Low-Dimensional Non-Linear Dynamical Systems and Generalized Entropy

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Low-dimensional non-linear maps are prototype models to study the emergence of complex behavior in nature. They may exhibit power-law sensitivity to initial conditions at the edge of chaos which can be naturally formulated within the generalized Tsallis statistics prescription which is characterized by the entropic index q. General scaling arguments provide a direct relation between the entropic index q and the scaling exponents associated with the extremal sets of the multifractal critical attractor. The above result comes in favor of recent conjectures that Tsallis statistics is the natural frame for studying systems with a fractal-like structure in the phase-space. Power-law sensitivity in high-dimensional dissipative and Hamiltonian systems are also discussed within the present picture.

I Introduction

Low-dimensional non-linear maps are the prototype models to study the emergence of complex behavior in dynamical systems. Their typical behavior include the occurrence of bifurcation instabilities, long-range correlated sequences, fractal structures and chaos, which are commonly observed in a great variety of systems ranging from fluids, magnetism, biology, social sciences and many others[1].

The study of the sensitivity to initial conditions of non-linear systems is one of the most important tools used to investigate the nature of the phase-space attractor. It is usually characterized by the Liapunov exponent λ, defined for the simple case of a one-dimensional dynamical variable x as

$$\Delta x(t) \sim \Delta x(0) e^{\lambda t} \qquad (\Delta x(0) \to 0, \quad t \to \infty) \qquad (1)$$

where $\Delta x(0)$ is the distance between two initially nearby orbits (in an equivalent point of view, it is the uncertainty on the precise initial condition). If $\lambda > 0$ the system is said to be strongly sensitive to the initial condition with the uncertainty on the dynamical variable growing exponentially in time and this characterizes a chaotic motion in the phase-space. On the other hand, if $\lambda < 0$ the system becomes strongly insensitive to the initial condition which is expected for any state whose dynamical attractor is an orbit with a finite period.

The problem of the sensitivity to initial conditions can be reformulated in an entropic language as a process of information loss (in the case of chaotic behavior) or recovery (for periodic attractors). Within this context, it is useful to introduce the Kolmogorov-Sinai entropy K. It is basically the rate of variation of the Boltzmann-Gibbs (BG) entropy $S = -\sum_{i=1}^{W} p_i \ln p_i$ where W is the total number of possible configurations and $\{p_i\}$ the associated probabilities[2]. Considering the evolution of an ensemble of identical copies of the system under investigation, p_i stands for the fractional number of points of the ensemble that are in the i cell of a suitable partition of the phase space in cells of size l. The Kolmogorov-Sinai entropy can be represented as

$$K \equiv \lim_{\tau \to 0} \lim_{l \to 0} \lim_{N \to \infty} \frac{1}{N\tau} [S(N) - S(0)] \quad , \qquad (2)$$

where $S(0)$ and $S(N)$ are the entropies of the system evaluated at times $t = 0$ and $t = N\tau$ (for maps $\tau = 1$). With the simplifying assumption that at time t there are $W(t)$ occupied cells with the same occupation number, we have from equation (2) that

$$W(t) = W(0) e^{Kt} \quad , \qquad (3)$$

which is equivalent to equation (1) for the sensitivity to initial condition and provides the well-known Pesin

equality, $K = \lambda$[3].

However, the above picture does not suitably describe the sensitivity to initial conditions at bifurcation points and at the threshold to chaos which are the marginal cases where $\lambda = 0$. At these points, the BG entropy does not vary at a constant rate and therefore does not provide a useful tool to characterize the rhythm of information loss or recovery. The failure of the above prescription to characterize these points is related to the fact that the extensive BG entropy can not properly deal with the underlying fractality (and therefore non-extensivity) of the phase-space attractor. In this work we will review some recent works which have shown that the Tsallis generalized q-entropies can give a proper description of these marginal points. Furthermore they have provided some enlightening relations between the q-entropic factor and the scaling properties of the dynamical attractor[4, 5, 6, 7, 8]

This work is organized as follows. In section 2, we numerically illustrate the behavior of BG entropy and sensitivity to initial conditions in the standard logistic map. In section 3, we show how the power-law sensitivity at bifurcation and critical points can be naturally derived within the generalized Tsallis entropy formalism characterized by the index q which is associated with the degree of non-extensivity. In section 4, we review the scaling properties of critical dynamical attractors that can be characterized as a multifractal measure. In section 5, we show how scaling arguments can be used to predict a direct relationship between the entropic index q and the scaling exponents associated with the extremal sets of the critical attractor. We also illustrate the accuracy of the predicted scaling relation using two distinct families of one-dimensional dissipative maps. In section 6, we discuss the emergence of power-law sensitivity in high dimensional dissipative and Hamiltonian systems. Finally, in section 7, we summarize and draw some perspectives on future developments.

II BG entropy and Sensitivity to initial conditions in the logistic map

From the Kolmogorov-Sinai entropic representation of the sensitivity to initial conditions problem, we learn that the exponential sensitivity to initial conditions is directly associated to the fact that the Boltzmann-Gibbs-Shannon entropy exhibits a constant asymptotic variation rate per unit time. Lets illustrate the above mentioned point using the standard logistic map

$$x_{t+1} = 1 - a x_t^2 \qquad (4)$$

with $x_t \in [-1, 1]$; $a \in [0, 2]$; $t = 0, 1, 2, ...$ The dynamical attractor as a function of a is shown in Fig. 1a. For small a it exhibits periodic orbits which bifurcate as a increases and the bifurcation points accumulate at a critical value $a_c = 1.40115518909...$ above which chaotic orbits emerge. The Liapunov exponent λ as a function of the parameter a is displayed in Fig. 1b. The predicted trend, i.e., $\lambda < 0$ ($\lambda > 0$) for periodic (chaotic) orbits is clearly observed. Notice that $\lambda = 0$ describes indistinctly the bifurcation points and chaos threshold. To numerically estimate the BG entropy, we perform a fine partitioning of the phase space. Then we follow the temporal evolution of a large number of initial conditions regularly distributed around $x = 0$ which corresponds to the extremal point of this map. Assuming equiprobability, we can directly estimate the BG entropy as a function of the number of iterations of the map by recording the number of distinct partitions visited by these systems and using that $S = \ln W$. Therefore, an exponential time dependence of W will be equivalent to a constant rate of variation of BG entropy. In Fig. 2 we show some numerical estimates of $W(N)$ for the logistic map at values of a for which the dynamical attractor is a fixed point and a chaotic orbit. Notice that the exponential time dependence is verified at the points were the Liapunov is expected to be finite. However, for marginal cases were $\lambda = 0$, as for example in a period doubling bifurcation point and at the chaos threshold (see Fig. 3), we observe a power-law time evolution of the phase space volume visited by the ensemble. Therefore, the BG entropy form fails in providing a good information measure that exhibits a constant variation rate at these marginal points.

An equivalent but numerically more precise study can be made directly on the sensitivity to initial conditions. The sensitivity function is defined as

$$\xi(t) \equiv \lim_{\Delta x(0) \to 0} \Delta x(t)/\Delta x(0) \equiv \frac{dx(t)}{dx(0)} \quad , \qquad (5)$$

from which we can directly follow the time evolution of the distance between two systems with nearby initial conditions. In Fig. 4, we show the sensitivity as a function of time for typical values of the non-linear parameter a which show trends similar to the ones observed for the phase-space volume visited $W(N)$.

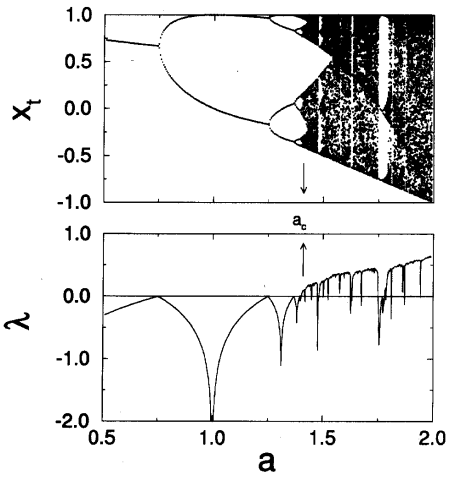

Figure 1. a) The dynamical attractor of the logistic map as a function of the parameter a. The attractor exhibits a series of bifurcations as a increases that accumulate at $a_c = 1.40115518909...$, above which chaotic orbits emerge; b) The Liapunov exponent λ versus a. Notice that $\lambda < 0$ for periodic orbits, $\lambda > 0$ for chaotic orbits and $\lambda = 0$ at bifurcation and critical points. Strong fluctuations of λ for $a > a_c$ reflects the presence of periodic windows at all scales.

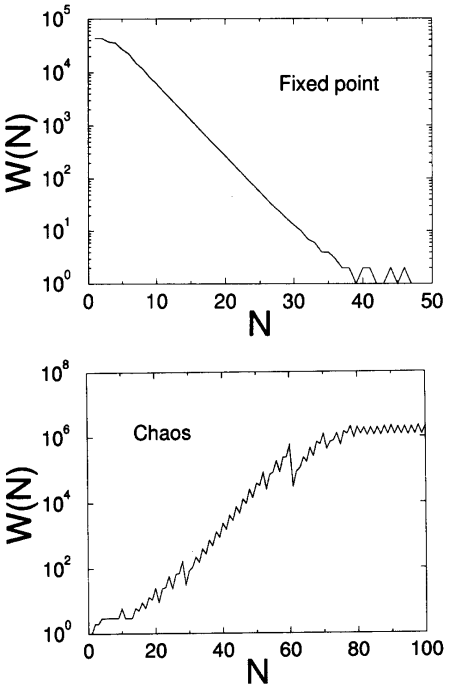

Figure 2. The temporal evolution of the phase space volume visited by an ensemble of logistic maps with initially nearby initial conditions for typical values of the nonlinear parameter a at which the Liapunov is finite. a) $a = 0.5$, corresponding to a fixed point attractor ($\lambda < 0$, exponentially converging orbits). Data were obtained from 10^5 initial conditions distributed in the interval $[-0.2, 0.2]$ using a partition with 10^7 boxes. b) $a = 1.45$, corresponding to a chaotic attractor ($\lambda > 0$, exponentially diverging orbits). Data were obtained from 10^7 initial conditions spread in the interval $[-10^{-4}, 10^4]$ using a partition with 10^7 boxes. The saturation for large times is due to the finite partition of the phase space.

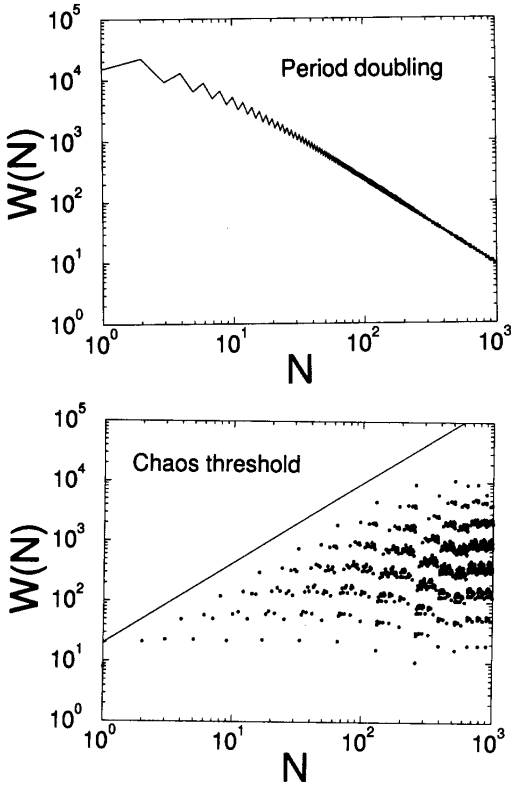

Figure 3. The temporal evolution of the phase space volume visited by an ensemble of logistic maps with initially nearby initial conditions for typical values of the nonlinear parameter a at which the Liapunov is zero. a) $a = 0.75$, corresponding to a period doubling bifurcation (power-law converging orbits). Data were obtained from 10^5 initial conditions distributed in the interval $[-0.2, 0.2]$ using a partition with 10^6 boxes. b) $a = 1.40115518909...$, corresponding to the onset of chaos (power-law diverging orbits). Data were obtained from 10^5 initial conditions spread in the interval $[-10^{-3}, 10^3]$ using a partition with 10^7 boxes. The pattern observed reflects the fractal-like structure of the critical attractor.

III Power-law sensitivity and generalized entropies

Power-law sensitivity, as observed at bifurcation and critical points, has been shown to be naturally derived from the assumption that a proper non-extensive entropy exhibits a constant variation rate at these points[4, 5]. Namely, using Tsallis entropy form

$$S_q = \frac{1 - \sum_{i=1}^{W} p_i^q}{q - 1} \qquad (6)$$

a generalized Kolmogorov-Sinai entropy can be defined as

$$K_q \equiv \lim_{\tau \to 0} \lim_{l \to 0} \lim_{N \to \infty} \frac{1}{N\tau} [S_q(t) - S_q(0)] \quad . \qquad (7)$$

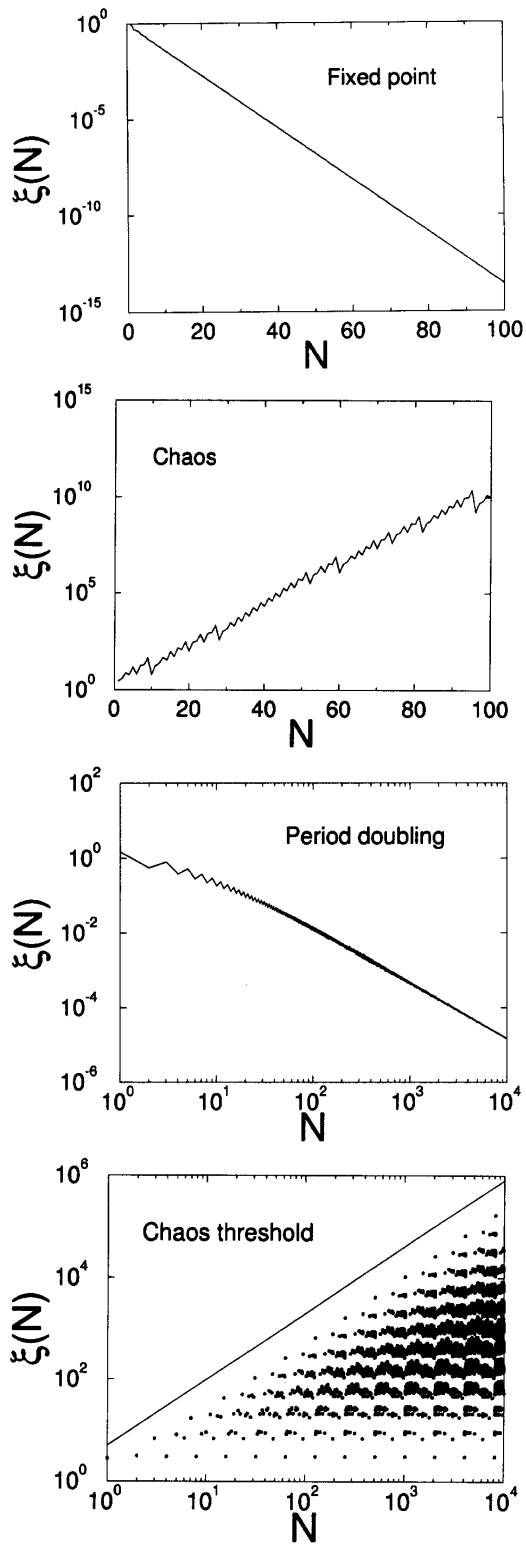

Figure 4. The sensitivity function at typical points of the logistic map. a) $a = 0.5$ which has a fixed point attractor; b) $a = 1.45$ which has a chaotic attractor; c) $a = 0.75$ which is a period doubling bifurcation; d) $a = 1.401155...$ corresponding to the chaos threshold. The trends are similar to the ones shown in Figures 2 and 3 for $W(N)$.

Assuming equiprobability and a constant K_q, it can be readily obtained that the volume on the phase space shall evolve in time as

$$W(t) = W(0)[1 + (1-q)K_q t]^{1/(1-q)} \qquad (8)$$

consistent with the asymptotic power-law behavior at marginal points where $\lambda = 0$. Assuming a generalized Pesin equality $K_q = \lambda_q$, we can also write the sensitivity function within the present formalism as

$$\xi(t) = [1 + (1-q)\lambda_q t]^{1/(1-q)} \quad . \qquad (9)$$

The above relation provides a direct relationship between the entropic index q and the sensitivity power-law exponent. For $q > 1$ the system becomes weakly insensitive to the initial conditions once the visited volume on the phase space slowly shrinks as the system evolves in time. This is the case for period doubling bifurcation points of the logistic map where $1/(1-q) = -3/2$ and therefore $q = 5/3$. On the other hand, for $q < 1$ the system becomes weakly sensitive to the initial conditions as $W(t)$ slowly grows with time. This is observed at the onset of chaos of the standard logistic map, where it was obtained $1/(1-q) = 1.325$ and therefore $q = 0.2445$[4].

The close relationship between the entropic index q of Tsallis entropies and the sensitivity to initial conditions at the onset of chaos of such non-linear low-dimensional dissipative maps provides a useful recipe to estimate the proper entropic index from the system dynamical rules. This relationship has been further used to investigate a recent conjecture that the non-extensive Tsallis statistics is the natural framework for studying systems with a fractal-like structure in the phase space[9]. The critical dynamical attractor of such non-linear dissipative systems can be associated with a multifractal measure whose scaling exponents can be obtained from traditional methods. Therefore, both the entropic index q and the scaling properties of the critical attractor can be estimated independently and their relation revealed.

IV Multifractal scaling of critical attractors

In order to completely describe the scaling behavior of critical dynamical attractors it is necessary to introduce a multifractal formalism[10]. The partition function $\chi_Q(N) = \sum_{i=1}^{N} p_i^Q$ is a central quantity within

this formalism, where p_i represents the probability (integrated measure) on the i-th box among the N boxes of the measure (we use Q instead of the standard notation q in order to avoid confusion with the entropic index q).

In chaotic systems p_i is the fraction of times the trajectory visits the box i. In the $N \to \infty$ limit, the contribution to $\chi_Q(N) \propto N^{-\tau(Q)}$, with a given Q, comes from a subset of all possible boxes, whose number scales with N as $N_Q \propto N^{f(Q)}$, where $f(Q)$ is the fractal dimension of the subset ($f(Q=0)$ is the fractal dimension d_f of the support of the measure). The content on each contributing box is roughly constant and scales as $P_Q \propto N^{-\alpha(Q)}$. These exponents are all related by a Legendre transformation

$$\tau(Q) = Q\alpha(Q) - f(Q) \quad (10)$$

$$\alpha(Q) = \frac{d}{dQ}\tau(Q). \quad (11)$$

The multifractal object is then characterized by the continuous function $f(\alpha)$, which reflects the different dimensions of the subsets with singularity strength α. $f(\alpha)$ is usually shaped like an asymmetric $\cap$. The α values at the end points of the $f(\alpha)$ curve are the singularity strength associated with the regions in the set where the measure is most concentrated ($\alpha_{min} = \alpha(Q=+\infty)$) and most rarefied ($\alpha_{max} = \alpha(Q=-\infty)$).

Halsey *et al* have shown how the singularity spectrum of measures possessing an exact dynamical rule can be obtained from a simple procedure[10] that considers a non-uniform grid of the phase space. First, one shall consider the original support with normalized measure and size. Then, one divides the region in N pieces, each one with measure p_i and size l_i. The proper values of N are dictated by the natural scaling factor inherent to the recursive relations. After that, it is computed the partition function

$$, (Q,\tau,l) = \sum_i \frac{(p_i)^Q}{(l_i)^\tau} \quad . \quad (12)$$

From the recursive structure of the measure, it can be shown that the proper $\tau(Q)$ is defined by $, (Q,\tau(Q),l) = 1$, and the singularity spectrum follows from the Legendre relations.

The end points of the $f(\alpha)$ curves of the critical attractor of one-dimensional dissipative maps can be inferred theoretically from well known scaling properties related to the most concentrated and most rarefied intervals in the attractor. Feigenbaum has shown that, after $N = \omega^n$ iterations (ω is a natural scaling factor inherent to the recursive relations), the size of these intervals scale respectively as $l_{-\infty} \sim [\alpha_F]^{-n}$ and $l_{+\infty} \sim [\alpha_F(z)]^{-zn}$, where α_F is a universal scaling factor [11] and z is the inflexion at the vicinity of the extremal point of the map. Since the measures in each box are simply $p_i = 1/N = \omega^{-n}$, the end points are expected to be

$$\alpha_{max} = \ln p_i / \ln l_{+\infty} = \frac{\ln \omega}{\ln \alpha_F(z)} \quad (13)$$

$$\alpha_{min} = \ln p_i / \ln l_{-\infty} = \frac{\ln \omega}{z \ln \alpha_F(z)} \quad (14)$$

V The entropic index and the extremal sets

Scaling arguments applied to the most rarefied and most concentrated regions of the attractor provide a precise relationship between the singularity spectrum extremals and the entropic index q[6, 7]. Considers an ensemble of identical systems whose initial conditions spread over a region of the order of the typical box size in the most concentrated region $l_{+\infty}$. In other words, we are considering that our uncertainty on the precise initial conditions is $\Delta x(t=0) \sim l_{+\infty}$. After N time steps these systems will be spread over a region whose size is at most of the order of the typical size of the boxes in the most rarefied region ($\Delta x(N) \sim l_{-\infty}$). Therefore, assuming power-law sensitivity on the initial conditions on the critical state, we can write Eq. 9 for large N as

$$\xi(N) \equiv \lim_{\Delta x()\to 0} \frac{\Delta x(N)}{\Delta x(0)} = \frac{l_{-\infty}}{l_{+\infty}} \sim N^{1/(1-q)}, \quad (15)$$

and using Eqs. (13-14) it follows immediately that

$$\frac{1}{1-q} = \frac{1}{\alpha_{min}} - \frac{1}{\alpha_{max}}. \quad (16)$$

The above relation indicates that the proper nonextensive statistics can be inferred from the knowledge of the scaling properties associated with the extremal sets of the dynamical attractor. This relation follows from very usual and general scaling arguments and therefore shall be applicable to a large class of nonlinear dynamical systems irrespective of the underlying topological and metrical properties.

The above relation has been numerically observed to hold with very high accuracy for the critical attractors of the family of generalized Logistic maps[6]

$$x_{t+1} = 1 - a|x_t|^z \quad ;(z > 1 \; ; \; 0 < a < 2 \; ; \; t = 0, 1, 2, ... \; ; \; x_t \in [-1, 1]) \; . \tag{17}$$

Here z is the inflexion of the map in the neighborhood of the extremal point $\bar{x} = 0$. These maps are well known[12, 13] to have the topological properties (such as the sequence of bifurcations while varying the parameter a) not dependent of z, but the metric properties (such as Feigenbaum's exponents and multifractal singularity spectra of the attractors) do depend on z. The scaling relation has also been checked to hold for the family of circular maps[8]

$$\theta_{t+1} = \Omega + K \left[\theta_t - \frac{1}{2\pi}\sin(2\pi\theta_t)\right]^{z/3} \quad , \quad mod(1), \tag{17}$$

with $0 < \Omega < 1 \; ; \; 0 < K < \infty$. For $K = 1$ these maps exhibit critical orbits for which the renormalized winding number $\omega = \lim_{t\to\infty}(\theta_{t+1} - \theta_t)$ equals to the golden mean[14]. The above two family of maps belong to distinct universality classes and therefore exhibit distinct scaling behavior for the same value of the inflexion z. The multifractal singularity spectra for these two families were numerically obtained and the extremal values of the singularity strength α_{min} and α_{max} estimated for a wide range of z values (see Fig. 5). From the power-law exponent of the sensitivity function the value of $1/(1-q)$ could be independently estimated. In the tables, we summarize the results obtained for both families which show that the proposed scaling relation is satisfied.

Table 1 - z-generalized family of logistic maps. Numerical values for several inflexions z of: *i*) the critical parameter a_c at the onset of chaos; *ii*) α_{min}; *iii*) α_{max}; *iv*) q as predicted by the scaling relation Eq. 16 and *v*) q from the sensitivity function.

z	a_c	α_{min}	α_{max}	q (Eq.(16))	q (Eq.(9))
1.10	1.1249885...	0.302	0.332	-2.34 ± 0.02	-2.33 ± 0.02
1.25	1.2095137...	0.355	0.443	-0.79 ± 0.01	-0.78 ± 0.01
1.50	1.2955099...	0.380	0.568	-0.15 ± 0.01	-0.15 ± 0.01
1.75	1.3550607...	0.383	0.667	0.10 ± 0.01	0.11 ± 0.01
2.00	1.4011551...	0.380	0.755	0.23 ± 0.01	0.24 ± 0.01
2.50	1.4705500...	0.367	0.912	0.39 ± 0.01	0.39 ± 0.01
3.00	1.5218787...	0.354	1.054	0.47 ± 0.01	0.47 ± 0.01
5.00	1.6455339...	0.315	1.561	0.61 ± 0.01	0.61 ± 0.01

Table 2 - z-generalized family of circle maps. Numerical values for several inflexions z of: *i*) the critical parameter Ω_c at the onset of chaos; *ii*) α_{min}; *iii*) α_{max}; *iv*) q as predicted by the scaling relation Eq. 16 and *v*) q from the sensitivity function.

z	Ω_c	α_{min}	α_{max}	q (Eq.(16))	q (Eq.(9))
3.0	0.606661063469...	0.632	1.895	0.05 ± 0.01	0.05 ± 0.01
3.5	0.629593799039...	0.599	2.097	0.16 ± 0.01	0.15 ± 0.01
4.0	0.648669091983...	0.572	2.289	0.24 ± 0.01	0.24 ± 0.01
4.5	0.664861001064...	0.542	2.440	0.30 ± 0.01	0.30 ± 0.01
5.0	0.678831756505...	0.516	2.581	0.36 ± 0.01	0.36 ± 0.01
5.5	0.691048981515...	0.491	2.701	0.40 ± 0.01	0.40 ± 0.01
6.0	0.701853340894...	0.473	2.838	0.43 ± 0.01	0.44 ± 0.01
7.0	0.720182442561...	0.438	3.065	0.49 ± 0.01	0.50 ± 0.01
8.0	0.735233625356...	0.410	3.280	0.53 ± 0.01	0.53 ± 0.01

VI Power-law sensitivity in higher dimensional dissipative and Hamiltonian systems

The predicted scaling relation between the entropic index q of generalized entropies and the scaling exponents related to the most extremal sets in the dynamical attractor provides an important clue for how to estimate the proper non-extensive entropy for systems with long-range spatio-temporal correlations. For these systems, one shall expect a power-law sensitivity to initial conditions whose exponent is directly related to q. Therefore, q can be estimated if we are able to follow a critical dynamical trajectory. Furthermore, if only the dynamical attractor is accessible, q can also be obtained from its multifractal singularity spectrum.

Usually, for systems with a large number of degrees of freedom, the scaling properties of the dynamical attractor in the phase space are hardly accessible due to computational limitations. However, a dynamical trajectory can be easily followed and the sensitivity to initial conditions estimated by computing the time evolution of the distance between two initially nearby orbits. Power-law sensitivity to initial conditions has been observed in a series of high dimensional dissipative systems which are naturally driven to a critical attractor, usually referred in the literature as self-organized critical systems. These systems range from the Bak-Sneppen model of biological evolution[15], the rice pile model[16] and coupled logistic maps[17]. Therefore, in all these dissipative extended model systems, there is a proper non-extensive generalized entropy that during the dynamical evolution exhibits a constant variation rate.

On the other hand, Hamiltonian systems are expected to be ergodic in the thermodynamical limit whenever the interactions are short-ranged. In other words, all trajectories becomes chaotic in the thermodynamical limit. However, Hamiltonian systems with just a few degrees of freedom may have a finite volume of the phase-space on which quasi-periodic orbits exist. In this case, one expects power-law sensitivity to initial conditions to take place. Usually, as further degrees of freedom are included and short-range interactions are present, the phase space volume with quasi-periodic orbits vanish. Let's illustrate the power-law sensitivity to initial conditions in the Hamiltonian map[18, 19]

$$x_i(t+1) = x_i(t) + y_i(t) \tag{18}$$

$$y_i(t+1) = y_i(t) + K \sin x_i(t+1) + CK \sin\left[x_i(t+1) - x_{i-1}(t+1)\right] + CK \sin\left[x_i(t+1) - x_{i+1}(t+1)\right] \tag{19}$$

where the indices go from 1 to N, periodic boundary conditions are assumed and x_i, y_i are taken modulo 2π. Here C is the coupling parameter between nearest neighbors. For $N = 1$ the system is regular and the Liapunov exponent is zero for any initial condition. For $N = 2$ it has been observed that, for $C = 0.5$ and $K = 0.15$, 40% of the phase space still has a zero Liapunov exponent (quasi-periodic orbits)[19]. In Fig. 6 we show some results for the sensitivity function for distinct initial conditions. Notice that, besides the exponentially diverging ones, some orbits exhibit a power-law (linear) time evolution. For these initial conditions S_q, with $q = 0$ is the proper dynamical entropy. The fraction of the phase-space with zero Liapunov exponent vanishes exponentially as further degrees of freedom are included and therefore, the system becomes ergodic (fully chaotic with positive maximum Liapunov exponent for any initial condition)[18]. However, recent results have indicated that Hamiltonian maps with long-range interactions may have zero Liapunov exponent,

i.e., power-law diverging orbits, even in the thermodynamic limit[20]. This fact may be related to the breakdown of the standard BG prescription in describing some statistical distributions of a variety of long-range interacting Hamiltonian systems[21, 22, 23, 24, 25].

VII Summary and perspectives

We briefly revised some recent results concerning the power-law sensitivity to initial conditions of dynamical systems at criticality and how it can be naturally formulated within the Tsallis nonextensive statistics prescription. The power-law sensitivity has been shown to provide a simple tool for estimating the proper entropic index q of critical systems tuned at criticality as well as of systems exhibiting a self-organized critical state.

The critical dynamical attractor of nonlinear dynamical systems usually presents a multifractal character. It has been shown that quite general scaling arguments applied to the most rarefied and most concentrated regions of the attractor provide a direct link between the entropic index q and the critical exponents associated with the scaling behavior of the extremal sets of the attractor. This result gives support for the recent conjecture that Tsallis statistics is the natural frame for studying systems with a fractal-like structure in the phase-space[9].

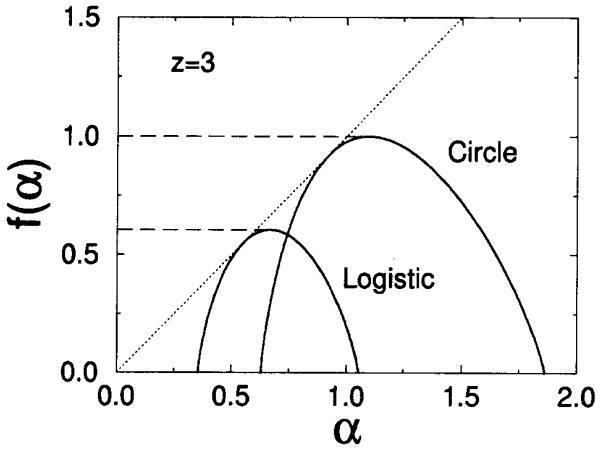

Figure 5. Multifractal singularity spectra of the critical attractor of generalized logistic and circle maps with $z = 3$. The maximum of the $f(\alpha)$ curves gives the fractal dimension of the support ($d_f = 0.605$ for the $(z = 3)$-logistic map and $d_f = 1$ for the circle map). The curves were obtained after $N = 2048$ (logistic) and $N = 2584$ (circle) iterations starting from the extremal point $\bar{x} = 0$ and following Halsey *et al* prescription[10]. For details see also[6, 7, 8].

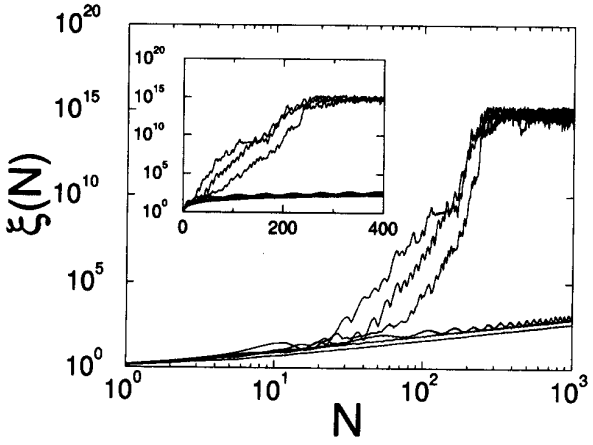

Figure 6. The sensitivity function for two coupled Hamiltonian maps as described in Eqs. 18-19 for $C = 0.5$ and $K = 0.15$ and several initial conditions. For these parameters there are regions (of the order of 0.40 of the four-dimensional phase-space) with zero Liapunov exponent at which power-law sensitivity is observed, as shown in the main figure. The inset shows the same data in Linear-Log scale which explicitly exhibit the exponential sensitivity of chaotic orbits.

The predicted scaling relation has been numerically checked to hold in one-dimensional dissipative maps independently of the topological and metrical properties of the dynamical attractor and is expected to hold for a very large class of non-linear dynamical systems. Particularly, it would be of great interest to verify its validity for Hamiltonian systems with few degrees of freedom where power-law sensitivity has been observed as well as for Hamiltonian systems with long-range interactions for which the non-extensive Tsallis statistics has successfully reproduced some unusual distribution functions. Further work along these directions would certainly be valuable.

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