The Partial Revival of the Quadrature Variance Product in the Dynamics of the Squeezed Vacuum Field in the Jaynes-Cummings Model

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Received February 4, 1997

We consider the time evolution of a squeezed vacuum cavity field interacting with an initially excited two-level atom via the Jaynes-Cummings model in the Rotating Wave approximation. For large enough initial squeezing, we recognize a new partial revival phenomenon in terms of the quadrature variance product. We show the time evolution of the field quasiprobability Q-function, which provides a quadrature space picture of the partial revival phenomenon. An interpretation of the phenomenon in terms of a certain kind of first momentum of the photon distribution function is given.

I Introduction

Since the seminal paper by Jaynes and Cummings [1], the so-called Jaynes-Cummings model (JCM) has been extensively studied in quantum optics in a variety of situations. Even though it is a simple model (allowing for exact analytical solutions), it has been a laboratory for investigations of many interesting non-classical features of light, such as squeezing [2] [3] [4] [5], antibunching [6], reversible spontaneous emission [7] [8], Rabi oscillations collapses [9] [10], and revivals [11] in the atomic transition probability. Here, we will be especially concerned with this latter phenomenon. The revival in the transition probability is a signature of the quantization of the electromagnetic field, and recently observations of such a phenomenon by using Rydberg atoms interacting with the radiation in a high-Q cavity have been reported [8] [12]. It is known that revivals or rephasings are determined by the initial photon number probability distribution, which we will call $P^{(1)}(n)$, and they happen whenever $P^{(1)}$ has well-defined peaks around some mean photon numbers. This phenomenon has been thoroughly investigated in the case of coherent states [11] and strongly squeezed coherent states [4] [5] with finite mean number of photons where there are very clear revivals and accordingly the $P^{(1)}$ distribution presents well-defined peaks.

In the case of squeezed vacuum states, there is no clear peak in $P^{(1)}$, and so there is no visible revival in the transition probability (nor in related quantities), resembling the case of initially chaotic field [10]. However, we will show that there is at least a partial revival in another quantity, namely the product of variances of the two field quadratures. This quantity is not governed by the photon number probability distribution $P^{(1)}$, but by a distribution we have called $P^{(2)}(n)$, a kind of first momentum of $P^{(1)}$. Unlike $P^{(1)}$, $P^{(2)}$ happens to have a better defined peak for the strongly squeezed vacuum state, and this is what allows us to see such a partial revival in the variance product, as it will become clear along the paper. We will illustrate this with specific cases of initial squeezed fields (the atom being initially in the excited state). Also, the Q-function will be used to have a phase-space (or quadrature-space) image of what is happening to the field as the system evolves in time. We will show that the partial revival in the variance product has a natural interpretation in terms of such a phase-space visualization.

This paper is organized as follows. In Section II, a very brief review of the JCM and the vacuum squeezed states is given. After settling the appropriate notation, we define the $P^{(2)}$ distribution. Then, the time

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evolution of the variance product is studied and we establish the conditions under which a partial revival phenomenon for this quantity occurs. In Section III, the Q-function is investigated at several times for a given initial squeezed-vacuum field. The purpose of this study is to analyse its behaviour according to that of the variance product, particularly in the region where the partial revival occurs. Some concluding remarks are given in Section IV.

II The Time Evolution of the Variance Product and the Partial Revival

We are going to study the time evolution of a system consisting of a single-mode radiation field within a perfectly non-dissipative cavity (which is initially in a squeezed vacuum state), and a single two-level atom (initially in the excited state). To describe the atom-field interaction, we will use the JCM in the rotating-wave approximation in the resonant regime, where the atom's transition frequency is equal to the mode's eigenfrequency ω. The Hamiltonian is:

$$H = \hbar\omega a^{\dagger}a + \frac{1}{2}\hbar\omega\sigma_z + \hbar g(a^{\dagger}\sigma_- + a\sigma_+), \tag{1}$$

where a and $a^{\dagger}$ are the photon annihilation and creation operators; σ_+ and σ_- are the atom's excitation and de-excitation operators; σ_z is the diagonal Pauli's operator in the basis $\{|e\rangle, |g\rangle\}$ (the excited and ground states of the atom, respectively); and g is the coupling constant between the atom and the field.

The Hamiltonian (1) is exactly diagonalizable. Its eigenstates are:

$$|\Psi_n^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|n,e\rangle \pm |n+1,g\rangle\right), \tag{2}$$

$$|\Psi_0\rangle = |0,g\rangle, \tag{3}$$

where $|n,\phi\rangle = |n\rangle\otimes|\phi\rangle$ ($\phi = e$ or g) are atom-field product states. The corresponding eigenenergies are (except for a factor of $\hbar$):

$$\omega_n^{\pm} = \omega(n+\frac{1}{2}) \pm g\sqrt{n+1}; \tag{4}$$

$$\omega_0 = -\frac{\omega}{2}. \tag{5}$$

For the initially coherent radiation field with an initially excited atom, Meystre and Zubairy [3] found subsequent squeezing. Also, the phase-dependent field fluctuations in the JCM for an atom interacting with a vacuum field has been studied by Knight [13], and it has been shown that the vacuum fluctuation level are aug- in the excited state. Here, we are going to show that for the field initially in a squeezed vacuum, such a periodicity cannot be manifested and the partial rephasing for a quasi-revival depends on the distribution $P^{(2)}(n)$.

The squeezed vacuum state of the cavity mode is defined by the action of the squeezing operator:

$$|0\rangle_s = \hat{S}(s)|0\rangle, \tag{6}$$

where

$$\hat{S}(s) = \exp\left[\frac{1}{2}s^* a^2 - \frac{1}{2}s(a^{\dagger})^2\right],$$

and s is a complex parameter. In terms of the Fock state basis, the squeezed vacuum state is written as:

$$|0\rangle_s = \sum_{n=0}^{\infty} c_n|n\rangle, \tag{7}$$

where

$$c_n = \begin{cases} 0 & n \text{ odd} \\ (-1)^{\frac{n}{2}}(\tanh s)^{\frac{n}{2}}\sqrt{\frac{(n-1)!!}{n!!}}c_0 & n \text{ even} \end{cases} \tag{8}$$

and c_0 is a normalization constant.

Next, we introduce the field quadrature operators a_1 and a_2:

$$a_1 = \frac{1}{2}(a + a^{\dagger}); \tag{9}$$

$$a_2 = \frac{1}{2i}(a - a^{\dagger}). \tag{10}$$

They are Hermitean conjugate operators, and satisfy the corresponding commutation relation:

$$[a_1, a_2] = \frac{i}{2},$$

which implies the following uncertainty relation between them:

$$\langle(\Delta a_1)^2\rangle\langle(\Delta a_2)^2\rangle \geq \frac{1}{16}. \tag{11}$$

The squeezed vacuum state $|0\rangle_s$ has minimum uncertainty for real or purely imaginary s, but as opposed to the vacuum state, the quadrature variances are unequal:

$$\langle(\Delta a_1)^2\rangle\langle(\Delta a_2)^2\rangle = \frac{1}{16};$$

$$\langle(\Delta a_1)^2\rangle = \frac{1}{4}e^{-2s};$$

$$\langle(\Delta a_2)^2\rangle = \frac{1}{4}e^{2s}.$$

What we are interested in here is the time evolution of the quadrature variances (and their product). Using eqs.(9,10), we can write the variances as functions of

$$\langle(\Delta a_1)^2\rangle = \frac{1}{2}\langle a^\dagger a\rangle + \frac{1}{4}\left(\langle(a^\dagger)^2\rangle + \langle a^2\rangle\right) - \frac{1}{4}\langle a^\dagger + a\rangle^2 + \frac{1}{4}; \tag{12}$$

$$\langle(\Delta a_2)^2\rangle = \frac{1}{2}\langle a^\dagger a\rangle - \frac{1}{4}\left(\langle(a^\dagger)^2\rangle + \langle a^2\rangle\right) + \frac{1}{4}\langle a^\dagger + a\rangle^2 + \frac{1}{4}. \tag{13}$$

To evaluate the mean values appearing above, we need to figure out the time evolution of the system. The initial state is:

$$|\Psi(0)\rangle = |e\rangle \otimes |0\rangle_s. \tag{14}$$

Using equations (2-5), and switching to the interaction picture, we find that the state at a time t is:

$$|\Psi(t)\rangle = \sum_n c_n\left[\cos(\sqrt{n+1}gt)|n,e\rangle - i\sin(\sqrt{n+1}gt)|n+1,g\rangle\right]. \tag{15}$$

Now we can calculate the mean values involved in the variances (12,13):

$$\begin{aligned}\langle a\rangle = \sum_{n=1}^{\infty} c_n c_{n-1}^* \quad & \left[\sqrt{n}\cos(\sqrt{n}gt)\cos(\sqrt{n+1}gt) + \right.\\ & \left.\sqrt{n+1}\sin(\sqrt{n}gt)\sin(\sqrt{n+1}gt)\right]; \end{aligned} \tag{16}$$

$$\begin{aligned}\langle a^2\rangle = \sum_{n=2}^{\infty} c_n c_{n-2}^* \quad & \left[\sqrt{n(n-1)}\cos(\sqrt{n-1}gt)\cos(\sqrt{n+1}gt) + \right.\\ & \left.\sqrt{n(n+1)}\sin(\sqrt{n-1}gt)\sin(\sqrt{n+1}gt)\right]; \end{aligned} \tag{17}$$

$$\langle a^\dagger a\rangle = \langle n\rangle = \sum_{n=0}^{\infty} |c_n|^2\left[n + \sin^2(\sqrt{n+1}gt)\right]; \tag{18}$$

$$\langle a^\dagger\rangle = \langle a\rangle^*; \tag{19}$$

$$\langle(a^\dagger)^2\rangle = \langle a^2\rangle^*. \tag{20}$$

Since the squeezed vacuum state has only even photon numbers in its expansion in the Fock state basis, we easily see that:

$$\langle a\rangle = \langle a^\dagger\rangle = 0, \tag{21}$$

for all times.

We can numerically calculate the product of the variances (12) and (13) as a function of time, by using eqs. (16-20). This is plotted in Fig.1a, for the initial conditions given by (14) with $s = 2$. In this figure we can clearly see first an increase of the product until its largest value around $gt = 7.9$, and then a tendency to decrease, with a partial revival at $gt \approx 18$. After this, it increases again and enters in a regime of small random oscillations near the maximum value.

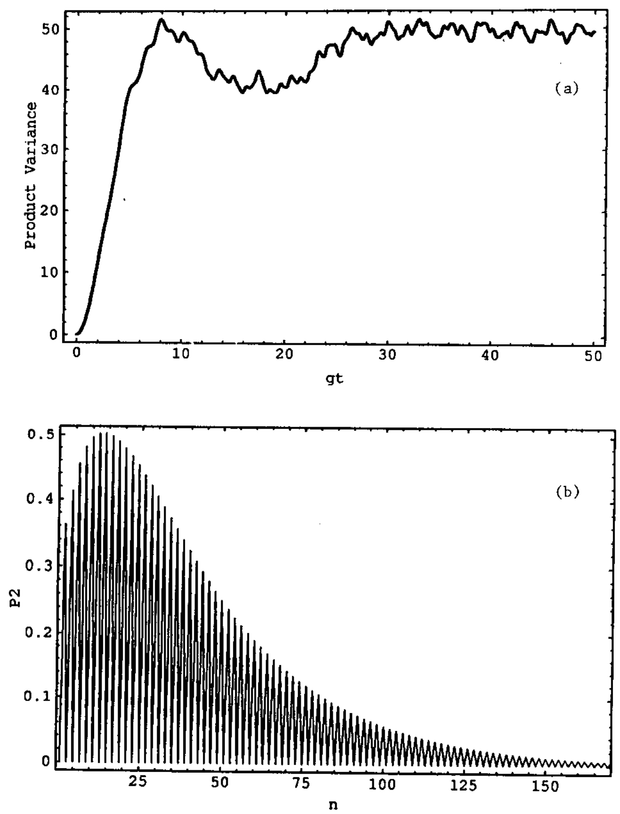

Figure 1a. Product of the quadrature variances as a function of the scaled time gt, for the atom initially in the excited state and the field in the squeezed vacuum state with squeezing parameter $s = 2$.
Figure 1b. $P^{(2)}(n)$ distribution for the squeezed vacuum state with $s = 2$.

To understand this phenomenon, let us look more closely at the expressions (12,13) of the quadrature variances.

The mean value of the photon number operator is, using eq.(18):

$$\bar{n}(t) = \langle a^\dagger a\rangle(t) = \bar{n}_0 + \sum_{n=0}^{\infty} |c_n|^2 \sin^2(\sqrt{n+1}gt), \quad (22)$$

where $\bar{n}_0$ is the mean photon number for the initial squeezed vacuum field, that is, $\bar{n}_0 = \sinh^2(s)$. Since we are in the rotating-wave approximation, and the atom was initially in the excited state, we have the selection rule:

$$\bar{n}_0 \leq \bar{n}(t) \leq \bar{n}_0 + 1 \quad (23)$$

Provided we have $\bar{n}_0 \gg 1$, we see by eq.(23) that $\bar{n}(t)$ does not vary much in time, so that we can make the approximation:

$$\langle a^\dagger a\rangle \approx \bar{n}_0 \quad (24)$$

Substituting eqs. (24), (21), (19) and (20) into eqs. (12,13), we have for the variances:

$$\langle(\Delta a_1)^2\rangle \approx \frac{\bar{n}_0}{2} + \frac{1}{4} + \frac{1}{2} Re\{\langle a^2\rangle\} \quad (25)$$

$$\langle(\Delta a_2)^2\rangle \approx \frac{\bar{n}_0}{2} + \frac{1}{4} - \frac{1}{2} Re\{\langle a^2\rangle\} \quad (26)$$

From the equations above, we notice that the time evolution of the quadrature variances are dominated by the term $\langle a^2\rangle(t)$, given by eq.(17).

Suppose now we have a field state where the coefficients c_n for small values of n do not contribute much for the total number of photons (such is the case for the squeezed vacuum states with high s). In this case, we can approximate the factors $\sqrt{n(n\pm 1)}$ by n. We obtain:

$$\langle a^2\rangle(t) \approx \sum_{n=2}^{\infty} n c_n c_{n-2}^* \quad \left[\cos(\sqrt{n-1}gt)\cos(\sqrt{n+1}gt) + \sin(\sqrt{n-1}gt)\sin(\sqrt{n+1}gt)\right] \quad (27)$$

We see from equation (27) that the oscillations in the variances (and in their product) are determined by the distribution $P^{(2)}(n)$:

$$P^{(2)}(n) = n c_n c_{n-2}^* \quad (28)$$

whereas those in the atomic transition probability are governed by $P^{(1)}(n) = |c_n|^2$. In particular, revival phenomena occur whenever these distributions have well-defined peaks. The $P^{(1)}$ distribution for the squeezed vacuum state has no clearly defined peak, and accordingly the transition probability shows no revival.

The $P^{(2)}$ distribution, in its turn, shows a very clear peak, as is shown in Fig.1b, even though this peak is very much spread in the n coordinate. So, one should expect a signature of revival in the variance product, as can indeed be seen in Fig.1a. This partial revival becomes more distinct as the squeezing parameter s is higher. We have plotted the variance product and $P^{(2)}$ for $s = 3$ in Figs. 2a and 2b; we notice at once that the peak in $P^{(2)}$ is much more pronounced in this case, and so is the quasi-revival in the variance product. The subsequent evolution is similar to the previous case (though not apparent in the time scale shown in the Fig.2a) in the sense that after the first partial revival the variance product suffers only small oscillations close to the maximum value.

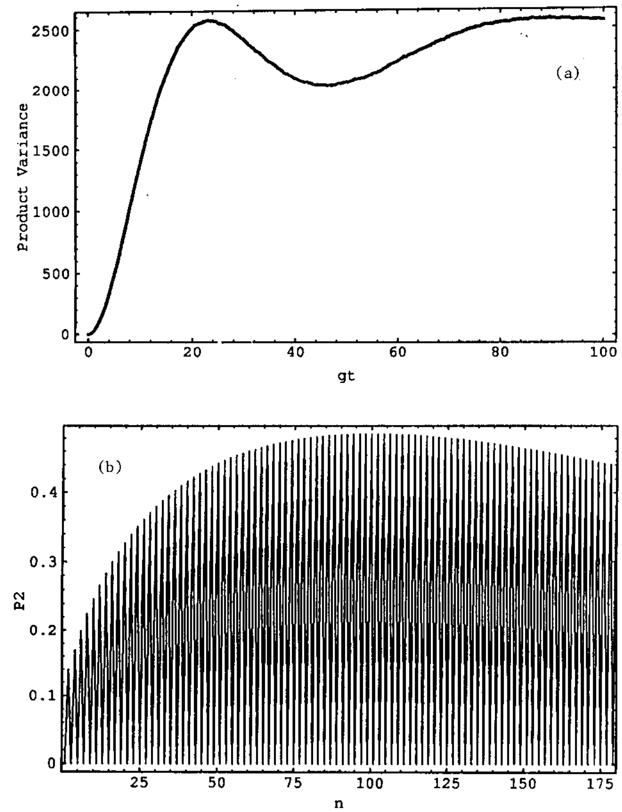

Figure 2. The same as in $Fig.1a$ and $1b$, but with $s = 3$.

III The Phase Space Representation

We are going to look now at the behaviour of the quasiprobability Q-function [14] [15], defined in the quadrature space of the field by:

$$Q(\alpha; t) = \langle \alpha | \rho_F(t) | \alpha \rangle \tag{29}$$

where $|\alpha\rangle$ is the coherent state representing a point in the phase space, and ρ_F is the reduced density operator obtained by tracing over the atomic variables. With the initial state given by eq.(14), and using eq.(15), by straightforward calculation we find that:

$$\begin{aligned} \rho_F(t) \quad = \quad & \sum_{m,n} c_n c_m^* \times \\ & [\sin(\sqrt{n+1}gt)\sin(\sqrt{m+1}gt)|n+1\rangle\langle m+1| + \\ & \cos(\sqrt{n+1}gt)\cos(\sqrt{m+1}gt)|n\rangle\langle m|] \end{aligned} \tag{30}$$

Substituting eq.(30) into eq.(29), we have an explicit expression for the Q-function:

$$
\begin{aligned}
Q(\alpha;t) \quad = \quad & e^{-|\alpha|^2} \sum_{m,n} c_n c_m^* \times \Bigg[\\
& \frac{\alpha^{m+1}(\alpha^*)^{n+1}}{\sqrt{(m+1)!(n+1)!}} \sin(\sqrt{n+1}gt)\sin(\sqrt{m+1}gt) + \\
& \frac{\alpha^m(\alpha^*)^n}{\sqrt{m!n!}} \cos(\sqrt{n+1}gt)\cos(\sqrt{m+1}gt)\Bigg]
\end{aligned} \tag{31}
$$

where the c_n for the squeezed vacuum state are given by eq.(8). Using eq.(31), we can calculate numerically the Q-function for any given time t.

Now, let us analyse the evolution of the Q-function for the initial state (14) with $s = 2$, comparing it with the product of the variances as a function of time. The original pure field state is represented by the elliptic contour plots centered at the origin in Fig.3a, with the minimum variance product of $\frac{1}{16}$. As time passes, this initial peak splits into other peaks, as shown in Fig.3b for the time $gt = 5$, where we have two clearly-defined peaks centered along the a_1 axis; this time corresponds to a variance product of 40 (see Fig.1a). The variance product reaches its maximum value at $gt = 7.9$, and the Q-function for this time is accordingly very broadly spread in phase space (Fig.3c), with two pairs of peaks along the two quadratures.

After this first regime of monotonically increasing variance product, we can see an oscillating reduction of it until the peaks have collapsed in one at the origin around $gt = 12.9$ (Fig.3d). Then, after some small oscillations added to a general tendency to decrease, the variance product has a local small maximum at $gt = 17.2$ where the split peaks in the a_2 direction can be seen in Fig.4e. Another collapse of the peak is seen at $gt \approx 18.8$ (Fig.4f), where now the variance product reaches a minimum, the smallest in this second regime; this region is the one recognized as showing the partial revival.

So, we see that the quasi-revival in the variance product happens when all the peaks of the phase space distribution, formed along the time evolution of the system, collapse, and the distribution comes as near as possible to reshaping the original elliptical form of the initial squeezed state. Of course, the reshaping is far from being perfect, and this is the reason why the variance product does not reach again its minimum of $\frac{1}{16}$ in the revival region.

Passed over this partial revival region, the variance product shows an oscillating increase before entering a regime of irregular small oscillations close to the maximum value of ≈ 50.

IV Conclusions

The time evolution of the product of variances of the two quadratures of the squeezed vacuum field interacting with a two-level atom has been studied. We have shown that for strong enough squeezing, the newly defined $P^{(2)}(n)$ (eq. 28) distribution determines the time evolution of the quadrature variance product, and that this distribution has a well-defined peak in the case of strongly squeezed vacuum state, though not that sharply around a given photon number. The existence of such a peak causes a partial revival phenomenon in the evolution of the variance product.

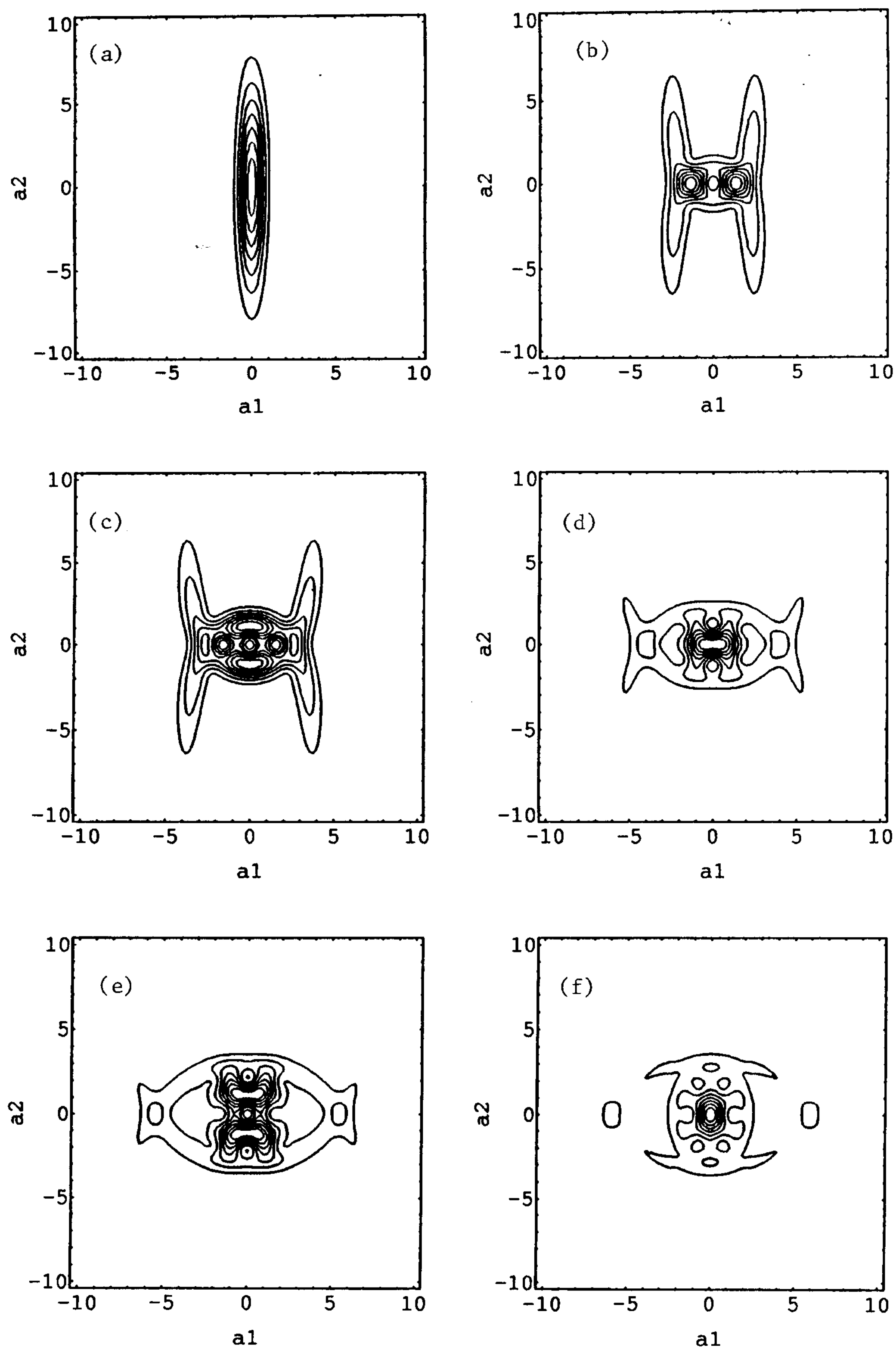

Figure 3. Plots of Q-function in the quadrature space, for the same initial conditions as in *Fig.*1: **3a.** Level contours of Q for $gt = 0.0$. **3b.** Level contours of Q for $gt = 5.0$. **3c.** Level contours of Q for $gt = 7.9$. **3d.** Level contours of Q for $gt = 12.9$. **3e.** Level contours of Q for $gt = 17.2$. **3f.** Level contours of Q for $gt = 18.8$.

The significance of this quasi-revival phenomenon is visually captured by following the evolution of the Q-function quasiprobability distribution. As the time evolves, the initial pure field state with a squeezed distribution peaked at the origin (with minimum variance product), splits into two peaks in the complex quadrature plane, causing a rapid increase in the variance product up to a certain maximum value. Next, we have observed a partial revival regime where the variance product decreases significantly, and reaches a local minimum, suggesting a large superposition with the original pure squeezed state. Along this process of splitting and collapsing of the peaks the border of the distribution becomes more and more spread, losing the original elliptic form. Finally, it reaches the regime where the quadrature variance product remains close to its maximum value, showing only small oscillations around this value.

Acknowledgements

It is a pleasure to thank S. M. Dutra and A. Vidiella-Barranco for helpful discussions and for providing some references the authors were not aware of. This work was supported in part by FAPESP, CNPq and FINEP.

References

[1] E. T. Jaynes and F. W. Cummings, Proc.E.E.E. **51** (1963) 89; for a recent review, see: B. W. Shore and P. L. Knight, J. Mod. Opt. **40** (1993) 1195.

[2] D. F, Walls, Nature, Lond. **306** (1983) 141; D. F. Walls and G. J. Milburn, 'Quantum Optics, Experimental Gravity and Measurement Theory', ed. by P. Meystre and M. O. Scully (N. Y. ,Plenum) p. 209; K. Zaheer and M. S. Zubairy, in 'Advances in Molecular and Optical Physics', vol. **28**, D. Bates and B. Benderson, eds. (Academic, New York, 1991) p.143.

[3] P. Meystre and M. S. Zubairy, Phys. Lett. A **89** (1982) 390.

[4] G. J. Milburn, Optica Acta **31** (1984) 671; K. A, Rustamov et al., Phys. A **158** (1989) 649; C. M. Savage, Quant. Opt. **2** (1989) 89; M. S, Kim et al, J. Mod. Opt. **37** (1990) 659; H. Moya-Cessa and A. Vidiella-Barranco, J. Mod. Opt. **39** (1992) 2481; see also, R. Loudon and P. L. Knight, J. Mod. Opt. **34** (1987) 709.

[5] M. Venkata Satyanarayana, P. Rice, R. Vyas and H. J. Carmichael, J. Opt. Soc. Am. B **6** (1988)228.

[6] H. J. Carmichael and D. F. Walls, J. Phys. B **9**(1976) L43; idem, J. Phys. B **9** (1976) 1199; H. J. Carmichael, Phys. Rev. Lett. **55**(1985) 2790.

[7] S. Haroche and D. Kleppner, Phys. Today (1989) 24; S. Haroche and J-M. Raimond, Scient. Am. (1993) 54.

[8] M. Brune et al., 'Quantum Rabi Oscillations : A Direct Test of Field Quantization in a Cavity', Phys. Rev. Lett, to appear; S. Haroche and J.-M. Raimond, ' Radiative properties of Rydberg states in resonant cavities', in 'Advances in Atomic and Molecular Physics', vol. **20**, D. Bates and B. Benderson, eds. (Academic, New York, 1985) p.347.

[9] F. W, Cummings, Phys. Rev. **140** (1965) A1051; P. Meystre et al, Nuovo Cimento B**25** (1975) 251; T. Von Forster, J. Phys. A**8** (1975) 95;

[10] P. L. Knight and P. M. Radmore, Phys Lett. A **90** (1982) 342.

[11] J. H. Eberly, J. J. Sanchez-Mondragon, N. B. Narozhny, Phys. Rev. Lett. **44** (1980) 1323; N. B. Narozhny, J. J. Sanchez-Mondragon, J. H. Eberly, Phys. Rev. A **23** (1981) 236.

[12] G. Rempe, H. Walther, N. Klein, Phys. Rev. Lett. **58** (1987) 353.

[13] P. L. Knight, Phys. Script. **T12** (1986) 51.

[14] M. Hillery, R.F. O'Connell, M. O. Scully and E. P. Wigner, Phys. Rep. **106** (1984) 121.

[15] J. Eiselt, H. Risken, Phys. Rev. A **43** (1991) 346.

[16] J. von Neumann, 'Mathematical Foundations of Quantum Mechanics' (Princeton: Princeton Univ. Press) (1955).

[17] S. J. D. Phoenix and P. L. Knight, Ann. Phys. **186** (1988) 381; J. Gea-Banacloche, Phys. Rev. Lett. **65** (1990) 3385; S. J. D. Phoenix and P. L. Knight, Phys. Rev. A **44** (1991) 6023.