangle + \langle RS\rangle + \langle RT\rangle - \langle QT\rangle = 2\sqrt{2}.$$

This is a larger value than was obtained in the classical inequality, which means that *within the framework of this entangled system*, quantum mechanics can produce greater statistical correlations than classical mechanics. In other words, classical systems must obey Bell-type inequalities, whereas quantum systems can at times violate them.

During the last several decades numerous Bell-type experiments have been made, as covered extensively by Bertlmann and Zeilinger [19], and they have consistently ruled in favor of quantum mechanics. As with most ideas connected with quantum mechanics, interpretations vary—but most involve some sort of elimination of "local reality." Two far-apart, isolated but entangled particles have an influence on each other. An example of this might be the following: Two electrons are emitted in a spin-singlet state, where their individual spin directions are unknown (according to the Copenhagen interpretation, actually *undefined* until an observer makes measurements on them), but whatever direction the spin of one points, the spin of the other must be in the opposite direction. When Alice randomly measures the direction of her electron, say, with respect to a *z* axis and gets $\uparrow$, this information is instantaneously conveyed somehow to Bob's electron, whose wave function immediately reduces to $\downarrow$. Einstein's spooky—and superluminal—action at a distance is real!

The small fraternity of "Bell inequality physicists" is sharply divided into two camps, those who support Bell's theorem (that violation of the inequality rules out classical behavior) and those who strongly oppose it. In fact, the two series of quantum fundamentals workshops argue rather consistently on opposite sides of the question—the Italian conferees [7] are primarily Bell supporters, while the Swedish conferees [6] are Bell opponents. Among the more compelling arguments against Bell's theorem is that it ignores relativistic and QED effects, so is simply no applicable to systems involving electron spin or photon polarization [20]. I adopt an alternative approach in this paper. In a sense this is attacking a theorem, which according to its detractors has already been shown to be irrelevant to many of the experimental results to which it has been applied. However, in a sense it is also a simpler alternative explanation, and, more important, it points out a slippage in the use of statistics that is rife in interpretations of quantum mechanics.

In this (nonlinear) parallel, the problem lies not in the quantum mechanics derivations but in the classical ones. Instead of a contest between quantum and classical mechanics, it is one between correlated versus uncorrelated statistics. In the so-called classical derivation above, the particles were presumably prepared in correlated pairs, but these correlations were then tacitly ignored, whereas the use of the Bell entangled state necessarily retained the complete correlations. Another way of looking at this—and here is the slippage referred to above—is that the classical correlations are indeed statistical, meaning that they apply only a large ensemble of particles, yet were effectively applied to individual measurements. Now, statistical correlated behavior on classical systems is quite well known—it simply requires nonlinear (sometimes emergent) behavior.

A codification of correlated statistics was introduced by Tsallis and his coworkers [21, 22], when they introduced "nonextensive" (meaning nonadditive) thermodynamics. Correlations can be expressed by a generalized, nonextensive entropy,

$$S_q = (1 - \sum_{i=1}^{W} p_i^q)/(q-1),$$

where the phase space of the system has been divided into *W* cells of equal measure, with p_i the probability of being in cell *i*. When the exponent *q* (termed the "entropic index") has a value of 1, this generalized entropy reduces, as it should, to the standard Boltzmann entropy,

$$S_1 = -\sum_{i=1}^{W} p_i \ln p_i.$$

As *q* differs more and more from 1, the deviation from standard distributions becomes greater and greater, indicating that "long-range" correlations become more and more important. In a nonlinear system, when such correlations are present, the entropy becomes nonextensive, with the total entropy becoming

$$\frac{S_q(A+B)}{k} = \frac{S_q(A)}{k} + \frac{S_q(B)}{k} + \frac{(1-q)S_q(A)S_q(B)}{k^2},$$

for a two-component system. When $q < 1$, the entropy of the combined system is greater (superextensive) than the sum of the individual entropies, and when $q > 1$, it is less (subextensive). Thus, physically $q > 1$ indicates that a system has long-range correlations that "interfere constructively," characteristic, e.g., of emergent systems. This concept of nonextensive entropy has found widespread applications in classical systems, ranging from the velocity distributions in tornadoes (a good example of an emergent system) to the energy distributions of cosmic rays, and, of course, it plays an important role in biological evolutions. It has also been the subject of several recent international conferences [23].

Pertinent to our discussion are the recent studies of systems "at the edge of quantum chaos [24]. For both the quantum kicked top and the logistic map [25], values of $q > 1$ could be applied, with $q \approx 2$ seeming reasonable for the logistic map. [One must proceed cautiously here because these values were obtained within the context of standard (i.e., linear) quantum mechanics. Nevertheless, the crossover into classical would be similar for nonlinear systems, and both systems need contain long-range correlations.]

It thus seems clear that classical systems can indeed exhibit behavior in which long-range correlations play an important role. N.B. This does not necessarily imply long-range forces or action-at-a-distance, as has been shown clearly in the behavior of cellular automata and emergent systems. As a result, it is the so-called "classical" version of the CHSH inequality—and most if not all of the other guises that Bell's inequalities can take on—that is suspect. (This includes versions that do not directly involve inequalities, such as the GHZ formulation, [26] but which require statistical arguments when experimental justification is sought. This is covered in more detail in a previous paper. [15]) Long-range correlations in classical systems can be "constructive," which raises the apparent upper limit imposed by such as the CHSH inequality. With a value of *q* in the vicinity of 2, one would expect something closer to an exponential rather than a Gaussian statistical distribution. If so, then Bell-type arguments are moot in ruling out the existence of "local reality" in quantum mechanics.

4 Conclusions

Because of space limitations, it is not possible to cover the other nonlinear parallels in any detail. However, the two covered in the previous sections should raise pertinent questions in the mind of the reader, even if they lack quantitative proof. Indeed, the basic object of this paper is to raise such questions rather than to attempt convincing, quantitative proof. In a sense we are still in a "quasi-botanical mode," seeking out and collecting specimens rather than offering detailed analysis. Nonlinear dynamics applies to essentially every other scientific discipline, so why is quantum mechanics exempt from nature's preferred feedback and nonlinearities? Perhaps, as Mielnik suggested, we have been unconsciously using a form of scientific "newspeak," which has prevented us from expressing any nonlinear "thoughtcrimes."

The founders and developers of early quantum mechanics did not have access to modern nonlinear dynamics and chaos theory, so they were forced to deal with the idiosyncrasies of quantum mechanics within a strictly linear, if perturbative framework. This framework has worked beautifully insofar as quantitative, practical applications are concerned. Indeed, until recent years scarcely anybody bothered to question the validity of the Copenhagen interpretation or worry that a point might be reached where the application of increasingly peculiar concepts could possibly break down. After all, who really cares if it requires thirteenth-order perturbation theory [27] or 1078 parameters in a variational calculation [28] to yield a precise value for the ground-state energy of the He atom?! With a glimpse of the possibility of quantum computing, however, this changes somewhat, for quantum computing depends critically on linear superpositions of qubits. (It is conceivable that, providing it does turn out that quantum mechanics does have fundamentally nonlinear elements—still a big uncertainty—there may be some nonlinear analog to superposition that does allow the possibility of quantum computers. But that remains an unanswered question for the future.) What does seem plausible, however, is that there have been many naive applications of statistics in interpretations of quantum mechanics. Statistics, including probabilities and wave-functions, rightfully apply only to large ensembles. When one speaks of a single wave-function of a single electron reducing to a specific expectation value, one must use extreme caution.

Finally, perhaps Einstein and Bohr were basically right in their debates. Chaos in quantum mechanics has nothing to do with hidden variables, but it directly provides the fundamental determinism so dear to Einstein's heart. On the other hand, for all practical purposes it yields indeterminate results that can only be interpreted statistically, as the Copenhagen interpretation insists. It is interesting to speculate how the Einstein-Bohr debates would have progressed had modern chaos theory been available.

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