# The NJL Interaction from $q$-Deformed Inspired Transformations *

V. S. Timóteo[1] and C. L. Lima[2]
[1] *Centro Superior de Educação Tecnológica, Universidade Estadual de Campinas, 13484-370, Limeira, SP, Brazil*
[2] *Instituto de Física, Universidade de São Paulo, CP 66318, 05315-970, São Paulo, SP, Brazil*

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From the mass term for $q$-deformed quark fields, we obtain effective contact interactions of the NJL type. The parameters of the model that maps a system of non-interacting deformed fields into quarks interacting via NJL contact terms is discussed.

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In hadron physics, the NJL model [1, 2] is a very simple effective model for strong interactions that describes important features like the dynamical mass generation, spontaneous chiral symmetry breaking, and chiral symmetry restoration at finite temperature.

$q$-Deformed algebras provide a nice framework to incorporate, in an effective way, interactions not originally contained in the Lagrangian of a particular system [3–5].

In recent works [6–9], we have been investigating possible applications of quantum algebras in hadronic physics. In general, we observed that when we deform the underlying algebra, the system is affected with correlations between its constituents. We have studied in detail the NJL model under the influence of a quantum $su(2)$ algebra.

In this work we approach the following question: is it possible to obtain an art of "canonical transformation" connecting the NJL model to a simpler non-interacting system? We verified that we can indeed obtain the same dynamics of the NJL interaction with non-interacting $q$-deformed quark fields.

We start by writing a mass term for the $q$-deformed quark fields

$$\begin{aligned} \mathcal{L}_q^{\text{mass}} &= -M\,\overline{\Psi}\Psi \\ &= -M\,\left(\,\overline{\Psi}_1\Psi_1+\overline{\Psi}_2\Psi_2\,\right) \\ &= -M\,\left(\,\overline{U}U+\overline{D}D\,\right) \end{aligned} \tag{1}$$

where $\Psi_{1,2}=U,D$ are the components of $\Psi$.

The $q$-deformed quark fields can be written in terms of the standard fields as

$$\Psi_1 = \psi_1+(q^{-1}-1)\;\psi_1\overline{\psi}_2\gamma_0\psi_2\;, \tag{2}$$
$$\Psi_2 = \psi_2+(q^{-1}-1)\;\psi_2\overline{\psi}_1\gamma_0\psi_1\;, \tag{3}$$

or

$$U = u+(q^{-1}-1)\;u\bar{d}\gamma_0 d\;, \tag{4}$$
$$D = d+(q^{-1}-1)\;d\bar{u}\gamma_0 u\;, \tag{5}$$

and $\psi_{1,2}=u,d$. Here both components are modified in the same way, so that the above expressions are different from he ones used in [4, 5], where only one component is affected. Extending the deformation to the two components is required to obtain a set of terms that will form an interaction of the NJL type. This implies that the anti-commutation relations for the deformed fields $\Psi$ will also be different from the ones in [4, 5]. Since obtaining the new anti-commutation relations is not in the scope of this work, we focus on the effective interactions contained in the non-interacting Lagrangian.

Using Eqs. (4) and (5), we can re-write the Lagragian Eq.(1) in terms of the non-deformed quark fields

$$\overline{U}U = \bar{u}u+Q\,\bar{u}ud^\dagger d+Q\,d^\dagger d\bar{u}u+Q^2\,\bar{d}d\bar{u}u\bar{d}d\;, \tag{6}$$
$$\overline{D}D = \bar{d}d+Q\,\bar{d}du^\dagger u+Q\,u^\dagger u\bar{d}d+Q^2\,\bar{u}u\bar{d}d\bar{u}u\;, \tag{7}$$

where $Q=(q^{-1}-1)$.

We can re-write the above equations as follows

$$\begin{aligned} \overline{U}U &= \left(1+2Qd^\dagger d\right)\bar{u}u+\frac{Q^2}{2}\left(\bar{d}d\bar{u}u\bar{d}d+\bar{d}d\bar{u}u\bar{d}d\right), \\ \overline{D}D &= \left(1+2Qu^\dagger u\right)\bar{d}d+\frac{Q^2}{2}\left(\bar{u}u\bar{d}d\bar{u}u+\bar{u}u\bar{d}d\bar{u}u\right), \end{aligned}$$

so that we identify the contact interactions between the quarks contained in the non-interacting deformed fields Lagrangian. The six-point contact terms are depicted in Fig. 1.

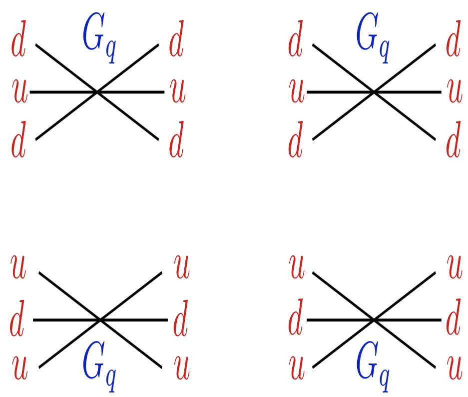


FIG. 1: Contact interactions generated by the mass term for the $q$-deformed fermion fields.

We can reduce the six-point interactions of Fig. 1 to four-point contact terms in a mean field approach [2], so that we

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have

$$\begin{aligned}
(\overline{U}U+\overline{D}D) &= \left(1+2Q\frac{\langle\psi^\dagger\psi\rangle}{A}\right)(\bar{u}u+\bar{d}d) \\
&+ \frac{Q^2}{2}\frac{\langle\overline{\psi}\psi\rangle}{A^2} \\
&\times \left(\bar{d}d\bar{u}u+\bar{d}d\bar{d}d+\bar{u}u\bar{d}d+\bar{u}u\bar{u}u\right), \qquad (8)
\end{aligned}$$

where $\langle\psi^\dagger\psi\rangle = \langle u^\dagger u\rangle = \langle d^\dagger d\rangle = \rho_v$, $\langle\overline{\psi}\psi\rangle = \langle\bar{u}u\rangle = \langle\bar{d}d\rangle = \rho_s$, and $A = A(T;q)$ has the same dimension of the condensate and will be determined later in this work. This procedure corresponds to have one fermion line closed and can be represented by the diagrams of Fig. 2.

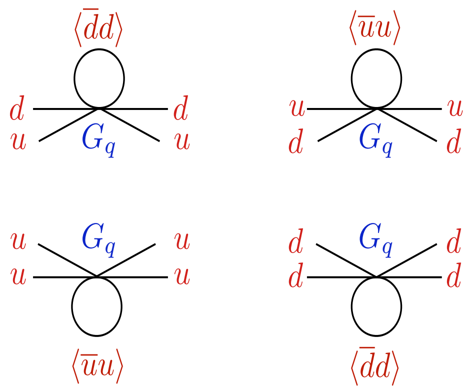


FIG. 2: Reduction of the six-point interactions of Fig. 1 to four-point interactions by closing one fernion line.

Now we can write the mass term for the $q$-deformed quark fields

$$\begin{aligned}
\mathcal{L}_q^{\mathrm{mass}} &= -M\overline{\Psi}\Psi \\
&= -M\left(1+2\left\langle\psi^\dagger\psi\right\rangle\Gamma\right)\overline{\psi}\psi \\
&- \frac{M}{2}\langle\overline{\psi}\psi\rangle\Gamma^2\,\overline{\psi}\psi\overline{\psi}\psi\,, \qquad (9)
\end{aligned}$$

with $\Gamma = Q/A$

Accordingly, the kinetic energy term for the deformed fields, $\overline{\Psi}\gamma^\mu\partial_\mu\Psi$, can be written in terms of the non-deformed ones as

$$\begin{aligned}
\overline{\Psi}\gamma^\mu\partial_\mu\Psi &= \overline{U}\gamma^\mu\partial_\mu U+\overline{D}\gamma^\mu\partial_\mu D \\
&= \bar{u}\gamma^\mu\partial_\mu u + Q\left(\bar{d}\gamma_0 d\bar{u}\gamma^\mu\partial_\mu u+\bar{u}\gamma^\mu\partial_\mu u\bar{d}\gamma_0 d\right) \\
&+ \bar{d}\gamma^\mu\partial_\mu d + Q\left(\bar{u}\gamma_0 u\bar{d}\gamma^\mu\partial_\mu d+\bar{d}\gamma^\mu\partial_\mu d\bar{u}\gamma_0 u\right) \\
&+ Q^2\left(\bar{d}\gamma_0 d\bar{u}\gamma^\mu\partial_\mu u\bar{d}\gamma_0 d\right) \\
&+ Q^2\left(\bar{u}\gamma_0 u\bar{d}\gamma^\mu\partial_\mu d\bar{u}\gamma_0 u\right) \qquad (10)
\end{aligned}$$

By using an extreme mean field approximation, namely, substituting everywhere in the kinetic energy contribution $\langle\psi^\dagger\psi\rangle = \langle u^\dagger u\rangle = \langle d^\dagger d\rangle \to \rho_v$, and $\langle\overline{\psi}\psi\rangle = \langle\bar{u}u\rangle = \langle\bar{d}d\rangle \to \rho_s$, we obtain

$$\begin{aligned}
\overline{\Psi}\gamma^\mu\partial_\mu\Psi &= \bar{u}\gamma^\mu\partial_\mu u\,(1+2\Gamma\rho_v) \\
&+ \bar{d}\gamma^\mu\partial_\mu d\,(1+2\Gamma\rho_v) \\
&+ (\bar{u}\gamma^\mu\partial_\mu u)\,\Gamma^2\rho_v + (\bar{d}\gamma^\mu\partial_\mu d)\,\Gamma^2\rho_v \\
&= (\bar{u}\gamma^\mu\partial_\mu u+\bar{d}\gamma^\mu\partial_\mu d)\,(1+\Gamma\rho_v)^2 \qquad (11)
\end{aligned}$$

This corresponds to a usual kinetic energy with a shifted momentum $p \to p\,(1+\Gamma\rho_v)^2$.

The treatment of the density dependence of the kinetic energy term is rather cumbersome and will be postponed to a further contribution. We will consider the influence of this momentum dependent kinetic energy term in an effective way. Therefore, we will study a class of Lagrangians of the type

$$\begin{aligned}
\mathcal{L}_q' &= \frac{1}{(1+\Gamma\rho_v)^2}\,\mathcal{L}_q = \overline{\psi}\gamma^\mu\partial_\mu\psi \\
&- M\left(1+2\left\langle\psi^\dagger\psi\right\rangle\Gamma\right)\frac{1}{(1+\Gamma\rho_v)^2}\,\overline{\psi}\psi \\
&- \frac{M}{2}\langle\overline{\psi}\psi\rangle\Gamma^2\frac{1}{(1+\Gamma\rho_v)^2}\,\overline{\psi}\psi\overline{\psi}\psi \qquad (12)
\end{aligned}$$

This representative of the full Lagrangian $\mathcal{L}_q = \overline{\Psi}\gamma^\mu\partial_\mu\Psi + \mathcal{L}_q^{\mathrm{mass}}$, when written in terms of the standard quark fields, can be identified with the NJL Lagrangian

$$\mathcal{L}_{\mathrm{NJL}} = \overline{\psi}\gamma^\mu\partial_\mu\psi - m_0\,\overline{\psi}\psi + G\,\overline{\psi}\psi\overline{\psi}\psi\,. \qquad (13)$$

The conditions for both Lagrangians, $\mathcal{L}_{\mathrm{NJL}}$ and $\mathcal{L}_{\mathrm{q}}'$, to be equivalent for any values of $T$ and $q$ are

$$M = \frac{(1+\Gamma\rho_v)^2}{(1+2\Gamma\rho_v)}m_0\,, \qquad (14)$$

and

$$G = -\frac{M}{2}\frac{\rho_s\Gamma^2}{(1+\Gamma\rho_v)^2}\,. \qquad (15)$$

If we insert Eq. (14) in Eq. (15), we obtain an equation for $\Gamma$

$$\Gamma^2 - 2\alpha\rho_v\,\Gamma - \alpha = 0\,, \qquad (16)$$

where

$$\alpha = -\frac{2G}{m_0\rho_s} > 0. \qquad (17)$$

This equation has two solutions

$$\Gamma_\pm = \alpha\rho_v\left(1 \pm \sqrt{1+\frac{1}{\alpha\rho_v^2}}\right)\,. \qquad (18)$$

The mass of the $q$-deformed fermion fields, $M$, has to be positive, so we associate the two solutions $\Gamma_-$ and $\Gamma_+$ with the two regimes $q<1$ and $q>1$, respectively. The quantity $A$ will be negative in both cases.

The scalar ($\rho_s$) and vector ($\rho_v$) densities were calculated from the NJL model at finite temperature:

$$\rho_s = -\frac{N_c N_f}{\pi^2}\int_0^\Lambda dp p^2 \frac{m}{E}\left[1-n-\bar{n}\right] , \tag{19}$$

$$\rho_v = \frac{N_c N_f}{\pi^2}\int_0^\Lambda dp p^2 \left[n-\bar{n}\right] , \tag{20}$$

where

$$n(\mathbf{p},T,\mu) = \frac{1}{1+\exp\left[\beta\left(E-\mu\right)\right]} , \tag{21}$$

and

$$\bar{n}(\mathbf{p},T,\mu) = \frac{1}{1+\exp\left[\beta\left(E+\mu\right)\right]} , \tag{22}$$

are the fermions and anti-fermions distribution functions respectively with $E=\sqrt{\mathbf{p}^2+m^2}$.

First solve the set of coupled gap equations for $m$, $\mu$, and $\rho_v$ (Eqs. 23 and 20, respectively) in the NJL model at finite temperature and chemical potential

$$\begin{cases} m = m_0 - 2G\rho_s , \\ \mu = \mu_0 - \frac{G}{N_c}\rho_v . \end{cases} \tag{23}$$

The next step is to calculate the scalar and vector densities entering in the equation for $\Gamma$ for a given value of the deformation parameter $q$. In this way we obtain $A(T;q)$, which in turn is used to obtain $M$. The numerical results are displayed in Figures 3 and 4, where we show the quantity $A$, in units of the condensate at zero temperature, as a function of both temperature and deformation for the $q>1$ and $q<1$ regimes.

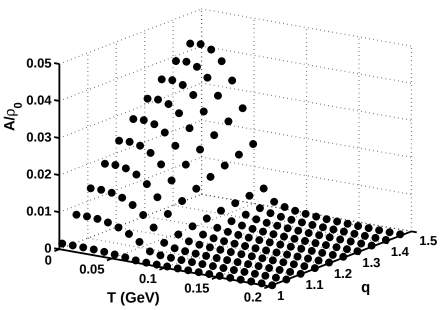


FIG. 3: The quantity $A$, in units of the chiral condensate at zero temperature $\rho_0=\rho_s(T=0)=-1.42\times10^{-2}$ GeV$^3$, as a function of temperature and $q$-deformation for the $q>1$ regime.

It is worth to note that the mass of the $q$-deformed fermion fields, $M$, does not depend on the deformation of the algebra while its temperature dependence is shown in 5.

The quantity $A(T;q)$ maps the simple non-interacting model into the NJL model. It represents, in an effective way, the correlations introduced by the quantum algebra, when we write the non-interacting Lagrangian in terms of the standard

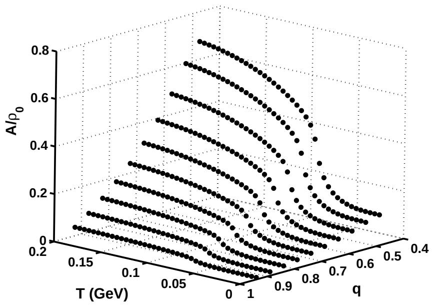


FIG. 4: The quantity $A$, in units of the chiral condensate at zero temperature $\rho_0=\rho_s(T=0)=-1.42\times10^{-2}$ GeV$^3$, as a function of temperature and $q$-deformation for the $q<1$ regime.

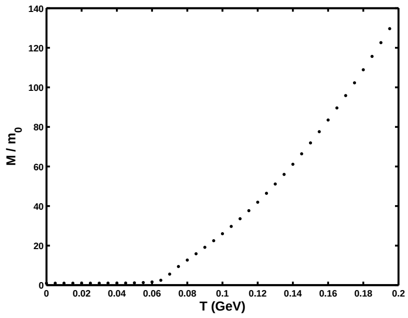


FIG. 5: The mass of the $q$-deformed quark fields, in units of the current quark mass $m_0=5$ MeV, as a function of $T$ for both $q>1$ and $q<1$ regimes. For small temperatures, $M=m_0$.

quark fields. These correlations, in a mean field approximation, are effectively represented by contact interactions of the NJL type. It is also important to mention that it inherits the phase transition. When the condensate and the dynamical mass vanishes with increasing $T$, the quantity $A$ also experiences the phase transition. This is an expected behavior, since it depends on the dynamical mass. For a given temperature, $T$, and deformation, $q$, there is a value of the mapping function, $A(T;q)$, that makes the Lagrangians Eq.(12) and Eq.(13) equivalent.

Summarizing, we have shown that it is possible to describe the dynamics of an interacting system of the NJL type with a simple non-interacting system by using a set of quantum algebra transformations and a mapping function.

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[1] Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122**, 345 (1961).
[2] U. Vogl and W. Weise, Prog. Part. Nucl. Phys. **27**, 195 (1991).
[3] D. Galetti and B. M. Pimentel, An. Acad. Bras. Ci. **67**, 7 (1995); S. S. Avancini, A. Eiras, D. Galetti, B. M. Pimentel, and C. L. Lima, J. Phys. A: Math. Gen. **28**, 4915 (1995); D. Galetti, J. T. Lunardi, B. M. Pimentel, and C. L. Lima, Physica A **242**, 501 (1997).
[4] M. Ubriaco, Phys. Lett. A **219**, 205 (1996).
[5] L. Tripodi and C. L. Lima, Phys. Lett. B **412**, 7 (1997).
[6] V. S. Timóteo and C. L. Lima, Phys. Lett. B **448**, 1 (1999).
[7] V. S. Timóteo and C. L. Lima, Mod. Phys. Lett. A **15**, 219 (2000).
[8] V. S. Timóteo and C. L. Lima, Braz. J. Phys. **36**, 208 (2006).
[9] V. S. Timóteo and C. L. Lima, Phys. Lett. B **635**, 168 (2006).