Phase Space Solutions in Scalar-Tensor Cosmological Models

José C. C. de Souza
Departamento de Física Matemática, Instituto de Física,
Universidade de São Paulo, C.P. 66318, 05315-970 São Paulo, SP, Brazil

and Alberto Saa
Departamento de Matemática Aplicada, IMECC – UNICAMP, C.P. 6065, 13083-859 Campinas, SP, Brazil
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An analysis of the solutions for the field equations of generalized scalar-tensor theories of gravitation is performed through the study of the geometry of the phase space and the stability of the solutions, with special interest in the Brans-Dicke model. Particularly, we believe to be possible to find suitable forms of the Brans-Dicke parameter ω and potential V of the scalar field, using the dynamical systems approach, in such a way that they can be fitted in the present observed scenario of the Universe.

I. SCALAR-TENSOR THEORIES OF GRAVITATION

In a homogeneous and isotropic space, described by the Friedmann-Lematre-Robertson-Walker metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-Kr^2} + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2)\right], \quad (1)$$

where a is the scale factor and K is the spatial curvature index, gravitation can be described by an action of the kind

$$S = \frac{1}{16\pi}\int \mathrm{d}^4\sqrt{-g}\left[\phi R - \frac{\omega(\phi)}{\phi}g^{ab}\nabla_a\phi\nabla_b\phi - V(\phi)\right] + S^m, \quad (2)$$

where S^m is the action of usual matter, g is the determinant of the metric tensor, ω is a coupling function (which we will eventually assume to be a constant, known as Brans-Dicke parameter) and $V(\phi)$ is the potential of the scalar field ϕ [2].

From (2), we obtain for the field equations:

$$H^2 = -H\left(\frac{\dot\phi}{\phi}\right) + \frac{\omega}{6}\left(\frac{\dot\phi}{\phi}\right)^2 + \frac{V(\phi)}{6\phi} - \frac{K}{a^2} + \frac{8\pi\rho^m}{3\phi}, \quad (3)$$

$$\begin{aligned}\dot H &= -\frac{\omega}{2}\left(\frac{\dot\phi}{\phi}\right)^2 + 2H\left(\frac{\dot\phi}{\phi}\right) \\ &\quad + \frac{1}{2(2\omega+3)\phi}\left[\phi\frac{\mathrm{d}V}{\mathrm{d}\phi} - 2V + \frac{\mathrm{d}\omega}{\mathrm{d}\phi}(\dot\phi)^2\right] \\ &\quad + \frac{K}{a^2} - \frac{8\pi}{(2\omega+3)\phi}[(\omega+2)\rho^m + \omega P^m], \quad (4)\end{aligned}$$

$$\begin{aligned}\ddot\phi + \left(3H + \frac{1}{2\omega+3}\frac{\mathrm{d}\omega}{\mathrm{d}\phi}\right)\dot\phi &= \frac{1}{2\omega+3}\left[2V - \phi\frac{\mathrm{d}V}{\mathrm{d}\phi} + \right. \\ &\quad \left. 8\pi(\rho^m - 3P^m)\right], \quad (5)\end{aligned}$$

where $H \equiv \dot a/a$ is the Hubble parameter and ρ^m and P^m are the energy density and the pressure of the material fluid.

As usual, we parameterize the equation of state for the fluid as $P^m = (\gamma-1)\rho^m$ with γ a constant chosen to indicate a variety of fluids that are predominantly responsible for the energy density of the Universe. We can see that through the energy conservation equation $\dot\rho^m + 3H(\rho^m + P^m) = 0$ we obtain $\rho^m = \rho_0/a^{3\gamma}$, with ρ_0 a constant.

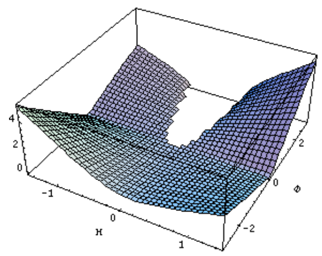

FIG. 1: Upper sheet of the of the phase space for a model with $\omega = 10$ (Brans-Dicke), corresponding to the positive sign in eq. (7) [2]. The "hole" in the surface indicates the region forbidden for the orbits of the solutions.

II. THE CASE FOR $V = \frac{1}{2}m^2\phi^2$ AND $K = 0$

In the referred paper [2], the author proceeds to show the phase space allowed for the orbits of solutions for these field equations in several cases with different potentials and parameters ω. For example, in the case of vacuum, flat space ($K = 0$) and potential $V = \frac{1}{2}m^2\phi^2$, equation (3) was rearranged as (making $m \equiv 1$)

$$\omega\dot\phi^2 - 6H\phi\dot\phi + (\frac{1}{2}\phi^2 - 6H^2\phi)\phi = 0, \quad (6)$$

which has the solutions

$$\dot\phi_\pm(H,\phi) = \frac{1}{\omega}\left[3H\phi \pm \sqrt{3(2\omega+3)H^2\phi^2 - \frac{1}{2}\omega\phi^2}\right]. \quad (7)$$

The assumption of flat space is required in order to reduce the dimensionality of the phase space.

We want to analyze qualitatively the geometry of the phase space $(H,\phi,\dot\phi)$, expecting to infer the form of the functions $\omega(\phi)$ and $V(\phi)$ to fit better the available data on the structure of the Universe.

The phase space for this situation is composed of a 2-d surface with two sheets, related to the lower and upper signs in eq. (7). Figures 1-3 show the phase space for the choice $\omega = 10$.

The fixed points for this dynamical system, obtained making $\dot H = \dot\phi = 0$, are de Sitter solutions, given by $H_0 = \pm\sqrt{\phi_0/12}$, with constant H_0 and ϕ_0.

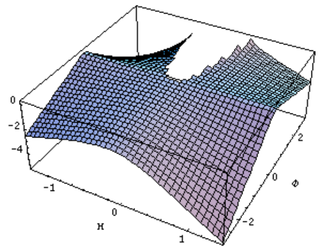

FIG. 2: Lower sheet of the phase space, now corresponding to the negative sign in eq. (7).

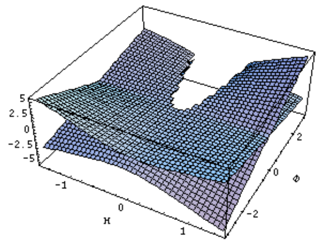

FIG. 3: The complete phase space composed of the upper and lower sheets linked to each other at the boundary of the forbidden region.

III. THE CASE FOR $V = \Lambda\phi$ AND $K = 0$

Following other works ([3]-[5]) which give a complete analysis of the phase space for Brans-Dicke model with a cosmological constant Λ (simply making $V(\phi) = \Lambda\phi$ in the action), we can illustrate the situation in which ω has a very large value and $\gamma = 0$. Therefore, the energy density of the fluid is a constant ρ_0. It should be emphasized that recent observational and simulation results seem to favor a scenario very similar to this one ([6],[11]-[14]). The solutions in this case are written as

$$\dot{\phi}_{\pm}(H,\phi) = \frac{1}{\omega}\left[3H\phi \pm \sqrt{9H^2\phi^2 - \omega[\phi^2(\Lambda - 6H^2) + 16\pi\rho_0]}\right]. \quad (8)$$

With this solutions, we are able to show the phase space for a particular choice of constants Λ, ω and ρ_0.

We proceed to find the dynamical equations system for this simple model, as done before.

Naming Δ the expression under the root in eq.(8), we can write the equation for $\dot{H}$:

$$\begin{aligned} \dot{H}_{\pm} &= -\frac{1}{2\omega\phi^2}\left[3H\phi \pm \sqrt{\Delta}\right]^2 + \frac{2H}{\omega}\left[3H \pm \frac{\sqrt{\Delta}}{\phi}\right] \\ &\quad -\frac{1}{2(2\omega+3)}\left(\frac{\Lambda}{2} + \frac{16\pi\rho_0}{\phi}\right). \end{aligned} \quad (9)$$

Now, equations (8) and (9) form the system for which the fixed points are the solutions $H_0 = \pm\sqrt{8\pi\rho_0/3\phi_0^2 + \Lambda/6}$.

Of special interest is the search for the most adequate functions $\omega(\phi)$ and $V(\phi)$, that may be more complicated than what was assumed until here.

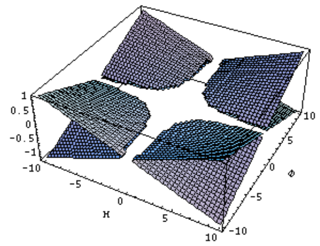

FIG. 4: Complete phase space for a Brans-Dicke model with a cosmological constant $\Lambda = 1$, energy density $\rho_0 = 2$ and constant parameter $\omega = 50000$, showing two sheets linked by the boundary of the forbidden region, as in the precedent case.

IV. CONCLUSIONS

The method of analyzing the geometry of the phase space have proved to be a useful tool in the search for the solutions of the field equations of generalized gravity models. Our aim is to achieve a complete analysis of the simple model presented before (including the stability of the solutions, via Lyapunov's direct method [1], in order to investigate further its *attraction basin*) and to apply more sophisticated functions to it.

V. ACKNOWLEDGMENTS

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