Exact Solutions of Brans-Dicke Cosmology and the Cosmic Coincidence Problem

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We present some cosmological solutions of Brans-Dicke theory, characterized by a decaying vacuum energy density and by a constant relative matter density. With these features, they shed light on the cosmological constant problems, leading to a presently small vacuum term, and to a constant ratio between the vacuum and matter energy densities. By fixing the only free parameter of our solutions, we obtain cosmological parameters in accordance with observations of the relative matter density, the universe age and redshift-distance relations.

Several authors have been considering the possibility of a varying cosmological term in order to fit the observed non-decelerated expansion of the universe and, at the same time, to explain the small value of the cosmological constant observed at present [1]-[13]. Such a variation of the vacuum density has found support in some quantum field approaches (see, for example, [6, 9]), in which context an induced variation of the gravitational coupling constant G may also be expected [5, 10, 11, 13].

Our goal in this contribution is to present some cosmological solutions with decaying vacuum in the realm of Brans-Dicke theory. For this purpose, we will consider an empirical variation law for G, given by the Weinberg relation $G \approx H/m_\pi^3$, where $H = \dot{a}/a$ is the Hubble parameter, and m_π is the energy scale of the QCD chiral symmetry breaking, the latest cosmological vacuum phase transition. Such a relation, originally based on the Eddington-Dirac large number coincidence, can find some support on theoretical, holographic arguments [11, 14]. Let us write it as

$$G = \frac{H}{8\pi\lambda}, \tag{1}$$

where the constant λ is positive and has the order of m_π^3.

With this variation law for G, we will form an empirical ansatz to be used in Brans-Dicke equations. It is fulfilled by the additional constraint

$$\rho = \frac{3\alpha H^2}{8\pi G}, \tag{2}$$

where $\rho = \rho_m + \rho_\Lambda$ is the total energy density, and α is a positive constant of the order of unity. As we know, in the context of scalar-tensor theories, even for zero spatial curvature, the total energy density is not necessarily equal to the critical density $\rho_c \equiv 3H^2/(8\pi G)$. Therefore, the above equation should also be considered an empirical coincidence, suggested by observation. Our point is that, since (1) and (2) are valid nowadays, they may be valid for any time, or at least in the limit of late times.

Let us find solutions for this limit, by considering a spatially flat FLRW space-time, with a cosmic fluid formed by dust matter (i.e., $p_m = 0$) plus a vacuum term with equation of state $p_\Lambda = -\rho_\Lambda$. The Brans-Dicke equations are then given by [15, 16]

$$\frac{d(\dot{\phi}a^3)}{dt} = \frac{8\pi}{3+2\omega}(\rho - 3p)\,a^3 = \frac{8\pi}{3+2\omega}(\rho + 3\rho_\Lambda)\,a^3, \tag{3}$$

$$\dot{\rho} = -3H(\rho + p) = -3H\rho_m, \tag{4}$$

$$H^2 = \frac{8\pi\rho}{3\phi} - \frac{\dot{\phi}}{\phi}H + \frac{\omega}{6}\frac{\dot{\phi}^2}{\phi^2}, \tag{5}$$

where $p = p_m + p_\Lambda$ is the total pressure, ω is the Brans-Dicke parameter, and the Brans-Dicke scalar field ϕ is related to the gravitational constant by $\phi = G_0/G$, G_0 being a positive constant of the order of unity.

From our ansatz (1)-(2), we obtain

$$\rho = 3\alpha\lambda H, \tag{6}$$

$$\phi = \frac{8\pi\lambda G_0}{H}, \tag{7}$$

$$\dot{\phi} = 8\pi\lambda G_0(1+q), \tag{8}$$

where $q = -a\ddot{a}/\dot{a}^2$ is the deceleration parameter.

By using (6)-(8), we can rewrite equations (3)-(5) in the form

$$(3+2\omega)\lambda G_0[\dot{q} + 3(1+q)H] = 3\alpha\lambda H + 3\rho_\Lambda, \tag{9}$$

$$\rho_m = \alpha\lambda(1+q)H, \tag{10}$$

$$\frac{\alpha}{G_0} = 2 + q - \frac{\omega}{6}(1+q)^2. \tag{11}$$

Equation (11) shows that q is a constant, and (9) reduces to

$$(3+2\omega)\lambda G_0(1+q)H = \alpha\lambda H + \rho_\Lambda. \tag{12}$$

Using (6) and (10), we obtain a decaying vacuum density, given by

$$\rho_\Lambda = \alpha\lambda(2-q)H. \tag{13}$$

Leading (1) into (13), and using $\rho_\Lambda = \Lambda/8\pi G$, we see that the cosmological term scales as

$$\Lambda = \alpha(2-q)H^2. \tag{14}$$

Equations (13) and (14) are curious results, because they can also be derived on the basis of theoretical reasoning

[4, 6, 8, 9, 11], with no use of the empirical relations we are considering here. We will see that relations (13) or (14) can be used to form a second ansatz, with a larger set of solutions.

Substituting (6) and (10) into (4), we obtain a first order differential equation for H, which solution is given by

$$H = \left(\frac{1}{1+q}\right)\frac{1}{t}. \qquad (15)$$

Here, an integration constant was made zero in order to obtain the divergence of H at $t = 0$. A second integration leads to a scale factor evolving as

$$a = At^{\frac{1}{1+q}}, \qquad (16)$$

where A is an arbitrary integration constant.

With the help of (1), one can write the critical density as $\rho_c = 3\lambda H$. Therefore, from (10) we obtain a relative matter density

$$\Omega_m \equiv \frac{\rho_m}{\rho_c} = \frac{\alpha(1+q)}{3}. \qquad (17)$$

This is perhaps the most interesting feature of the present solutions. It means that, in the limit of late times we are considering here, the relative matter density is a constant. This is a consequence of the conservation of the total energy, expressed by equation (4). The vacuum decay is only possible if associated to a process of matter production (a general feature of vacuum states in non-stationary space-times). On the other hand, since in our ansatz $\rho = \alpha\rho_c$ (see (2)), the constancy of Ω_m is a possible solution to the cosmic coincidence problem, that is, the unexpected approximate coincidence between the matter density and the dark energy density. We will see below that (17) is in accordance with present observations.

Substituting ρ_Λ from (13) into (12), one has

$$\frac{\alpha}{G_0} = \frac{(3+2\omega)(1+q)}{3-q}. \qquad (18)$$

Comparing the values of α/G_0 given by (11) and (18), we obtain a relation between ω and q:

$$(3+2\omega)(1+q) = \left[2+q-\frac{\omega}{6}(1+q)^2\right](3-q). \qquad (19)$$

We can also, by eliminating ω from (11) and (18), derive α/G_0 as a function of q:

$$\frac{\alpha}{G_0} = \frac{12(2+q)+3(1+q)^2}{(1+q)(3-q)+12}. \qquad (20)$$

With all these results we can estimate corresponding values for the cosmological parameters. For example, for $q = 0$ (that is, for $a = At$) we obtain, from equation (19), $\omega = 6/5$; from (20) we have $\alpha/G_0 = 9/5$; from (15) it follows $Ht = 1$; and from (17) one obtains $\Omega_m = \alpha/3$.

As $\alpha \approx 1$, we see that in this case $\Omega_m \approx 0.3$, a result corroborated by astronomical estimations [17]. On the other hand, an age parameter $Ht \approx 1$ has been suggested by globular clusters observations [18]. Finally, a coasting expansion with $q \approx 0$ is consistent with observations of distance-redshift relations for supernova Ia and compact radius sources [19–21].

As to the Brans-Dicke parameters ω and G_0 are concerned, they are positive and of the order of unity. This could be considered a bad result, in view of the high lower limits imposed to ω by astronomical tests in the Solar System. Nevertheless, let us remember that we are concerned here to a vacuum density variation (and to a corresponding induced G variation) at the cosmological scale, variations that cannot be ruled out by observations at the local scale (where the metric is stationary, and where any spatial dependence of G, related to some spatial dependence of the vacuum density, is negligible). In this sense, the Brans-Dicke theory we are using here, with constant ω, must be considered an effective description, valid only in the cosmological limit [13]. A more general approach can be based on scalar-tensor theories in which ω depends on the scale, being very high in the weak field approximation of Solar System.

Another point to be considered is the constant character of the deceleration parameter in our solutions. Although the present observations indicate a non-decelerating expansion, an earlier decelerated phase is usually expected in order to allow structure formation. In spite of the claim of some authors [21] about the possibility of a coasting expansion (i.e., with $q \approx 0$) even for earlier times, a more conservative viewpoint would be to consider our empirical ansatz (1)-(2) valid only in the limit of late times, as we have been done.

We can also replace our original ansatz by a more general one (in the sense of presenting a larger set of solutions). We have seen that our results (13) and (14) (with constant q) have been justified in different theoretical approaches to the vacuum energy in curved space-times (for instance, in [6, 9]). We can therefore substitute $\Lambda = \beta H^2$ (or, equivalently, $\rho_\Lambda = \beta\lambda H$, where β is a constant) for our empirical relation (2), retaining the Weinberg relation (1). As this last one may also be understood on the basis of theoretical arguments (as in [11], for example), this new ansatz should be considered more justified from a theoretical viewpoint.

The set of solutions we can find leading the new ansatz into Brans-Dicke equations (3)-(5) will be shown in a forthcoming publication. Let us just mention that we re-obtain, as a particular case, the same solutions we have obtained with the former ansatz. In addition, we have three other cosmological solutions, for which the Brans-Dicke parameter is $\omega = -1$, and the vacuum energy density is negative. In two of these additional solutions, the deceleration parameter is always highly positive or always highly negative, being them therefore ruled out by observations.

The third new solution has an initial singularity, an early decelerating phase followed by an accelerated one, and a "big-rip" singularity, with a, H and ρ_m diverging at a finite time in the future (but with Ω_m remaining finite). So, it is an interesting solution from a theoretical perspective. Unfortunately, for q in the range given by the present observations ($-1 < q < 1$), the age parameter for this solution is less than $Ht \approx 0.5$, outside the observed limits.

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