Statistical Physics of Random Searches

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We apply the theory of random walks to quantitatively describe the general problem of how to search efficiently for randomly located objects that can only be detected in the limited vicinity of a searcher who typically has a finite degree of "free will" to move and search at will. We illustrate Lévy flight search processes by comparison to Brownian random walks and discuss experimental observations of Lévy flights in the special case of biological organisms that search for food sites. We review recent findings indicating that an inverse square probability density distribution $P(\ell) \sim \ell^{-2}$ of step lengths ℓ can lead to optimal searches. Finally we survey the explanations put forth to account for these surprising findings.

I Introduction

What is the most efficient strategy for searching randomly located objects whose exact locations are not known *a priori*? This question has been recently studied by physicists [1, 2]. The general problem of how to search efficiently is a challenging one, because on the one hand the searchers typically have a certain degree of "free will" to move and search according to their choice. On the other hand, they are subject to certain physical and biological constraints which restrict their behavior.

A classic example of efficient search strategies relates to animal foraging. On the one hand, the animal's brain is sufficiently complex to allow a broad range of behavioral choices and "freedom," but on the other hand the animal must adapt and restrict its behavior to increase the chances of survival, e.g., if an animal does not eat food for a certain maximum time then it will die.

The richness of the problem stems partially because of the "ignorance" of the locations of the randomly located "target sites." However, even if the positions of all target sites were completely known in advance by a "demon" as resourceful as Laplace's [3], the problem of what sequential order to visit the sites in order to reduce the energy costs of locomotion is itself rather challenging: the famous "travelling salesman" optimization problem [4]. The ignorance of the target site locations, however, introduces yet another level of difficulty and renders the problem unsuited to deterministic search algorithms that do not use some element of randomness. Indeed, only a statistical approach to the search problem can deal adequately with the element of ignorance. Such a statistical approach is analogous to how only the conceptual framework of a statistical mechanical entropy can adequately model the ignorance involved in the relationship between a single macroscopic ("thermodynamic") state and the very large number of corresponding microstates of a system in thermodynamic equilibrium [5]. Similarly, it has been argued that statistical physics is ideally suited to the study of complex phenomena of this nature [6]. Indeed, the general problem of how to search efficiently for randomly-located target sites can be quantitatively described [1, 2, 6] using ideas developed in the study of random walks [7, 8, 9].

Quantifying the statistical properties of search patterns is of practical relevance not only in physics but also in theoretical ecology, industry, and conceivably even to problems such as the search for missing children. Very recently, such concepts have even found application in information technology (e.g., *information foraging theory* [10]).

II Random walks and random searches

When a searcher wanders in search of the target sites, the resulting motion can be described quantitatively as a list of the visited sites, in sequential (temporal) order. Such motion is typically random. Specifically, the motion has some degree of stochastic noise, just like a random walk created by a hypothetical "drunk" who takes steps forwards and backwards randomly with equal probability.

Brownian motion can be thought of as a kind of random walk, but the particle can move on a continuous scale. The name "Brownian" refers to Robert Brown who, in 1827, observed the irregular motion of pollen grains suspended in water [11]. Brownian motion was not satisfactorily explained until 1905 when Albert Einstein published his classic paper [12]. Although Brownian random walks were the first to be studied, there also exist non-Brownian random-walks.

Specifically, random walks can be classified either as Brownian (B) random walks or Lévy (L) walks:

(B) The step lengths ℓ_j have a characteristic scale, usually defined by the first or second moment (mean and variance respectively) of the step length density distribution $P(\ell)$. An essential feature of such random walks is that their square displacement increases linearly with the number of steps taken.

(L) The step lengths have no characteristic scale, by which we mean that the moments diverge and the distribution has self-affine properties: $P(\lambda\ell) \sim \lambda^{-\mu} P(\ell)$. The square displacement of Lévy random walks, also called Lévy flights, can grow quadratically with the number of steps, so their behavior is dominated by extremely long but rare step lengths.

The majority of studied probability distributions lead to Brownian motion as a consequence of the Central Limit Theorem (CLT). If the steps are (i) large in number ($\gg 10^2$) and (ii) independent, i.e., free of correlations, then any probability distribution with finite moments will lead to Brownian motion. Lévy distributions have diverging lower moments, therefore the CLT is not applicable and superdiffusive behavior is possible. Moreover, Lévy walks result in a set of visited sites that form a fractal.

It is often possible to estimate experimentally the probability density distribution $P(\ell)$ of the step or flight lengths ℓ taken by a searcher. Until recently it has often been assumed [7, 8, 9] that such a histogram of flight lengths $P(\ell_j)$ has a well defined second moment. Hence arise Gaussian, Poisson and other classical distributions that lead to Brownian behavior. Indeed, it has generally been assumed *a priori* that searchers perform movements in their environments that correspond to normal diffusion.

Recently, however, it has been questioned if this assumption is unnecessarily restrictive, and whether its predictions can be supported by existing experimental data [1,2,13,14,15]. To address this question, one can assume the more general Lévy distribution [8, 9, 21],

$$P(\ell_j) \sim \ell_j^{-\mu} , \tag{1}$$

with $1 < \mu \leq 3$ where, in fact, Gaussian behavior is a special case for $\mu \geq 3$ [16]. Values $\mu \leq 1$ do not correspond to normalizable probability distributions. Apart from its intrinsic mathematic merit, as being the largest class of stable distributions, Lévy distributions have in the last decade found useful applications in biology [8, 9], and studies of biological search processes specifically [2,13,14,15]. (Note that P. Lévy originally had studied Lévy statistics since 1937, see ref. [8].) Whereas Brownian motion corresponds to normal diffusion, Lévy flights, in contrast, correspond to anomalous super-diffusive motion [8, 9, 17] (Fig. 1). Lévy flights have also been found to be associated with enhanced diffusion in chaotic systems (see, for instance, [18]).

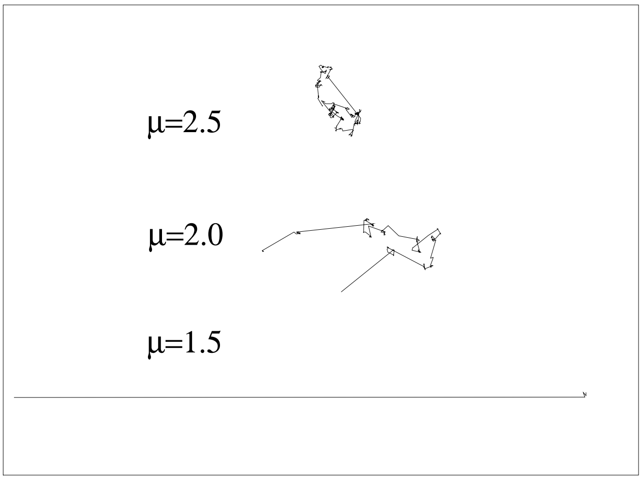


Figure 1. 2-D random walks for $\mu = 2.5$, 2.0, and 1.5 respectively with identical total lengths of 10^3 units. Microorganisms, mammals, birds, and insects show episodes of approximately straight locomotion randomly interrupted by re-orientation events.

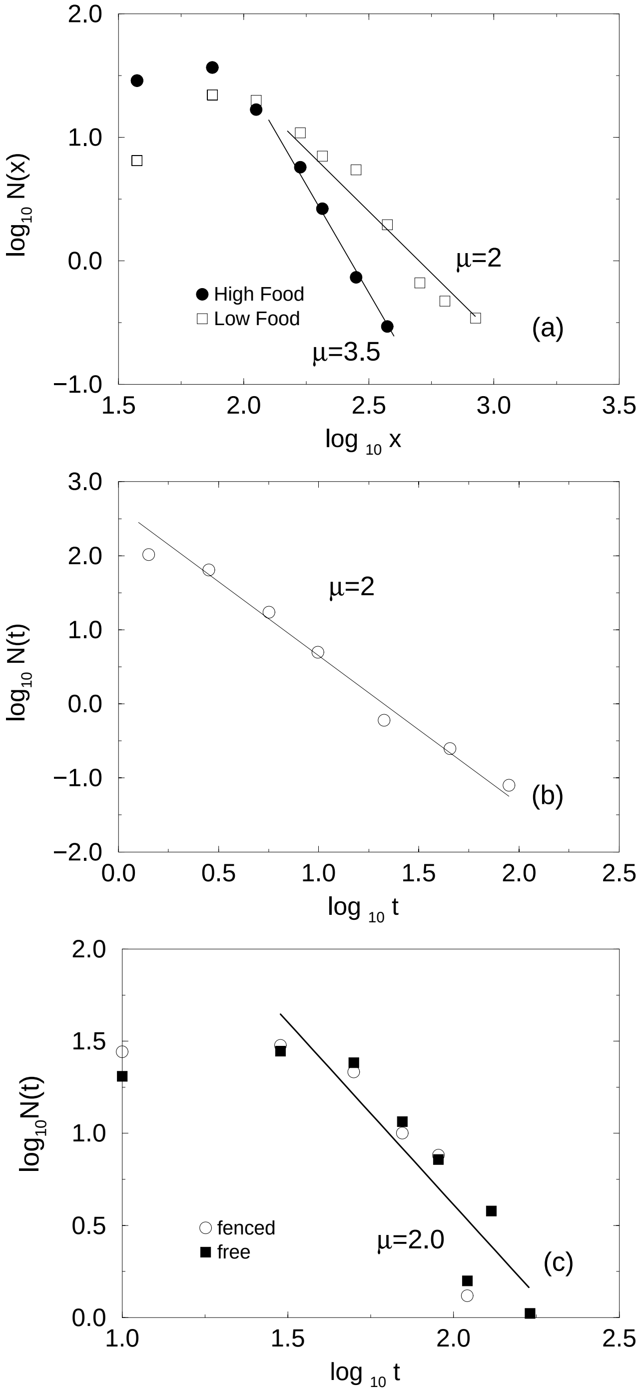


Figure 2. (a) Double log plot of the flight length percentage distributions for searching bumble bees, digitized from ref.[22]. Note the value $\mu \approx 2$ for normal (low) nectar concentration. The value $\mu \approx 3.5$ for ($\approx 10\times$) higher nectar concentrations in which long flights become very rare (see text) is also consistent with the theory. (b) double log plot of the histograms of flight times (in 1 h intervals) for the Wandering Albatross[2]. (c) Double log plot of the searching time (secs) percentage distributions for deer in wild areas and fenced areas.

Search processes can be found in biological phenomena [19]. Extensive experimental data exist for the special case of animal search processes, in which an animal optimizes its search for, say, food [2,13,14,15]. Evolution has through natural selection led over time to highly efficient—even optimal—biological search strategies. According to *optimal foraging theory*, animals seek to maximize the returns (in calories, nutrients etc.) on their labor in deciding how best to forage [20]. Since physical as well as neurophysiological and evolutionary factors come into play, searching is a rich problem that continues to present multi-faceted and interdisciplinary challenges.

Microorganisms, insects, birds, and mammals have been found to follow a Lévy distribution of flight lengths or times (assumed to be proportional or at least correlated statistically) [1,2,13,14,15] (Fig. 2). Moreover, the exponent μ appears to be the same in many instances [1]. When the nectar concentration is normal (low), the flight length distribution of bumble bees [1, 22] decays like Eq. (1) with $\mu \approx 2$ (Fig. 2(a)). Similarly, the value $\mu \approx 2$ is also found for the searching time distribution of the Wandering Albatross [2] (Fig. 2(b)) and deer (Fig. 2(c)) in both wild and fenced areas [1, 23]. Even the value $2 \leq \mu \leq 2.5$ found for amoebas [14] supports the hypothesis that $\mu_{\text{opt}} = 2$ might be a universal value of the exponent in Lévy flight searches. What, might we ask, drives animals to this type of behavior and what benefits, if any, do they thus derive from Lévy flight motion?

III Lévy flight search patterns

Why flight lengths might follow a Lévy distribution rather than a Gaussian or Poisson distribution is of general interest. The reasons behind the experimentally observed Lévy flights in biological searches have never been fully understood, but a number of studies have shed some light. Levandowsky et al.[13, 14] have suggested reasons why microorganisms may perform Lévy flights in three dimensions (3-D), showing that a Lévy distribution is advantageous since the probability of returning to a previously-visited site is smaller than for a Gaussian distribution, irrespective of the value of μ chosen [24]. A related explanation proposed by M. F. Shlesinger (see [2]) argues that foragers may perform Lévy flights because the number of new visited sites is much larger for n Lévy walkers than for n Brownian walkers [25]. The n Lévy walkers diffuse so rapidly that the competition for the resources (target sites) among themselves is greatly reduced relative to the competition encountered by the n Brownian walkers, who typically remain close to the origin, hence to each other. A Lévy flight strategy is also a good solution for the related problem where N radar stations search for M targets [26]. Yet another proposed hypothesis is that the fractal properties of the set of sites visited by a Lévy walker are related to scale invariant properties of the underlying ecosystem [2]. Specifically, a fractal distribution of target sites may explain the observed Lévy flights [2]. Very recently, there has been a study of how the search efficiency depends on the value μ of the Lévy exponent [1]. This study finds that there is an optimum value $\mu_{\text{opt}} = 2$ which can lead to optimal searches when the target sites are randomly and sparsely distributed. Below, we discuss this latest development in greater detail.

By studying how the search efficiency varies with μ, one can compare different classes of search strategies characterized by unique values of μ. In the first case of "non-destructive search", the forager can visit the same target site many times. Nondestructive search is more realistic (see below) and can occur in either of two cases: (i) if the target sites become temporarily depleted, or (ii) if the forager becomes satiated and leaves the area. In the second case of "destructive search", the target site found by the forager becomes undetectable in subsequent flights. Consider the following idealized model that captures some of the essential dynamics of searches in the limiting case in which predator-prey relationships are ignored, and learning is minimized. Assume that target sites are distributed randomly, and the forager behaves as follows (see Fig. 3):

(1) If there is a target site located within a "direct vision" distance r_v, then the forager moves on a straight line to the nearest target site.

(2) If there is no target site within a distance r_v, then the forager chooses a direction at random and a distance ℓ_j from the probability distribution, Eq. (1). It then incrementally moves to the new point, constantly looking for a target within a radius r_v along its way. If it does not detect a target, it stops after traversing the distance ℓ_j and chooses a new direction and a new distance ℓ_{j+1}, otherwise it proceeds to the target as in step (1).

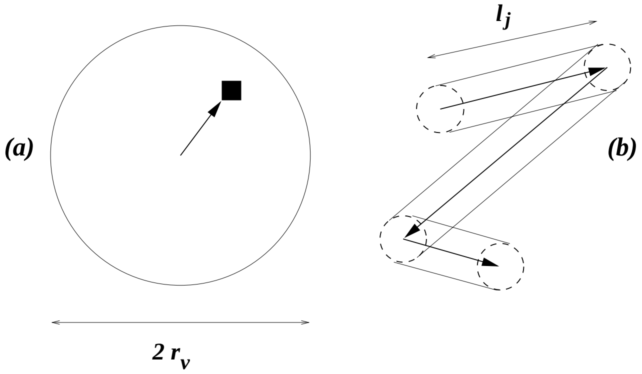


Figure 3. Search strategy: (a) If there is a target site (full square) located within a "direct vision" distance r_v, then the forager moves on a straight line to it. (b) If there is no target site within a distance r_v, then the forager chooses a random direction and a random distance ℓ_j from the Lévy probability distribution $P(\ell_j) \sim \ell_j^{-\mu}$, and then proceeds as explained in the text.

One can solve this model as follows: let λ be the mean free path of the forager between successive target sites (for 2-D, $\lambda \equiv (2 r_v \rho)^{-1}$ where ρ is the target site area density). The mean flight distance is

$$\langle \ell \rangle \approx \frac{\int_{r_v}^{\lambda} dx x^{1-\mu} + \lambda \int_{\lambda}^{\infty} x^{-\mu} dx}{\int_{r_v}^{\infty} x^{-\mu} dx} = \left(\frac{\mu-1}{2-\mu}\right)\left(\frac{\lambda^{2-\mu} - r_v^{2-\mu}}{r_v^{1-\mu}}\right) + \frac{\lambda^{2-\mu}}{r_v^{1-\mu}} \quad . \tag{2}$$

The second term of this "mean field" calculation is an approximation because it assumes that the distances between successive sites are identically equal to λ, so that there are no flights longer than λ. A new target site is always encountered a maximum distance λ away from the previous target site, effectively resulting in a truncated Lévy distribution [27]. A more rigorous treatment that considers not only the mean value but also a Poisson distribution of the free paths does not seem to alter the results significantly, as further discussed below.

One defines the search efficiency function $\eta(\mu)$ to be the ratio of the number of target sites visited to the total distance traversed by the forager. Since the total distance travelled is approximately equal to the product of the total number of flights and the mean flight length$\langle \ell \rangle$, therefore

$$\eta = \frac{1}{N \langle \ell \rangle} \ , \tag{3}$$

where N is the mean number of flights taken by a Lévy forager in order to travel between two successive target sites.

Consider first the case of *destructive search*, when the target site is "eaten" or destroyed by the searching animal and becomes unavailable in subsequent flights. The mean number of flights N_d taken to travel an average distance λ between two successive target sites scales as

$$N_d \sim (\lambda / r_v)^{\mu - 1} \tag{4}$$

for $1 < \mu \leq 3$. Here $\mu - 1$ is the fractal dimension of the set of sites visited by a Lévy random walker[28]. Note that $N_d \sim (\lambda / r_v)^2$ for $\mu > 3$ (Brownian case). Consider the common case in which the target sites are "sparsely" distributed, defined by $\lambda \gg r_v$. Substituting Eqs. (2) and (4) into (3) one finds that the mean efficiency η has no maximum, with lower values of μ leading to more efficient searches. Note that when $\mu = 1 + \epsilon$ with $\epsilon \to 0^+$, the fraction of flights with $\ell_j < \lambda$ becomes negligible, and effectively the forager moves along straight lines until it detects a target site.

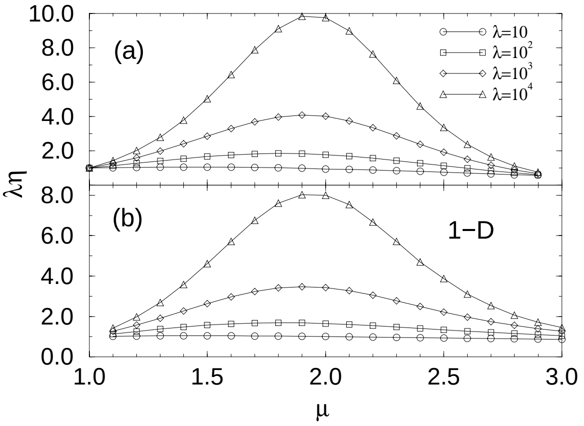
Consider next the case of *nondestructive search* for sparsely distributed target sites. Since previously-visited sites can then be revisited, the mean number N_d of flights between successive target sites in Eq. (4) overestimates the true number N_n for the nondestructive case. It can be shown that $N_n \sim N_d^{1/2}$ holds generally, so it follows that

$$N_n \sim (\lambda / r_v)^{(\mu - 1)/2} \tag{5}$$

for $1 < \mu \leq 3$. Notice that $N_n \sim \lambda / r_v$ for $\mu > 3$ (Brownian case). Indeed, we have recently proved that Eq. 5 is in fact a rigorous result [17]. This result has also been systematically tested using simulations and found to become better and better as (λ / r_v) increases (compare also Figs. 4(a) and (b)). Note that if $\lambda \gg r_v$ then $N_d \gg N_n$. Substituting Eqs. (2) and (5) into (3) and differentiating with respect to μ, one finds that the optimal efficiency $\eta = 1/(N_n \langle \ell \rangle)$ is achieved at

$$\mu_{\text{opt}} = 2 - \delta \ , \tag{6}$$

where $\delta \sim 1/[\ln{(\lambda / r_v)}]^2$. So in the absence of *a priori* knowledge about the distribution of target sites, an optimal strategy for a forager is to choose $\mu_{\text{opt}} = 2$ when λ / r_v is large but not exactly known.



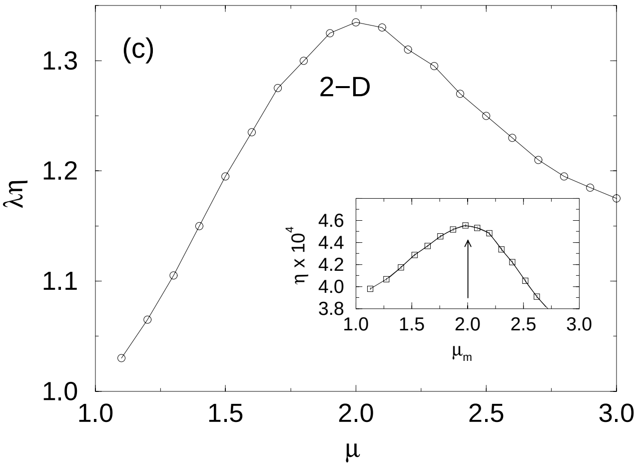


Figure 4. The product of the search efficiency η and the mean free path λ vs. μ in 1-D for different λ, found (a) from Eqs. (2) and (3) ($r_v = 1$) for the case of nondestructive search and (b) from simulations. (c) $\eta\lambda$ found from simulations in 2-D, with $\lambda = 5000$ ($r_v = 1$). In each case, $\mu_{\text{opt}} \approx 2$ emerges as an optimal value of the Lévy flight exponent. Inset: the food is distributed in patches of food-rich areas in an otherwise empty environment, obtained by computing $\mu_{\text{m}} = -d \log N(\ell) / d \log \ell$ from the histograms $N(\ell)$ of flight lengths. Only flights with $\log_{10} \ell$ <4.5 are considered in order to eliminate the effects of the periodic boundaries. Again, $\mu_{\text{m}} \approx 2$ seems to optimize the search efficiency.

IV Discussion

The above results are independent of the dimension of the search space. This is analogous to the behavior of Brownian random walks whose mean square displacement is proportional to the number of steps in any dimension [7]. Furthermore, Eqs. (4) and (5) describe the correct scaling properties even in the presence of short-range correlations in the directions and lengths of the flights. Short-range correlations can alter the width of the distribution $P(\ell)$, but cannot change μ, so these findings remain unchanged. Hence, learning, predator-prey relationships, and other short-term memory effects become unimportant in the long-time long-distance limit.

Note also that for both destructive and nondestructive searches, Brownian behavior, corresponding to $\mu \geq 3$, is significantly less efficient than Lévy flight motion. This finding suggests that a power law distribution of flight lengths may be *essential* for optimal searches when the target sites are sparsely and randomly distributed. (One may argue that animals would possibly starve to death while adopting a Gaussian strategy of foraging.)

For completeness, consider also the case in which the target sites are plentiful, i.e., $\lambda \leq r_v$. Then $\langle \ell \rangle \approx \lambda$ and $N_d \approx N_n \approx 1$. Hence, η becomes independent of μ. This behavior does not correspond to Lévy flight searches but is more similar to a Brownian random walk. The independence of η on μ is a direct consequence of the extreme rarity of long flights with $\ell_j > r_v$.

These theoretical results have been supported with numerical simulations which do not depend on approximations. Indeed, 1-D and 2-D simulations have been performed of the above model to study how η varies with μ for the case of nondestructive searches. For the case of nondestructive searches, Fig. 4(a) shows the simulation results for various values of λ and $r_v = 1$. For 1-D, the position of the maximum in η for the simulation agrees with the analytical results (Fig. 4(b)), and approaches $\mu_{\rm opt} = 2$ as $\lambda \to \infty$. The simulation results for 2-D nondestructive search also show maxima near $\mu_{\rm opt} = 2$. Fig. 4(c) shows simulated search in a system of size $10^4 \times 10^4$ with $r_v = 1$, periodic boundary conditions, and $\lambda / r_v = 5 \times 10^3$. Moreover, for destructive search with $\lambda \gg r_v$, simulations show that $\mu \to 1$ optimizes the efficiency as predicted. In contrast, if the target sites are densely distributed such that $\lambda \approx r_v$, then, as expected, we find no significant effect of varying μ. These findings agree with the theoretical predictions and raise the possibility that Lévy flight searches with $\mu < 3$ may be confined to instances of low global target site concentration, since the principal advantage of choosing small μ — long flights — becomes negligible when there are ample target sites. Indeed, bees appear not to apply Lévy flight foraging for artifically high nectar concentrations (see also Figs. 2(a) and 2(b)).

We also note that nondestructive search is more realistic than destructive search because in nature, flowers, berries, krill, fish, etc. are usually found in patches or clumps which are rarely completely depleted. Thus an animal can revisit the same food patch many times, and a patch can restore itself by regrowth. Simulations of destructive searches in patchy target site distributions give results consistent with nondestructive searches for uniformly distributed target sites. As an illustrative example, Fig. 4c (inset) shows $\eta(\mu)$ found from simulating a patchy distribution of food. There are many small randomly distributed food-rich regions, each with radius R, outside of which there is no food to be found. To speed up the simulations, it was assumed that the forager performs a Lévy Flight only outside the region of radius R, and that it instead performs Brownian motion within, finding food at each site along the way separated by the local mean free path $\lambda = R/3$. The system size used was $10^5 \times 10^5$. A low patch area to system area ratio of $\pi \times 10^{-6}$ is achieved using a patch radius of $R = 10$, and 10^4 patches. Here, $\mu_{\rm m} \equiv -d \log N(\ell) / d \log \ell$ was measured from the histogram $N(\ell)$ of flight lengths instead of using the parameter μ from the model. Such a histogram represents an experimentally observable distribution of flight lengths. Note that $\mu_{\rm m}$ is consistent with the theoretically predicted value $\mu_{\rm opt}$.

Acknowledgements

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