Regularizations: A Unique Prescription for All Situations

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A detailed investigation on the possible role played by the intrinsic arbitrariness of the perturbative evaluation of physical amplitudes and their symmetry relations, initiated in a first work, is continued. Previously announced results are detailed presented. The very general calculational method, concerning the divergences manipulations and calculations, adopted to discuss the questions of ambiguities and symmetry relations, in purely fermionic divergent Green functions, is applied to explicit evaluate three-point functions. Two of such functions, the well known Scalar-Vector-Vector and Axial-Vector-Vector triangle amplitudes are considered in details. Given the fact that within the adopted strategy, all the arbitrariness intrinsic to the problem are maintained in the final results, and that it is possible to map them in to the ones of traditional techniques, clean and sound conclusions can be extracted. In particular, we can map our results in to the Dimensional Regularization ones as well as in to those corresponding to surface's terms evaluation. The first above cited amplitude can be treated within the Dimensional Regularization while the second do not and, consequently, it is usually treated by the surface's terms evaluation strategy. Within the adopted strategy both problems can be equally treated. We show that when we require consistency in the interpretations of the intrinsic indefinitions present in the perturbative amplitudes, which means to treat all physical amplitudes on the same way, no room is left for the ambiguities. As a natural consequence, the physical amplitudes are obtained symmetry preserving, where they must be, and anomalous, where they need to in spite of being non-ambiguous.

Keywords: Perturbative calculations; Divergences; Anomalies; Ambiguities

I. INTRODUCTION

It is certainly present in the mind of all physicists the idea that it would be desirable to describe in an adequate way the dynamics of the fundamental interacting particles from the point of view of a simple and well defined theoretical apparatus. The expected status is wondered as better as simpler and more minimum is the set of ingredients and symmetries we need to use. When such framework becomes reliable, it should be possible to give an accurate description of all experimentally established phenomenology, and also to make additional predictions about new phenomena not observed yet. To reach this expected situation the theoretical apparatus constructed needs to be completely well defined so that following an unique prescription the predictions we make throughout the solution of the theory could be considered as a consequence of the assumed principles, and that such predictions could be stated free from ambiguities. On the way to be followed to achieve this goal some philosophical guides are frequently taken into account. The major one is perhaps the following: If there is a choice, which is an exclusive attribute of the observer in the theoretical apparatus or physical laws, then the final results should be independent on such choice. This means that the final results should not be dependent on the arbitrariness involved in the application of the physical laws or equivalents, otherwise we have ambiguities. While this description has not been achieved it becomes a goal to be reached by additional investigations, until all the predictions are completely independent on the arbitrariness involved.

The reasoning line above works like an important motivation for the investigations we will consider in this manuscript in the context of the Quantum Field Theory (QFT) perturbative calculations. There are intrinsic arbitrariness on the way to be walked in order to make predictions and, in the context of the traditional methods, the results may emerge ambiguous. Once it is unacceptable that the physical consequences of any theory become dependent on the involved arbitrariness, an ambiguities free and consistent treatment of the physical amplitudes must be searched. Having this in mind, in previous contributions [1–3] we have made detailed investigations on the question of arbitrariness and their possible associated ambiguities in the evaluation of perturbative (divergent) physical amplitudes and/or in the preservation of symmetry relations. A very general calculational strategy [4] has been used on the referred investigations. The adoption of a regularization technique or equivalent philosophy was avoided in the intermediary steps such that one of the main arbitrariness involved in this type of calculations (the choice of the regularization) were preserved until the final results. Besides, the (arbitrary) choices for the internal lines momentum of the divergent amplitudes have been taken as the most general ones. Proceeding on this way we have obtained very general expressions for the physical amplitudes preserving the arbitrariness usually chosen at the first steps of the perturbative calculations.

As a consequence it was possible to analyze, in a very transparent way, the role of the regularizations on the ambiguities elimination and in the symmetry preservation when the calculated amplitudes are divergent quantities, specially when the degree of the divergences are higher than the logarithmic one. It was identified a set of properties required for a regularization in order to eliminate the possible ambiguities and to preserve symmetries which are very general and, in certain way, very simple. Such properties are automatically satisfied by the Dimensional Regularization (DR) technique [5] and may be implemented in a superposition of regulating functions like in the Pauli-Villars technique [6]. However, the most remarkable aspect of the investigation resides on the anomalous triangle amplitudes. The common association of the symmetry violations with the intrinsic arbitrariness involved on the (linearly) divergent amplitudes evaluation [7, 8] were clearly showed as a non-consistent procedure. This is due to the fact that in a consistent interpretation of the involved arbitrariness, which maps the DR results (where the method applies) the ambiguities are automatically eliminated. The source of the freedom usually used to justify the choices made in the anomalous amplitudes are clearly not allowed just because would imply violating fundamental symmetries or to treat the amplitudes in a case by case way attributing different values for the same mathematical object in two different physical amplitudes which sometimes are related by an (unambiguous) identity. The DR and the surfaces terms evaluation are classified as belonging to different classes of regularizations which are not compatible [1]. Once we have excluded the arbitrariness associated to the choice for the internal lines momentum as an ingredient of the analysis, discarding then the usual procedure, a crucial question has emerged: If all the amplitudes are looked in the same footing, concerning the divergences, which are the sources of symmetry violating terms in the anomalous amplitudes? A brief discussion about this point was made in the final section of the above mentioned publication with the promise that detailed results would be presented elsewhere. In this contribution we will present a detailed investigation of the referred aspects now performing also the explicit evaluation of the three-point functions. In order to have the ingredients for the present discussion we take two similar physical amplitudes, concerning the divergent integrals involved. They are the Scalar-Vector-Vector (SVV) and Axial-Vector-Vector (AVV) triangle amplitudes. For the first cited one we can apply the DR technique while for the second we do not. We will show that the crucial questions relative to symmetry properties of both amplitudes are deeply related. When an unique interpretation is adopted for the same mathematical structures appearing, very interesting aspects concerning the regularization procedures will emerge. The possible implications for the interpretation of the perturbative origin of violating terms in the AVV anomalous triangle is extensively discussed.

II. THE PROBLEM

In the Ref. [2] we have considered the $S \to VV$ process in a simple model where the vector field is coupled to a $1/2$ spin fermionic field. The corresponding amplitude can be written in the form

$$T_{\mu\nu}^{S\to VV} = T_{\mu\nu}^{SVV}\left(k_1,k_2,k_3\right) + T_{\nu\mu}^{SVV}\left(l_1,l_2,l_3\right), \tag{1}$$

where

$$T_{\mu\nu}^{SVV}\left(k_1,k_2,k_3\right) = \int \frac{d^4k}{(2\pi)^4} Tr\left\{\hat{1}\frac{1}{\not k+\not k_3-m}\gamma_\mu\frac{1}{\not k+\not k_1-m}\gamma_\nu\frac{1}{\not k+\not k_2-m}\right\}, \tag{2}$$

is the direct channel and $T_{\nu\mu}^{SVV}\left(l_1,l_2,l_3\right)$ is the crossed channel, required for bose final state symmetrization, and the set of momentum (k_i,l_i) are the (arbitrary) internal loop momentum.

Identities at the traces level can be used to identify relations among Green's functions

$$\begin{aligned}(k_3-k_1)^\mu T_{\mu\nu}^{S\to VV}(k_1,k_2,k_3) &= T_\nu^{VS}\left(k_2,k_1\right) - T_\nu^{VS}\left(k_2,k_3\right), &(3)\\ (k_1-k_2)^\nu T_{\mu\nu}^{S\to VV}(k_1,k_2,k_3) &= T_\mu^{VS}\left(k_2,k_3\right) - T_\mu^{VS}\left(k_1,k_3\right), &(4)\end{aligned}$$

which, for the $S\to VV$ process, imply in

$$\begin{aligned}p^\mu T_{\mu\nu}^{S\to VV} &= T_\nu^{VS}\left(k_2,k_1\right) - T_\nu^{VS}\left(k_2,k_3\right) + T_\nu^{VS}\left(l_2,l_3\right) - T_\nu^{VS}\left(l_1,l_3\right), &(5)\\ p'^\nu T_{\mu\nu}^{S\to VV} &= T_\mu^{VS}\left(k_2,k_3\right) - T_\mu^{VS}\left(k_1,k_3\right) + T_\mu^{VS}\left(l_2,l_1\right) - T_\mu^{VS}\left(l_2,l_3\right), &(6)\end{aligned}$$

where we have defined the two-point function structure

$$T_\mu^{VS}(k_1,k_2) = \int \frac{d^4k}{(2\pi)^4} Tr\left\{\gamma_\mu\frac{1}{\not k+\not k_1-m}\hat{1}\frac{1}{\not k+\not k_2-m}\right\}. \tag{7}$$

In the above expressions, the physical (external) momentum are related to the internal arbitrary ones as: $p=(k_3-k_1)=(l_1-l_2)$ and $p'=(k_1-k_2)=(l_3-l_1)$.

The Eqs. (5) and (6) means that when the amplitudes $T_{\mu\nu}^{SVV}$ and T_{ν}^{VS} are evaluated, in spite of the divergent character involved, within the context of a specific method, and the indicated contractions with the external momentum are taken, in the result it must be possible to identify the difference between the scalar-vector (SV) two-point functions having arbitrary internal momentum routing as indicated in the right hand side of the Eqs. (5) and (6). The Ward identities, on the other hand, implied by the vector current conservation of the adopted model, require the following properties for the calculated amplitudes

$$p^{\mu}T_{\mu\nu}^{S\to VV} = p'^{\nu}T_{\mu\nu}^{S\to VV} = 0. \qquad (8)$$

For the $A\to VV$ process, on the other hand, we have considered in Ref.[2] also a simple model where a pseudoscalar and an axial field, in addition to a vector one, are coupled to a $1/2$ spin fermionic field, such that the corresponding amplitude can be written as

$$T_{\lambda\mu\nu}^{A\to VV} = T_{\lambda\mu\nu}^{AVV}(k_1,k_2,k_3) + T_{\lambda\nu\mu}^{AVV}(l_1,l_2,l_3), \qquad (9)$$

where, the direct channel, $T_{\lambda\mu\nu}^{AVV}$ is defined as

$$T_{\lambda\mu\nu}^{AVV}(k_1,k_2,k_3) = \int \frac{d^4k}{(2\pi)^4} Tr\left\{ i\gamma_{\lambda}\gamma_5 \frac{1}{\not k+\not k_3-m}\gamma_{\mu}\frac{1}{\not k+\not k_1-m}\gamma_{\nu}\frac{1}{\not k+\not k_2-m}\right\}, \qquad (10)$$

and $T_{\lambda\nu\mu}^{AVV}(l_1,l_2,l_3)$ represents the crossed channel. Here, identities at the trace level can be used to identify the following relations among Green's functions

$$(k_3-k_2)^{\lambda}T_{\lambda\mu\nu}^{AVV} = -2imT_{\mu\nu}^{PVV} + T_{\mu\nu}^{AV}(k_1,k_2) - T_{\nu\mu}^{AV}(k_3,k_1), \qquad (11)$$

$$(k_3-k_1)^{\mu}T_{\lambda\mu\nu}^{AVV} = T_{\lambda\nu}^{AV}(k_1,k_2) - T_{\lambda\nu}^{AV}(k_3,k_2), \qquad (12)$$

$$(k_1-k_2)^{\nu}T_{\lambda\mu\nu}^{AVV} = T_{\lambda\mu}^{AV}(k_3,k_2) - T_{\lambda\mu}^{AV}(k_3,k_1), \qquad (13)$$

and, consequently, for the $A\to VV$ process we get

$$\begin{aligned} q^{\lambda}T_{\lambda\mu\nu}^{A\to VV} &= -2imT_{\mu\nu}^{P\to VV} \\ &+ T_{\mu\nu}^{AV}(k_1,k_2) - T_{\nu\mu}^{AV}(k_3,k_1) + T_{\nu\mu}^{AV}(l_1,l_2) - T_{\mu\nu}^{AV}(l_3,l_1), \qquad (14)\\ p^{\mu}T_{\lambda\mu\nu}^{A\to VV} &= T_{\lambda\nu}^{AV}(k_1,k_2) - T_{\lambda\nu}^{AV}(k_3,k_2) + T_{\lambda\nu}^{AV}(l_3,l_2) - T_{\lambda\nu}^{AV}(l_3,l_1), \qquad (15)\\ p'^{\nu}T_{\lambda\mu\nu}^{A\to VV} &= T_{\lambda\mu}^{AV}(k_3,k_2) - T_{\lambda\mu}^{AV}(k_3,k_1) + T_{\lambda\mu}^{AV}(l_1,l_2) - T_{\lambda\mu}^{AV}(l_3,l_2). \qquad (16)\end{aligned}$$

where $q = p+p'$. In the above expressions we have introduced also the Pseudoscalar-Vector-Vector (PVV) three-point function

$$T_{\mu\nu}^{PVV}(k_1,k_2,k_3) = \int \frac{d^4k}{(2\pi)^4} Tr\left\{ \gamma_5 \frac{1}{\not k+\not k_3-m}\gamma_{\mu}\frac{1}{\not k+\not k_1-m}\gamma_{\nu}\frac{1}{\not k+\not k_2-m}\right\}, \qquad (17)$$

which corresponds to the direct channel for the $P\to VV$ process as well as the Axial-Vector (AV) two-point function defined as

$$T_{\mu\nu}^{AV}(k_1,k_2) = \int \frac{d^4k}{(2\pi)^4} Tr\left\{ i\gamma_{\mu}\gamma_5 \frac{1}{\not k+\not k_1-m}\gamma_{\nu}\frac{1}{\not k+\not k_2-m}\right\}. \qquad (18)$$

The conservation of the vector current and the proportionality between the axial-vector and the pseudoscalar one, which are properties of the adopted model, state the following constraints for the calculated amplitude

$$p^{\mu}T_{\lambda\mu\nu}^{A\to VV} = p'^{\nu}T_{\lambda\mu\nu}^{A\to VV} = 0, \qquad (19)$$

$$q^{\lambda}T_{\lambda\mu\nu}^{A\to VV} = -2imT_{\mu\nu}^{P\to VV}. \qquad (20)$$

In order to satisfy, in a simultaneously way, the five Ward identities for the $S\to VV$ and $A\to VV$ process, the two-point structures, SV and AV, play crucial role. These mathematical structures have their own relations among other Green's functions of the perturbative calculation as well as their own Ward identities which means that, in evaluating them, some consistency constraints must be automatically maintained. In this line of reasoning we first note that these amplitudes are related through an unambiguous relation [1] which is,

$$T_{\mu\nu}^{AV}(k_1,k_2) = \frac{1}{2m}\varepsilon_{\mu\nu\alpha\beta}(k_1-k_2)^{\beta}\left(T^{\alpha}\right)^{VS}. \qquad (21)$$

The contractions with the external momentum $p' = k_1 - k_2$, on the other hand, can be put in terms of other Green's functions as we have done in the case of the considered three-point functions. The corresponding result for the SV structure is

$$(k_1 - k_2)^\mu T_\mu^{SV} = T^S(k_2) - T^S(k_1). \quad (22)$$

For the AV amplitude we get

$$(k_1 - k_2)^\mu T_{\mu\nu}^{AV} = -2imT_\nu^{PV} + T_\nu^A(k_2) - T_\nu^A(k_1), \quad (23)$$

$$(k_1 - k_2)^\nu T_{\mu\nu}^{AV} = T_\mu^A(k_2) - T_\mu^A(k_1). \quad (24)$$

In the above expressions we have introduced the scalar one-point function, defined as

$$T^S(k_1) = \int \frac{d^4k}{(2\pi)^4} Tr\left\{\frac{1}{\not k + \not k_1 - m}\right\}, \quad (25)$$

and the amplitudes T_ν^{PV} and T_μ^A which are defined in completely similar way like the previous considered ones. The conservation of the vector current in the model, on the other hand, implies the properties

$$(k_1 - k_2)^\mu T_\mu^{SV} = 0, \quad (26)$$

$$(k_1 - k_2)^\nu T_{\mu\nu}^{AV} = 0. \quad (27)$$

The proportionality between the axial and the pseudoscalar currents states

$$(k_1 - k_2)^\mu T_{\mu\nu}^{AV} = -2imT_\nu^{PV}. \quad (28)$$

At this point it is interesting to add some comments. There are three types of constraints to be considered for the amplitudes. The first is those what we have called relations among Green's functions which are consequence of the use of algebraic identities at the interior of the traces operation. The unique aspect related to the integration is the assumption of the linearity as a valid procedure. The second type is a relation between two amplitudes identified by comparing the expressions after the traces are taken like the relation (21) which allows us to relate an amplitude having an even number of γ_5 Dirac matrices with amplitudes possessing an odd number of such matrix. No specific properties involving the integration operation is involved in the derivation of such type of relations. The third kind of constraint is the one related to the symmetry properties (Ward identities) which are derived also without any mention to the eventual divergent character of the involved amplitudes. Then it is completely reasonable to expect that they must be preserved in a simultaneously way within the context of any consistent procedure adopted to evaluate all the involved amplitudes. The referred identities can be used, in this context, to test the consistency of the procedure used to regularize the divergences or to manipulate and calculate the divergent amplitudes. Having this in mind the next step we must perform is the evaluation of the amplitudes. For these purposes we first take the Dirac traces which allows us to put the amplitudes as a combination of Feynman integral. After this we treat the divergent integrals so obtained according to the adopted regularization method or equivalent philosophy. In the present contribution we adopt the calculational method described in Refs. [1–3, 9, 10] whose original reference is [4]. According to this strategy the Feynman integrals involved in the one-point function becomes

$$I_1 = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k+k_1)^2 - m^2} = I_{quad}(m^2) + k_1^\alpha k_1^\beta \triangle_{\alpha\beta}. \quad (29)$$

On the other hand, those involved in two-point functions evaluation become

$$I_2 = \int \frac{d^4k}{(2\pi)^4} \frac{1}{\left[(k+k_1)^2 - m^2\right]\left[(k+k_2)^2 - m^2\right]}$$

$$= \left[I_{log}(m^2)\right] - \left(\frac{i}{16\pi^2}\right)\left[Z_0\left((k_2 - k_1)^2; m^2\right)\right], \quad (30)$$

$$(I_2)_\mu = \int \frac{d^4k}{(2\pi)^4} \frac{k_\mu}{\left[(k+k_1)^2 - m^2\right]\left[(k+k_2)^2 - m^2\right]}$$

$$= -\frac{1}{2}(k_1 + k_2)^\xi (\Delta_{\xi\mu}) - \frac{1}{2}(k_1 + k_2)_\mu (I_2), \quad (31)$$

where we have introduced (in shorthand notation) the two-point function structures [11]

$$Z_k(\lambda_1^2, \lambda_2^2, q^2; \lambda^2) = \int_0^1 dz\, z^k ln\left(\frac{q^2 z(1-z) + (\lambda_1^2 - \lambda_2^2)z - \lambda_1^2}{-\lambda^2}\right), \quad (32)$$

and the basic divergent objects

$$\Delta_{\mu\nu} = \int_\Lambda \frac{d^4k}{(2\pi)^4} \frac{4k_\mu k_\nu}{(k^2 - m^2)^3} - \int_\Lambda \frac{d^4k}{(2\pi)^4} \frac{g_{\mu\nu}}{(k^2 - m^2)^2}, \quad (33)$$

$$I_{log}(m^2) = \int_\Lambda \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - m^2)^2}, \quad (34)$$

$$I_{quad}(m^2) = \int_\Lambda \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - m^2)}. \quad (35)$$

With these ingredients we can write the results for the amplitudes

$$T^S(k_1) = 4m\left\{\left[I_{quad}(m^2)\right] + k_1^\beta k_1^\alpha \left(\triangle_{\beta\alpha}\right)\right\}, \quad (36)$$

$$T_\mu^{VS}(k_1, k_2) = -4m(k_1 + k_2)^\beta \left(\triangle_{\beta\mu}\right), \quad (37)$$

$$T_{\mu\nu}^{AV}(k_1, k_2) = -2\varepsilon_{\mu\nu\alpha\beta}(k_1 - k_2)^\beta (k_1 + k_2)^\xi \left(\triangle_\xi^{\ \alpha}\right). \quad (38)$$

It is now easy to verify all the considered constraints involving these amplitudes. The relation (21) is immediate as well as the relations (22)-(24). The symmetry properties (26) and (27), on the other hand are not automatically satisfied. They suggest that there is only one reasonable way to maintain the Ward identities which is the consistency requirement (for detail see please Refs. [1–3])

$$\Delta_{\mu\nu}^{reg} = 0. \quad (39)$$

The consequences of such imposition are

$$T^{AV}_{\mu\nu} = T^{VS}_{\mu} = 0. \tag{40}$$

Note that these results are consistent with the calculations made within the context of DR where T^{SV}_{μ} is obtained identically vanishing. Then since the relation (21) is unambiguous, the unique consistent result is $T^{AV}_{\mu\nu} = 0$. The important aspect for the present discussion is that, the Ward identities for the processes $A \to VV$ and $S \to VV$, which involve three-point functions, become

$$\begin{aligned} p^{\mu} T^{A\to VV}_{\lambda\mu\nu} &= p'^{\nu} T^{A\to VV}_{\lambda\mu\nu} = 0, & (41) \\ q^{\lambda} T^{A\to VV}_{\lambda\mu\nu} &= -2imT^{P\to VV}_{\mu\nu}, & (42) \end{aligned}$$

and

$$p^{\mu} T^{S\to VV}_{\mu\nu} = p'^{\nu} T^{S\to VV}_{\mu\nu} = 0, \tag{43}$$

as a consequence of the consistency relation (CR) (39).

The CR (39) is a necessary condition to the preservation of the vector currents associated to the vector Lorentz indexes in the $S \to VV$ process. However, in adopting this point of view as universal, which means to apply the same procedure to the evaluation of all amplitudes in all theories and models we apparently have no violation in the Ward identities associated to the amplitude corresponding to the $A \to VV$ process which, as it is well known, is anomalous. This situation is completely different of that which we denominate the traditional procedure where the result (38) is substituted in the Eqs. (11), (12) and (13) in order to obtain

$$\begin{aligned} (k_3-k_2)^{\lambda} T^{AVV}_{\lambda\mu\nu} &= -2im[T^{PVV}_{\mu\nu}] \\ &\quad - 2\varepsilon_{\mu\nu\alpha\beta}\left[(k_1-k_2)^{\beta}(k_1+k_2)^{\xi} + (k_3-k_1)^{\beta}(k_1+k_3)^{\xi}\right]\left(\triangle^{\alpha}_{\xi}\right), \end{aligned} \tag{44}$$

$$(k_3-k_1)^{\mu} T^{AVV}_{\lambda\mu\nu} = -2\varepsilon_{\lambda\nu\alpha\beta}\left[(k_1-k_2)^{\beta}(k_1+k_2)^{\xi} - (k_3-k_2)^{\beta}(k_2+k_3)^{\xi}\right]\left(\triangle^{\alpha}_{\xi}\right), \tag{45}$$

$$(k_1-k_2)^{\nu} T^{AVV}_{\lambda\mu\nu} = 2\varepsilon_{\lambda\mu\alpha\beta}\left[(k_3-k_1)^{\beta}(k_1+k_3)^{\xi} - (k_3-k_2)^{\beta}(k_2+k_3)^{\xi}\right]\left(\triangle^{\alpha}_{\xi}\right). \tag{46}$$

Now, the object Δ is interpreted as a surface's term assuming the value

$$\Delta_{\mu\nu} = -\frac{i g_{\mu\nu}}{32\pi^2}, \tag{47}$$

which allows the surviving of the ambiguous combination of the loop (arbitrary) internal momenta (for details please see Refs. [1–3]). The nonphysical pieces are then parametrized in terms of the external momenta, for example

$$\begin{cases} k_1 = ap' + bp \\ k_2 = bp + (a-1)p' \\ k_3 = ap' + (b+1)p, \end{cases} \tag{48}$$

and

$$\begin{cases} l_1 = cp + dp' \\ l_2 = dp' + (c-1)p \\ l_3 = cp + (d+1)p'. \end{cases} \tag{49}$$

The arbitrary parameters a, b, c, and d are then chosen in order to satisfy the vector Ward identities and the axial is obtained violated since there is no choice which allows the preservation of all symmetry relations simultaneously. Clearly this procedure make use of a non-zero value for the AV (and SV) structure. If the same value for the object Δ is adopted the expressions (5) and (6) for the Ward identities corresponding to the $S \to VV$ process become

$$\begin{aligned} p^{\mu} T^{S\to VV}_{\mu\nu} &= 4m(k_3-k_1)^{\alpha}(\triangle_{\alpha\nu}) + 4m(l_1-l_2)^{\alpha}(\triangle_{\alpha\nu}), \\ &= 8mp^{\alpha}(\triangle_{\alpha\nu}), \end{aligned} \tag{50}$$

$$\begin{aligned} p'^{\nu} T^{S\to VV}_{\mu\nu} &= 4m(l_3-l_1)^{\alpha}(\triangle_{\alpha\mu}) + 4m(k_1-k_2)^{\alpha}(\triangle_{\alpha\mu}), \\ &= 8mp'^{\alpha}(\triangle_{\alpha\mu}), \end{aligned} \tag{51}$$

which imply in the violation of both symmetry relations since no ambiguous combinations involving the internal loop momentum appear to be chosen in a posterior step.

So, given the above discussion, we have arrived at the point: apparently if a universal procedure is adopted no reasonable situation is achieved. If the CR (39) is adopted the desirable consistency is obtained for all involved amplitudes but it seems to forbidden any violation in Ward identities including the anomalous amplitudes. If the interpretation for the object Δ is the one usually adopted in the perturbative description of the AVV triangle anomaly, adopting (47), we get a freedom to make convenient choices in the anomalous amplitudes but we will lead to a very large lack of violations in symmetry relations in non anomalous amplitudes.

In what follows we show that is possible to adopt an universal point of view for the perturbative (divergent) amplitudes obtaining symmetry preservation in all non anomalous physical amplitudes and a consistent description of anomalies in a natural way.

III. EXPLICIT EVALUATION OF THE THREE-POINT FUNCTIONS

In view of the arguments put in the preceding section we will explicitly calculate the SVV and AVV triangles within our calculational strategy. All the calculations will be performed in the most general way, adopting arbitrary choices for the internal lines momenta and maintaining the external lines off the mass shell. In a posterior step the Ward identities will be verified in an exact and completely algebraic way. We start by the SVV triangle, which was defined in Eq. (2). After performing the Dirac traces the result can be written as

$$\begin{aligned} T_{\mu\nu}^{SVV} &= 4m\int\frac{d^4k}{(2\pi)^4}\{(k+k_1)_\mu(k+k_2)_\nu+(k+k_1)_\nu(k+k_2)_\mu \\ &\quad +(k+k_1)_\mu(k+k_3)_\nu+(k+k_1)_\nu(k+k_3)_\mu \\ &\quad +(k+k_2)_\nu(k+k_3)_\mu-(k+k_2)_\mu(k+k_3)_\nu\} \\ &\quad \times\frac{1}{\left[(k+k_1)^2-m^2\right]\left[(k+k_2)^2-m^2\right]\left[(k+k_3)^2-m^2\right]} \\ &+ g_{\mu\nu}\left[T^{SPP}\right], \end{aligned} \tag{52}$$

where we have defined the three-point function

$$T^{SPP}=\int\frac{d^4k}{(2\pi)^4}Tr\left\{\widehat{1}\frac{1}{\not k+\not k_3-m}\gamma_5\frac{1}{\not k+\not k_1-m}\gamma_5\frac{1}{\not k+\not k_2-m}\right\}. \tag{53}$$

In the present discussion the above decomposition does not play an important role. It serves, however, to show, one more time, that the Green's functions of the perturbative calculations are always related.

The next step is to identify the set of Feynman integrals we need to evaluate in order to complete the calculation, which can be represented as

$$\left(I_3;I_3^\mu;I_3^{\mu\nu}\right)=\int\frac{d^4k}{(2\pi)^4}\frac{(1;k^\mu;k^\mu k^\nu)}{\left[(k+k_1)^2-m^2\right]\left[(k+k_2)^2-m^2\right]\left[(k+k_3)^2-m^2\right]}. \tag{54}$$

The most severe degree of divergence is the logarithmic one, as it should be expected. Solving the integrals according to our prescription, we get

$$(I_3) = \left(\frac{i}{16\pi^2}\right)(\xi_{00}), \tag{55}$$

$$(I_3)_\mu = \left(\frac{i}{16\pi^2}\right)\{-(k_3-k_1)_\mu(\xi_{10})-(k_2-k_1)_\mu(\xi_{01})\}-k_{1\mu}(I_3), \tag{56}$$

$$\begin{aligned} (I_3)_{\mu\nu} &= \left(\frac{i}{16\pi^2}\right)\left\{\frac{g_{\mu\nu}}{4}\left[I_{\log}\left(m^2\right)\right]+\frac{1}{4}(\Delta_{\mu\nu})-\frac{g_{\mu\nu}}{2}(\eta_{00})\right. \\ &\quad +(k_2-k_1)_\mu(k_2-k_1)_\nu(\xi_{02})+(k_3-k_1)_\mu(k_3-k_1)_\nu(\xi_{20}) \\ &\quad \left.+(k_2-k_1)_\mu(k_3-k_1)_\nu(\xi_{11})+(k_2-k_1)_\nu(k_3-k_1)_\mu(\xi_{11})\right\} \\ &- k_{1\mu}(I_3)_\nu-k_{1\nu}(I_3)_\mu+k_{1\nu}k_{1\mu}(I_3), \end{aligned} \tag{57}$$

where we have introduced (in shorthand notation) the three-point function structures

$$\xi_{nm}\left(p,p'\right)=\int_0^1 dz\int_0^{1-z} dy\frac{z^n y^m}{Q\left(p,y;p',z\right)}, \tag{58}$$

with $Q\left(p,y;p',z\right)=p^2y(1-y)+p'^2z(1-z)+2\left(p\cdot p'\right)yz-m^2$, and

$$\begin{aligned}\eta_{00}\left(p,p';m^2\right) &= \frac{1}{2}\left[Z_0(\left(p+p'\right)^2;m^2)\right]-\left[\frac{1}{2}+m^2\left(\xi_{00}\right)\right]\\ &+\frac{1}{2}p'^2\left(\xi_{10}\right)+\frac{1}{2}p^2\left(\xi_{01}\right).\end{aligned} \tag{59}$$

Inserting the results for the integrals, Eqs.(30), (31), (55), (56) and (57), we get

$$\begin{aligned}T^{SVV}_{\mu\nu} &= 4m\,g_{\mu\nu}\left[I_{\log}\left(m^2\right)\right]+4m\left(\Delta_{\mu\nu}\right)\\ &+\frac{im}{4\pi^2}\Big\{g_{\mu\nu}\left(-2\eta_{00}\right)+\left(k_2-k_1\right)_\mu\left(k_2-k_1\right)_\nu\left(4\xi_{02}-2\xi_{01}\right)\\ &+\left(k_3-k_1\right)_\mu\left(k_3-k_1\right)_\nu\left(4\xi_{20}-2\xi_{10}\right)\\ &+\left(k_3-k_1\right)_\nu\left(k_2-k_1\right)_\mu\left(4\xi_{11}-\xi_{00}\right)\\ &+\left(k_3-k_1\right)_\mu\left(k_2-k_1\right)_\nu\left(4\xi_{11}+\xi_{00}-2\xi_{01}-2\xi_{10}\right)\Big\}\\ &+g_{\mu\nu}\left[T^{SPP}\right],\end{aligned} \tag{60}$$

where

$$\begin{aligned}T^{SPP} &= 4m\Big\{-\left[I_{\log}(m^2)\right]+\frac{i}{16\pi^2}\left[Z_0\left(\left(k_3-k_2\right)^2;m^2\right)\right]\\ &-\frac{i}{32\pi^2}\left[\left(k_3-k_2\right)^2-\left(k_3-k_1\right)^2-\left(k_2-k_1\right)^2\right]\left(\xi_{00}\right)\Big\}.\end{aligned} \tag{61}$$

To arrive at these results the identity

$$\left(k+k_i\right)\cdot\left(k+k_j\right)=\frac{1}{2}\left[\left(k+k_i\right)^2-m^2\right]+\frac{1}{2}\left[\left(k+k_j\right)^2-m^2\right]+\frac{1}{2}\left[2m^2-\left(k_i-k_j\right)^2\right]. \tag{62}$$

has been used in the T^{SPP} term.

Let us now consider the AVV triangle, using rigorously the same steps and results. First we evaluate the AVV amplitude defined in the Eq.(10), which after performing the traces we write

$$T^{AVV}_{\lambda\mu\nu}=-4\left\{-F_{\lambda\mu\nu}+N_{\lambda\mu\nu}+M_{\lambda\mu\nu}+P_{\lambda\mu\nu}\right\}, \tag{63}$$

where we have introduced the definitions

$$P_{\lambda\mu\nu}=g_{\mu\nu}\varepsilon_{\alpha\beta\lambda\xi}\int\frac{d^4k}{(2\pi)^4}\frac{(k+k_1)^\alpha(k+k_2)^\beta(k+k_3)^\xi}{[(k+k_1)^2-m^2][(k+k_2)^2-m^2][(k+k_3)^2-m^2]}, \tag{64}$$

$$\begin{aligned}F_{\lambda\mu\nu} &= \int\frac{d^4k}{\left(2\pi\right)^4}\{\varepsilon_{\nu\beta\lambda\xi}\left(k+k_1\right)_\mu\left(k+k_2\right)^\beta\left(k+k_3\right)^\xi\\ &+\varepsilon_{\mu\beta\lambda\xi}\left(k+k_1\right)_\nu\left(k+k_2\right)^\beta\left(k+k_3\right)^\xi\\ &+\varepsilon_{\mu\alpha\nu\beta}\left(k+k_1\right)^\alpha\left(k+k_2\right)^\beta\left(k+k_3\right)_\lambda\\ &+\varepsilon_{\mu\alpha\nu\xi}\left(k+k_1\right)^\alpha\left(k+k_3\right)^\xi\left(k+k_2\right)_\lambda\Big\}\\ &\times\frac{1}{\left[\left(k+k_1\right)^2-m^2\right]\left[\left(k+k_2\right)^2-m^2\right]\left[\left(k+k_3\right)^2-m^2\right]},\end{aligned} \tag{65}$$

$$\begin{aligned} N_{\lambda\mu\nu} &= \frac{\varepsilon_{\mu\alpha\nu\lambda}}{2}\left\{\int \frac{d^4k}{(2\pi)^4}\frac{(k+k_1)^\alpha}{[(k+k_2)^2-m^2][(k+k_1)^2-m^2]}\right. \\ &\quad +\int \frac{d^4k}{(2\pi)^4}\frac{(k+k_1)^\alpha}{[(k+k_1)^2-m^2][(k+k_3)^2-m^2]} \\ &\quad +[2m^2-(k_3-k_2)^2] \\ &\quad \left.\times\int \frac{d^4k}{(2\pi)^4}\frac{(k+k_1)^\alpha}{[(k+k_1)^2-m^2][(k+k_2)^2-m^2][(k+k_3)^2-m^2]}\right\}, \end{aligned} \tag{66}$$

$$M_{\lambda\mu\nu} = m^2\varepsilon_{\mu\nu\alpha\lambda}\int \frac{d^4k}{(2\pi)^4}\frac{(k+k_2)^\alpha-(k+k_1)^\alpha+(k+k_3)^\alpha}{\left[(k+k_1)^2-m^2\right]\left[(k+k_2)^2-m^2\right]\left[(k+k_3)^2-m^2\right]}. \tag{67}$$

The most severe degree of divergence the linear one is contained in the $N_{\lambda\mu\nu}$ term. The performed reorganization came from the use of the identity (62). Using the results (30), (31), (55), (56) and (57) for the involved integrals we get

$$P_{\lambda\mu\nu} = 0, \tag{68}$$

$$M_{\lambda\mu\nu} = -\left(\frac{i}{16\pi^2}\right)\varepsilon_{\mu\alpha\nu\lambda}m^2\left\{(k_2-k_1)^\alpha(\xi_{00}-\xi_{01})+(k_3-k_1)^\alpha(\xi_{00}-\xi_{10})\right\}, \tag{69}$$

$$\begin{aligned} N_{\lambda\mu\nu} &= \frac{1}{4}\varepsilon_{\mu\alpha\nu\lambda}(k_2-k_1)^\alpha\left\{-\left[I_{\log}(m^2)\right]+\left(\frac{i}{16\pi^2}\right)\left[Z_0\left((k_1-k_2)^2;m^2\right)\right]\right. \\ &\qquad \left.-\left(\frac{i}{16\pi^2}\right)\left[2m^2-(k_3-k_2)^2\right](2\xi_{01})\right\} \\ &+ \frac{1}{4}\varepsilon_{\mu\alpha\nu\lambda}(k_3-k_1)^\alpha\left\{-\left[I_{\log}(m^2)\right]+\left(\frac{i}{16\pi^2}\right)\left[Z_0\left((k_1-k_3)^2;m^2\right)\right]\right. \\ &\qquad \left.-\left(\frac{i}{16\pi^2}\right)\left[2m^2-(k_3-k_2)^2\right](2\xi_{10})\right\} \\ &- \frac{1}{4}\varepsilon_{\mu\alpha\nu\lambda}\left[(k_1+k_2)^\beta+(k_3+k_1)^\beta\right]\left(\Delta^\alpha_{\ \beta}\right), \end{aligned} \tag{70}$$

$$\begin{aligned} F_{\lambda\mu\nu} &= \left(\frac{i}{16\pi^2}\right)(k_3-k_1)^\xi(k_2-k_1)^\beta\left\{\varepsilon_{\nu\beta\lambda\xi}[(k_2-k_1)_\mu(\xi_{02}+\xi_{11}-\xi_{01})\right. \\ &\qquad +(k_3-k_1)_\mu(\xi_{20}+\xi_{11}-\xi_{10})] \\ &\qquad +\varepsilon_{\mu\beta\lambda\xi}[(k_2-k_1)_\nu(\xi_{02}+\xi_{11}-\xi_{01}) \\ &\qquad +(k_3-k_1)_\nu(\xi_{20}+\xi_{11}-\xi_{10})] \\ &\qquad +\varepsilon_{\mu\beta\nu\xi}[(k_3-k_1)_\lambda(\xi_{11}-\xi_{20}+\xi_{10}) \\ &\qquad \left.+(k_2-k_1)_\lambda(\xi_{02}-\xi_{11}-\xi_{01})]\right\} \\ &- \frac{1}{4}\varepsilon_{\mu\nu\lambda\xi}\left[(k_3-k_1)^\xi+(k_2-k_1)^\xi\right]\left[\left[I_{\log}(m^2)\right]+\left(\frac{i}{16\pi^2}\right)(-2\eta_{00})\right] \\ &+ \frac{1}{4}\varepsilon_{\nu\beta\lambda\sigma}(k_2-k_3)^\beta\left(\Delta^\sigma_{\ \mu}\right)+\frac{1}{4}\varepsilon_{\mu\beta\lambda\sigma}(k_2-k_3)^\beta\left(\Delta^\sigma_{\ \nu}\right)+ \\ &+ \frac{1}{4}\varepsilon_{\mu\sigma\nu\beta}\left[(k_2-k_1)^\beta+(k_3-k_1)^\beta\right]\left(\Delta^\sigma_{\ \lambda}\right). \end{aligned} \tag{71}$$

The complete solution to AVV can be written in the following form

$$
\begin{aligned}
T_{\lambda\mu\nu}^{AVV} = & \left(\frac{i}{4\pi^2}\right)(k_3-k_1)^{\xi}(k_1-k_2)^{\beta}\Big\{\varepsilon_{\nu\lambda\beta\xi}[(k_3-k_1)_\mu(\xi_{20}+\xi_{11}-\xi_{10}) \\
& \qquad +(k_2-k_1)_\mu(\xi_{11}+\xi_{02}-\xi_{01})] \\
& \qquad +\varepsilon_{\mu\lambda\beta\xi}[(k_3-k_1)_\nu(\xi_{11}+\xi_{20}-\xi_{10}) \\
& \qquad +(k_2-k_1)_\nu(\xi_{02}+\xi_{11}-\xi_{01})] \\
& \qquad +\varepsilon_{\mu\nu\beta\xi}[(k_3-k_1)_\lambda(\xi_{11}-\xi_{20}+\xi_{10}) \\
& \qquad +(k_2-k_1)_\lambda(\xi_{02}-\xi_{01}-\xi_{11})]\Big\} \\
& -\left(\frac{i}{16\pi^2}\right)\varepsilon_{\mu\nu\lambda\beta}(k_3-k_1)^{\beta}\Big\{\left[Z_0\left((k_1-k_3)^2;m^2\right)\right]-\left[Z_0\left((k_2-k_3)^2;m^2\right)\right] \\
& \qquad +\left[2(k_3-k_2)^2-(k_3-k_1)^2\right](\xi_{10}) \\
& \qquad -(k_2-k_1)^2(\xi_{01})+\left[1-2m^2(\xi_{00})\right]\Big\} \\
& -\left(\frac{i}{16\pi^2}\right)\varepsilon_{\mu\nu\lambda\beta}(k_2-k_1)^{\beta}\Big\{\left[Z_0\left((k_1-k_2)^2;m^2\right)\right]-\left[Z_0\left((k_2-k_3)^2;m^2\right)\right] \\
& \qquad +\left[2(k_3-k_2)^2-(k_1-k_2)^2\right](\xi_{01}) \\
& \qquad -(k_3-k_1)^2(\xi_{10})+\left[1-2m^2(\xi_{00})\right]\Big\} \\
& +\varepsilon_{\mu\nu\beta\sigma}\left[(k_2-k_1)^{\beta}+(k_3-k_1)^{\beta}\right]\left(\Delta^{\sigma}{}_{\lambda}\right) \\
& -\varepsilon_{\nu\lambda\beta\sigma}(k_2-k_3)^{\beta}\left(\Delta^{\sigma}{}_{\mu}\right)+\frac{\varepsilon_{\mu\lambda\beta\sigma}}{4}(k_2-k_3)^{\beta}\left(\Delta^{\sigma}{}_{\nu}\right) \\
& +\varepsilon_{\mu\nu\lambda\alpha}\left[(k_1+k_2)^{\beta}+(k_3+k_1)^{\beta}\right]\left(\Delta^{\alpha}{}_{\beta}\right). \qquad (72)
\end{aligned}
$$

Note the presence of the potentially ambiguous term, the last one in the above expression. The last ingredient involved in the Ward identities that we want to verify is the PVV triangle, defined in the Eq. (17), which, after the traces calculations and using the results (55) and (56) in order to complete the calculation, becomes

$$T_{\mu\nu}^{PVV}=\left(\frac{1}{4\pi^2}\right)m\varepsilon_{\mu\nu\alpha\beta}(k_2-k_1)^{\alpha}(k_3-k_1)^{\beta}(\xi_{00}). \qquad (73)$$

With these ingredients we have arrived at the position where the Ward identities for the AVV and SVV triangle amplitudes can be verified.

IV. WARD IDENTITIES VERIFICATION FOR THE THREE-POINT FUNCTIONS

In the preceding section we have derived two properties for the SVV triangle; Eqs. (5) and (6). It is time to ask : Is the obtained result for the explicit calculation, a reasonable one concerning such properties? In order to verify such relations we need to evaluate the contractions of the amplitude with the external momenta. Such verification requires a reasonable algebraic effort. However, a considerable simplification can be obtained if we note some properties of the $\xi_{nm}(p,p')$ and $Z_k\left(p^2;m\right)$ functions. They are

$$p^2(\xi_{11}) - (p\cdot p')(\xi_{02}) = \frac{1}{2}\left\{-\frac{1}{2}Z_0\left(q^2;m^2\right) + \frac{1}{2}Z_0\left(p'^2;m^2\right) + p^2(\xi_{01})\right\}, \tag{74}$$

$$p^2(\xi_{20}) - (p\cdot p')(\xi_{11}) = \frac{1}{2}\left\{-\left[\frac{1}{2} + m^2(\xi_{00})\right] + \frac{p'^2}{2}(\xi_{01}) + \frac{3p^2}{2}(\xi_{10})\right\}, \tag{75}$$

$$p'^2(\xi_{02}) - (p\cdot p')(\xi_{11}) = \frac{1}{2}\left\{-\left[\frac{1}{2} + m^2(\xi_{00})\right] + \frac{p^2}{2}(\xi_{10}) + \frac{3p'^2}{2}(\xi_{01})\right\}, \tag{76}$$

$$p'^2(\xi_{11}) - (p\cdot p')(\xi_{20}) = \frac{1}{2}\left\{-\frac{1}{2}Z_0\left(q^2;m^2\right) + \frac{1}{2}Z_0\left(p^2;m^2\right) + p'^2\xi_{10}\right\}, \tag{77}$$

$$p^2(\xi_{10}) - (p\cdot p')(\xi_{01}) = \frac{1}{2}\left\{-Z_0\left(q^2;m^2\right) + Z_0\left(p'^2;m^2\right) + p^2(\xi_{00})\right\}, \tag{78}$$

$$p'^2(\xi_{01}) - (p\cdot p')(\xi_{10}) = \frac{1}{2}\left\{-Z_0\left(q^2;m^2\right) + Z_0\left(p^2;m^2\right) + p'^2(\xi_{00})\right\}. \tag{79}$$

Having this in mind we first contract the SVV explicit expression, Eq.(60), with $(k_1-k_2)_\nu$ to get

$$\begin{aligned}
(k_1-k_2)^\nu T^{SVV}_{\mu\nu} &= \left(\frac{im}{4\pi^2}\right)(k_3-k_1)_\mu\Big\{-4\Big[(k_2-k_1)^2(\xi_{11}) - (k_1-k_2)\cdot(k_3-k_1)(\xi_{20}\Big])\\
&\qquad +2\Big[(k_1-k_2)^2(\xi_{01}) - (k_1-k_2)\cdot(k_3-k_1)(\xi_{10})\Big]\\
&\qquad +(k_2-k_1)^2(2\xi_{10}-\xi_{00})\Big\}\\
&+ \left(\frac{im}{4\pi^2}\right)(k_2-k_1)_\mu\Big\{-4\Big[(k_1-k_2)^2(\xi_{02}) - (k_1-k_2)\cdot(k_3-k_1)(\xi_{11})\Big]\\
&\qquad -(k_1-k_2)\cdot(k_3-k_1)(\xi_{00})\\
&\qquad +2(k_1-k_2)^2(\xi_{01}) + 2(\eta_{00})\Big\}\\
&+ 4m(k_1-k_2)_\mu\left[I_{\log}(m^2)\right] + 4m(k_1-k_2)^\nu\Delta_{\mu\nu}\\
&+ (k_1-k_2)^\mu\left[T^{SPP}\right],
\end{aligned} \tag{80}$$

where it becomes clear the useful character of the properties (74)-(79), which lead us to

$$\begin{aligned}
(k_1-k_2)^\nu T^{SVV}_{\mu\nu} &= 4m(k_1-k_2)_\mu\Big\{\left[I_{\log}(m^2)\right] - \left(\frac{i}{16\pi^2}\right)\left[Z_0\left((k_3-k_2)^2;m^2\right)\right]\\
&\qquad +\left(\frac{i}{16\pi^2}\right)(k_1-k_2)\cdot(k_3-k_1)(\xi_{00})\Big\}\\
&+ (k_1-k_2)_\mu\left[T^{SPP}\right]\\
&+ 4m(k_1-k_2)^\nu(\Delta_{\mu\nu}).
\end{aligned} \tag{81}$$

Looking at the SPP explicit expression, Eq. (61), this means that we get

$$(k_1-k_2)^\nu T^{SVV}_{\mu\nu} = 4m(k_1-k_2)^\alpha(\triangle_{\alpha\mu}), \tag{82}$$

which implies in the identification

$$(k_1-k_2)^\mu T^{SVV}_{\mu\nu} = T^{VS}_\nu(k_2,k_3) - T^{VS}_\nu(k_1,k_3). \tag{83}$$

Following essentially the same steps we can also verify the contraction with $(k_3-k_1)_\mu$:

$$\begin{aligned}(k_3-k_1)^\mu T^{SVV}_{\mu\nu} &= \left(\frac{im}{4\pi^2}\right)(k_3-k_1)_\nu\Big\{4\Big[(k_3-k_1)^2(\xi_{20})-(k_1-k_2)\cdot(k_3-k_1)(\xi_{11})\Big]\\
&\qquad +(k_1-k_2)\cdot(k_3-k_1)(\xi_{00})\\
&\qquad -2(k_3-k_1)^2(\xi_{10})-2(\eta_{00})\Big\}\\
&+\left(\frac{im}{4\pi^2}\right)(k_2-k_1)_\nu\Big\{-2\Big[(k_3-k_1)^2(\xi_{10})-(k_1-k_2)\cdot(k_3-k_1)(\xi_{01})\Big]\\
&\qquad +4\Big[(k_3-k_1)^2(\xi_{11})-(k_1-k_2)\cdot(k_3-k_1)(\xi_{02})\Big]\\
&\qquad +(k_3-k_1)^2(\xi_{00}-2\xi_{01})\Big\}\\
&+4m(k_3-k_1)_\nu\left[I_{\log}(m^2)\right]+(k_3-k_1)_\nu\left[T^{SPP}\right]\\
&+4m(k_3-k_1)^\alpha(\Delta_{\alpha\nu}),\end{aligned} \tag{84}$$

Considering the result (61) and the properties (74)-(79) we see that

$$(k_3-k_1)^\mu T^{SVV}_{\mu\nu}=4m(k_3-k_1)^\mu(\triangle_{\mu\nu}). \tag{85}$$

Comparing with (37), the above result means that

$$(k_3-k_1)^\mu T^{SVV}_{\mu\nu}=T^{VS}_\nu(k_1,k_2)-T^{VS}_\nu(k_3,k_2). \tag{86}$$

If we compare the Eqs. (83) and (86) with the results previously obtained, Eqs. (5) and (6), we see that the identities among the involved Green's functions are preserved before any assumptions about the arbitrariness. At this point only the correctness of the intermediary steps has been tested. In order to give a definite significance for the SVV amplitude, a definition for the $\triangle_{\mu\nu}$ piece, which is arbitrary, is required. This means to choose a regularization method or an equivalent philosophy. Such a choice must be taken using as a guide the symmetry properties of the amplitude. The requirement of the vector current conservation leaves no room for other choices than to select a regularization such that $\Delta^{reg}_{\mu\nu}=0$. As we have pointed out before, the adoption of the DR leads us to the desirable result once the required property is automatically fulfilled by the method.

Let us now consider the most interesting case of three-point functions concerning the symmetry properties maintenance. The contractions of the Eq. (72) with the external momenta, after the use of the identities (74)-(79), lead us to

$$\begin{aligned}(k_3-k_1)^\mu T^{AVV}_{\lambda\mu\nu} &= (k_3-k_1)^\mu \Gamma^{AVV}_{\lambda\mu\nu}\\
&+\left(\frac{i}{8\pi^2}\right)\varepsilon_{\nu\beta\lambda\xi}(k_3-k_1)^\xi(k_1-k_2)^\beta,\end{aligned} \tag{87}$$

$$\begin{aligned}(k_1-k_2)^\nu T^{AVV}_{\lambda\mu\nu} &= (k_1-k_2)^\nu \Gamma^{AVV}_{\lambda\mu\nu}\\
&-\left(\frac{i}{8\pi^2}\right)\varepsilon_{\mu\beta\lambda\xi}(k_3-k_1)^\xi(k_1-k_2)^\beta,\end{aligned} \tag{88}$$

$$\begin{aligned}(k_3-k_2)^\lambda T^{AVV}_{\lambda\mu\nu} &= (k_3-k_2)^\lambda \Gamma^{AVV}_{\lambda\mu\nu}\\
&-\left(\frac{i}{4\pi^2}\right)\varepsilon_{\mu\nu\xi\beta}(k_3-k_1)^\xi(k_1-k_2)^\beta\left[2m^2(\xi_{00})\right].\end{aligned} \tag{89}$$

where we have defined

$$\begin{aligned}\frac{\Gamma^{AVV}_{\lambda\mu\nu}}{4} &= \frac{1}{4}\varepsilon_{\mu\nu\beta\xi}\left[(k_2-k_1)^\beta+(k_3-k_1)^\beta\right]\left(\Delta^\xi_{\ \lambda}\right)\\
&-\frac{1}{4}\varepsilon_{\nu\lambda\beta\xi}(k_2-k_3)^\beta\left(\Delta^\xi_\mu\right)-\frac{1}{4}\varepsilon_{\mu\lambda\beta\xi}(k_2-k_3)^\beta\left(\Delta^\xi_{\ \nu}\right)\\
&+\frac{1}{4}\varepsilon_{\mu\nu\lambda\alpha}\left[(k_1+k_2)^\beta+(k_3+k_1)^\beta\right]\left(\Delta^\alpha_{\ \beta}\right).\end{aligned} \tag{90}$$

The above expressions can be rewritten in terms of other structures if we observe the Eq. (38) and Eq. (37). As an example, note that

$$(k_3-k_2)^{\lambda}\Gamma^{AVV}_{\lambda\mu\nu} = 2\varepsilon_{\mu\nu\alpha\beta}\left[(k_1-k_3)^{\beta}(k_1+k_3)^{\xi}+(k_2-k_1)^{\beta}(k_1+k_2)^{\xi}\right]\left(\triangle^{\alpha}_{\xi}\right), \tag{91}$$

which means that

$$(k_3-k_2)^{\lambda}\Gamma^{AVV}_{\lambda\mu\nu} = T^{AV}_{\mu\nu}(k_1,k_2)-T^{AV}_{\nu\mu}(k_3,k_1). \tag{92}$$

Again, all the arbitrariness involved are still present. The expression (72), and, in consequence, the Eqs. (87)-(89), are completely independent of the specific assumptions for the divergent character of the amplitude. The next step involves some arbitrary choices for the undefined quantities present in the expressions. What becomes evident at this point is that the relations (12) and (13) among the involved Green's functions have not been obtained satisfied. The terms which breaks the identities comes from finite contributions. The violations cannot be avoided by choosing properties for the divergent integrals. This seems to indicate that no regularization can be constructed in order to preserve the three identities simultaneously by the simple reason that the violating terms cannot be affected by the eventual properties of the regulating distribution. The main point is that the arbitrariness involved, the value for the $\triangle^{\alpha}_{\xi}$ piece and the ambiguous combinations of the internal lines momenta, do not play any relevant role in the establishment of the violations.

V. FINAL REMARKS AND CONCLUSIONS

Applying a very general calculational method, concerning the divergences treatment, we considered the aspects ambiguities and symmetry relations using for this purpose two well-known physical amplitudes; the *SVV* and *AVV* triangle amplitudes. From the point of view of traditional techniques, the first cited one can be treated within the scope of DR and therefore consistently handled. The second one is a γ_5-odd amplitude and the recourse of the DR is not available. However, from the point of view of the adopted strategy, both amplitudes can be equally treated and exhibit very similar aspects concerning the divergence character. This is due to the fact that in the adopted strategy it is avoided the explicit use of a regularization on the intermediary steps. The routing for the internal lines momenta is adopted as completely arbitrary. Only very general properties for an eventual regulating distribution are assumed in order to perform an adequate reorganization of the divergent Feynman integrals. The terms which present a dependence on the internal lines momenta are left only in finite integrals. The undefined parts of the amplitudes are organized in a small number of standard objects but, in fact, they are not calculated. With this procedure all the intrinsic arbitrariness of perturbative calculations remain still present in the final results. Due to this fact it becomes possible to map the obtained results into those corresponding to any traditional regularization philosophies. Two of such treatments are particularly important for the performed investigations: The DR and the explicit evaluation of surface's term. The DR is usually applied for the *SVV* treatment and the corresponding results can be easily recovered from ours. The surface's term evaluation is traditionally applied to the *AVV* amplitude in order to justify the perturbative origin of the triangle anomalies and the corresponding results can be equally recovered from our results. However, these two mappings are obtained through different interpretations for the undefined objects. Precisely this aspect allows us to get very interesting conclusions about the nature of the anomaly phenomenon. In order to make the conclusions clean and sound let us summarize the main steps.

First we stated identities relating the *SVV* and *AVV* external momentum contracted amplitudes with other two and three-point functions which are:

$$(k_3-k_1)^{\mu}T^{SVV}_{\mu\nu} = T^{VS}_{\nu}(k_2,k_1)-T^{VS}_{\nu}(k_2,k_3), \tag{93}$$

$$(k_1-k_2)^{\nu}T^{SVV}_{\mu\nu} = T^{VS}_{\mu}(k_2,k_3)-T^{VS}_{\mu}(k_1,k_3), \tag{94}$$

$$(k_3-k_2)^{\lambda}T^{AVV}_{\lambda\mu\nu} = -2imT^{PVV}_{\mu\nu}+T^{AV}_{\mu\nu}(k_1,k_2)-T^{AV}_{\nu\mu}(k_3,k_1), \tag{95}$$

$$(k_3-k_1)^{\mu}T^{AVV}_{\lambda\mu\nu} = T^{AV}_{\lambda\nu}(k_1,k_2)-T^{AV}_{\lambda\nu}(k_3,k_2), \tag{96}$$

$$(k_1-k_2)^{\nu}T^{AVV}_{\lambda\mu\nu} = T^{AV}_{\lambda\mu}(k_3,k_2)-T^{AV}_{\lambda\mu}(k_3,k_1). \tag{97}$$

The two-point functions that appeared on the right hand side of the above equation have been evaluated from the point of view of the adopted strategy leading to

$$T^{VS}_{\mu}(k_1,k_2) = -4m(k_1+k_2)^{\beta}\left(\triangle_{\beta\mu}\right), \tag{98}$$

$$T^{AV}_{\mu\nu}(k_1,k_2) = -2\varepsilon_{\mu\nu\alpha\beta}(k_1-k_2)^{\beta}(k_1+k_2)^{\xi}\left(\triangle^{\alpha}_{\xi}\right), \tag{99}$$

where it must be noted the relation between them

$$T^{AV}_{\mu\nu}(k_1,k_2) = \frac{1}{2m}\varepsilon_{\mu\nu\alpha\beta}(k_1-k_2)^{\beta}\left(T^{\alpha}\right)^{VS}. \tag{100}$$

which is non-ambiguous in spite of the potentially ambiguous character of both involved amplitudes. The above relation is valid at the traces level, i.e., independent of the calculational aspects. The expressions (98) and (99) are the most general ones for these mathematical structures. They preserve all the intrinsic arbitrariness involved, which are present in the ambiguous combinations of the internal lines momentum and in the undefined $\triangle_{\beta\mu}$ piece. It is clear that, in order to give a definite value for the amplitudes some (arbitrary) choices must be taken in to account. Which are then the best ones? The only guides we have are the eventual physical requirements to be imposed to the amplitudes in spite of the divergent character. Once no other result than the identically zero one is acceptable for the *SV* two-point function, which can be immediately obtained in the DR method, we have no option at our disposal than to select a regularization such that $\Delta_{\mu\nu}^{reg} = 0$, and in consequence $T_{\mu}^{VS} = T_{\mu\nu}^{AV} = 0$. Given this choice, the Eqs. (93)-(97) become

$$p^{\prime\nu} T_{\lambda\mu\nu}^{A\to VV} = p^{\mu} T_{\lambda\mu\nu}^{A\to VV} = 0, \tag{101}$$

$$q^{\lambda} T_{\lambda\mu\nu}^{A\to VV} = -2im T_{\mu\nu}^{P\to VV}, \tag{102}$$

and

$$p^{\prime\nu} T_{\mu\nu}^{S\to VV} = p^{\mu} T_{\mu\nu}^{S\to VV} = 0. \tag{103}$$

This is certainly what we wish to obtain for the last two equations but in the case of the AVV triangle amplitude at first sight it can be a trouble. This is due to the fact that, with this choice, the Eqs. (101)-(102) apparently imply that no violations in symmetry relations for the AVV amplitude may occur. At least this is what we should conclude if we follow the traditional line of reasoning for the perturbative justification of the AVV triangle anomaly. What should we make in order to reconcile the situation? At this point one can say that both considered problems must not necessarily be constrained. This means to adopt different choices for the indefinitions, i.e., in the treatment of the SVV problem we adopt $\Delta_{\mu\nu}^{reg} = 0$, obtaining the desirable result, while in the AVV amplitude we adopt $\Delta_{\mu\nu}^{reg} \neq 0$. It is precisely the present status of the problem: for the SVV amplitude we adopt the DR point of view ($\Delta_{\mu\nu}^{reg} = 0$) and for the AVV anomaly's justifications we adopt the surface's terms evaluation analysis ($\Delta_{\mu\nu}^{reg} \neq 0$). In some sense this option represents a very frustrating situation because both problems (and many others) are deeply related. The last sentence can be clearly viewed if we observe the Eqs. (93)-(97), (98)-(99) and the Eq. (100). They indicate that the potentially ambiguous terms present in the AVV structure are, in fact, SV two-point functions. This is immediately verified if we take the term between curly brackets on the left hand side of the Eq. (10) and perform the traces. The answer can be written in a similar form to the decomposition (63),

$$t_{\lambda\mu\nu}^{AVV} = -4\left\{-f_{\lambda\mu\nu} + n_{\lambda\mu\nu} + m_{\lambda\mu\nu} + p_{\lambda\mu\nu}\right\}, \tag{104}$$

where, after the integration, only $n_{\lambda\mu\nu}$ will acquire a linear divergence's degree. It is explicitly given by

$$n_{\lambda\mu\nu} = \varepsilon_{\mu\alpha\nu\lambda} \frac{(k+k_2)\cdot(k+k_3)(k+k_1)^{\alpha}}{[(k+k_1)^2-m^2]\,[(k+k_2)^2-m^2]\,[(k+k_3)^2-m^2]}, \tag{105}$$

which can be conveniently reorganized as

$$\begin{aligned} n_{\lambda\mu\nu} = {} & \frac{\varepsilon_{\mu\alpha\nu\lambda}}{4}\left\{\frac{2k^{\alpha}+(k_1+k_2)^{\alpha}}{[(k+k_1)^2-m^2]\,[(k+k_2)^2-m^2]} + \frac{2k^{\alpha}+(k_1+k_3)^{\alpha}}{[(k+k_1)^2-m^2]\,[(k+k_3)^2-m^2]}\right\} \\ & + \frac{\varepsilon_{\mu\alpha\nu\lambda}}{4}\left\{\frac{(k_1-k_2)^{\alpha}}{[(k+k_1)^2-m^2]\,[(k+k_2)^2-m^2]} + \frac{(k_1-k_3)^{\alpha}}{[(k+k_1)^2-m^2]\,[(k+k_3)^2-m^2]}\right. \\ & \left. + [2m^2-(k_2-k_3)^2]\frac{2(k+k_1)^{\alpha}}{[(k+k_1)^2-m^2]\,[(k+k_2)^2-m^2]\,[(k+k_3)^2-m^2]}\right\}. \end{aligned} \tag{106}$$

The first two terms contain now all the linear divergence and the ambiguous combination of the arbitrary internal lines momentum. Given the identity (62) it is expected that such terms are related to SV two-point functions. In fact, it is easy to verify that

$$4m\frac{2k_{\alpha}+(k_i+k_j)_{\alpha}}{[(k+k_i)^2-m^2]\,[(k+k_j)^2-m^2]} = Tr\left[\hat{1}\frac{1}{(\not{k}+\not{k}_i)-m}\gamma_{\alpha}\frac{1}{(\not{k}+\not{k}_j)-m}\right]. \tag{107}$$

After the integration in the momentum k, the right hand side can be identified with the SV two-point function defined in Eq. (7). The important aspect involved resides on the fact that all the undefined pieces present in the AVV amplitude are linked with the value of the SV physical amplitude. Consequently, we can make use of the eventual physical constraints, to be imposed on the SV amplitude, to guide us in taking the consistent choices for the corresponding arbitrariness present in the AVV amplitude. So, it seems not reasonable to assume the zero value for the SV mathematical structure when we are dealing with the SVV problem and, for the identical mathematical structures, to attribute a non-zero value when we are dealing with the AVV anomaly justification. After these important remarks, it become very clear that an alternative point of view for the problem must be searched. In order to construct it, we first note that the Eqs. (101)-(102) does not imply that the anomalies are eliminated by the choice $\Delta_{\mu\nu}^{reg} = 0$. In fact, the origin of the AVV triangle anomaly phenomenon is connected with the neutral electromagnetic pion decay through the Sutherland-Veltman paradox [12]. There are four properties to be satisfied by the AVV amplitude. Three of them are precisely the Ward identities which we considered and the fourth one is the kinematical limit: $\lim_{q_{\lambda}\to 0} q^{\lambda} T_{\lambda\mu\nu}^{A\to VV} = 0$, which is related to the pion decay. The manipulations performed until the Eqs. (101) - (102) do not allow us to verify simultaneously all four ingredients. Only the explicit calculation can reveal us all of them. So, following this point of view, we first have to explicit evaluate the three-point functions and after this to verify the Ward identities and the kinematical limit. The corresponding results are

$$
\begin{aligned}
(k_3-k_1)^{\mu}T^{AVV}_{\lambda\mu\nu} &= T^{AV}_{\lambda\nu}(k_1,k_2)-T^{AV}_{\lambda\nu}(k_3,k_2), \\
&+\left(\frac{i}{8\pi^2}\right)\varepsilon_{\nu\beta\lambda\xi}(k_3-k_1)^{\xi}(k_1-k_2)^{\beta}, && (108)\\
(k_1-k_2)^{\nu}T^{AVV}_{\lambda\mu\nu} &= T^{AV}_{\lambda\mu}(k_3,k_2)-T^{AV}_{\lambda\mu}(k_3,k_1) \\
&-\left(\frac{i}{8\pi^2}\right)\varepsilon_{\mu\beta\lambda\xi}(k_3-k_1)^{\xi}(k_1-k_2)^{\beta}, && (109)\\
(k_3-k_2)^{\lambda}T^{AVV}_{\lambda\mu\nu} &= T^{AV}_{\mu\nu}(k_1,k_2)-T^{AV}_{\nu\mu}(k_3,k_1) \\
&+\left(\frac{i}{4\pi^2}\right)\varepsilon_{\mu\xi\nu\beta}(k_3-k_1)^{\xi}(k_1-k_2)^{\beta}\left[2m^2\xi_{00}\right]. && (110)
\end{aligned}
$$

and

$$
\begin{aligned}
(k_3-k_1)^{\mu}T^{SVV}_{\mu\nu} &= T^{VS}_{\nu}(k_2,k_1)-T^{VS}_{\nu}(k_2,k_3), && (111)\\
(k_1-k_2)^{\nu}T^{SVV}_{\mu\nu} &= T^{VS}_{\mu}(k_2,k_3)-T^{VS}_{\mu}(k_1,k_3). && (112)
\end{aligned}
$$

Now, if we adopt the same point of view for both problems, i.e., $\Delta^{reg}_{\mu\nu}=0$, which implies in to assume $T^{SV}_{\mu}=T^{AV}_{\mu\nu}=0$, we get

$$
\begin{aligned}
(k_1-k_2)^{\nu}T^{SVV}_{\mu\nu} &= (k_3-k_1)^{\mu}T^{SVV}_{\mu\nu}=0, && (113)\\
(k_3-k_1)^{\mu}T^{AVV}_{\lambda\mu\nu} &= \left(\frac{i}{8\pi^2}\right)\varepsilon_{\nu\beta\lambda\xi}(k_3-k_1)^{\xi}(k_1-k_2)^{\beta}, && (114)\\
(k_1-k_2)^{\nu}T^{AVV}_{\lambda\mu\nu} &= -\left(\frac{i}{8\pi^2}\right)\varepsilon_{\mu\beta\lambda\xi}(k_3-k_1)^{\xi}(k_1-k_2)^{\beta}, && (115)\\
(k_3-k_2)^{\lambda}T^{AVV}_{\lambda\mu\nu} &= -2im\left[T^{PVV}_{\mu\nu}\right]. && (116)
\end{aligned}
$$

The inclusion of the crossed channel gives us

$$
\begin{aligned}
p^{\mu}T^{S\to VV}_{\mu\nu} &= p'^{\nu}T^{S\to VV}_{\mu\nu}=0, && (117)\\
p^{\mu}T^{A\to VV}_{\lambda\mu\nu} &= \left(\frac{i}{4\pi^2}\right)\varepsilon_{\nu\beta\lambda\xi}p^{\xi}p'^{\beta}, && (118)\\
p'^{\nu}T^{A\to VV}_{\lambda\mu\nu} &= -\left(\frac{i}{4\pi^2}\right)\varepsilon_{\mu\beta\lambda\xi}p^{\xi}p'^{\beta}, && (119)\\
q^{\lambda}T^{A\to VV}_{\lambda\mu\nu} &= -2im\left[T^{P\to VV}_{\mu\nu}\right]. && (120)
\end{aligned}
$$

Then it becomes clear what we mean about the problem. The Ward identities for the SVV amplitude are satisfied, as they should, and those corresponding to the AVV one present violations. The violating terms can be related to a kinematical situation of the AVV amplitude. We can write the Ward identities as

$$
\begin{aligned}
(k_3-k_1)^{\mu}T^{AVV}_{\lambda\mu\nu} &= (k_3-k_1)^{\mu}T^{AVV}_{\lambda\mu\nu}(0), && (121)\\
(k_2-k_1)^{\nu}T^{AVV}_{\lambda\mu\nu} &= (k_2-k_1)^{\nu}T^{AVV}_{\lambda\mu\nu}(0), && (122)\\
(k_3-k_2)_{\lambda}T^{AVV}_{\lambda\mu\nu} &= -2im\left[T^{PVV}_{\mu\nu}\right], && (123)
\end{aligned}
$$

where

$$
T^{AVV}_{\lambda\mu\nu}(0)=-\left(\frac{i}{8\pi^2}\right)\varepsilon_{\mu\nu\lambda\xi}\left[(k_3-k_1)^{\xi}-(k_1-k_2)^{\xi}\right], \quad (124)
$$

which is the anomalous term. The expressions above reflects now what we expect for the anomaly phenomenon. The calculated AVV amplitude violates three of the four symmetry properties. The fundamental character of the phenomenon resides in the fact that there is no dependence on the specific regularization we eventually want to use. The symmetry breaking involved represents a particular type of arbitrariness: once it is not possible to satisfy simultaneously all symmetry properties (in accordance to the Sutherland-Veltman paradox [12]), it is completely justifiable to take the best possible choice due to physical reasons. It is evident that we must to choose what is required by the phenomenology. This means to choose the expression for the AVV triangle with a correct low energy behavior or to make the redefinition

$$
\left(T^{A\to VV}_{\lambda\mu\nu}(p,p')\right)_{phys}=T^{A\to VV}_{\lambda\mu\nu}(p,p')-T^{A\to VV}_{\lambda\mu\nu}(0), \quad (125)
$$

where

$$T^{A\to VV}_{\lambda\mu\nu}(0) = -\left(\frac{i}{4\pi^2}\right)\varepsilon_{\mu\nu\lambda\xi}\left(p^{\xi} - p'^{\xi}\right).$$

Consequently, we get for the AVV physical amplitude $\left(T^{A\to VV}_{\lambda\mu\nu}\right)_{phy}$,

$$p'^{\nu}\left(T^{A\to VV}_{\lambda\mu\nu}\right)_{phy} = p^{\mu}\left(T^{A\to VV}_{\lambda\mu\nu}\right)_{phy} = 0, \tag{126}$$

$$q^{\lambda}\left(T^{A\to VV}_{\lambda\mu\nu}\right)_{phy} = -2im\left[T^{P\to VV}_{\mu\nu}\right] - \left(\frac{i}{2\pi^2}\right)\varepsilon_{\mu\nu\alpha\beta}p^{\alpha}p'^{\beta}. \tag{127}$$

Note that, with the redefinition imposed by phenomenological reasons, the vector Ward identities are now preserved in the AVV amplitude. This is different from the usual argumentation which is the redefinition of the calculated amplitude in order to recover the $U(1)$ gauge symmetry. The origin of the violations do not reside in ingredients which are exclusive of the perturbative approach as the divergences and ambiguities are. No regularization can avoid the violation of at least one of the four symmetry properties because the violating terms come from finite parts of the amplitude.

The most important points of the conclusions are: (a) Concerning the ambiguities: they cannot play any role in a consistent interpretation of perturbative physical amplitudes. Any result for a perturbative calculation which exhibit dependence on the arbitrariness involved cannot be taken seriously; (b) Concerning the regularizations it is possible to summarize all we can say about them in a simple sentence: if a method is consistent it is not necessary to take it explicitly in any place so that no role is left to be played also for these tools. These conclusions may be very important for the discussions of many problems and controversies stated recently in the literature where the ambiguities are called to play relevant role [10].

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