Some Recent Developments in Models with Absorbing States

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We describe some of the recent results obtained for models with absorbing states. First, we present the nonequilibrium absorbing-state Potts model and discuss some of the factors that might affect the critical behaviour of such models. In particular we show that in two dimensions the further neighbour interactions might split the voter critical point into two critical points. We also describe some of the results obtained in the context of synchronization of chaotic dynamical systems. Moreover, we discuss the relation of the synchronization transition with some interfacial models.

I   Introduction

Nonequilibrium statistical mechanics is nowadays a very active research field [1] and there are several reasons for that. First, the most interesting phenomena in Nature take place out of equilibrium. The best example is provided by living matter. More generally, systems which are open (traversed by fluxes of energy, entropy or matter) may reach stationary states which cannot be described by equilibrium statistical physics. Second, the properties of systems in equilibrium are by now rather well understood as equilibrium statistical physics is a well established theory. The puzzling problem of universal behavior observed in the vicinity of second order phase transitions is beautifully explained by the renormalization group approach [2]. It is thus natural to try to extend our understanding of equilibrium systems to nonequilibrium ones.

Indeed, the situation is not so clear for nonequilibrium statistical mechanics for which no general theory has been developed yet. This is particularly true for the case of nonequilibrium phase transitions where, according to the values of some control parameters, the system can change, continuously or not, from one stationary state to another one [1].

At the microscopic level, models for nonequilibrium phase transitions are usually defined in terms of a master equation [3]. Most of the physics is contained in the transition rates. One of the key differences between equilibrium and nonequilibrium systems is related with the detailed-balance condition which is generally not obeyed in nonequilibrium. As a result, it is often impossible to find an analytical solution to the master equation, even for the stationary state. This is why a lot of results in this field are obtained by numerical simulations. At a coarse-grained level, the physics is often described in terms of a generalized Langevin equation. Unfortunately, there is no general method which allows us to perform, in a controlled way, the coarse-graining process and usually the determination of the form taken by the Langevin equation is based only on general symmetry arguments [4-7]. This procedure is particularly ambiguous for systems with noise [8]. When the noise is multiplicative different interpretations of the stochastic process described by the master equation are possible, leading to further confusion [9, 10].

There are many examples of nonequilibrium phase transitions. One of the simplest examples is provided by an Ising like model with competing dynamics. The system is in contact with two heat baths at different temperatures, each generating a microscopic dynamics obeying detailed-balance (see [11-13] for more details). As a result of this competition, effective long range interaction develops in the system and its critical properties are similar to the ones of equilibrium models. Another generic way to obtain nonequilibrium phase transition is to induce dynamical anisotropy. This can be done in several ways. Examples are given by the so-called driven lattice gases [14], where particles diffusing on a lattice are driven by an external field oriented in a particular direction. At low temperature, the system exhibits a nonequilibrium phase transition and the ordered phase is characterized by strong anisotropy. As a result, the critical properties belong to a new universality class, not related with equilibrium phase transitions.

The question of the characterization of the possible universality classes for nonequilibrium phase transitions is thus an important one which is much debated. The effects of violating detailed-balance on the universal static and dy-

namic scaling behavior has been investigated independently by Grinstein et al. [15] and Täuber et al. [16]. It turns out that the standard critical dynamics universality classes are rather robust and that detailed-balance can be effectively restored at criticality in some cases. Nevertheless, this is not always true and a complete characterization of the nonequilibrium universality classes is clearly a difficult and open question.

An important type of nonequilibrium phase transition is the so-called *absorbing state phase transition* that takes place when a system during its evolution reaches a configuration in which it remains trapped forever. Such a state is called an absorbing state and a given system may have one or more such states. Absorbing states are present in numerous systems encountered in physics, chemistry and biology. A lot of effort has been devoted to the study of such systems and there are some comprehensive reviews on this subject [17, 18].

In the present paper we describe some recently obtained results that are not yet covered in these reviews and that are related with our own research in this field. In particular, we describe a certain d-dimensional model with q absorbing states whose dynamics can be considered as a modification of the Metropolis algorithm for the equilibrium Potts model [19]. We describe the role of q and d on the critical behaviour of this model. We also show that some other details of the dynamics, as e.g., positivity of certain transition rates [20] or the range of interactions [22], influence the critical behaviour. Although its dynamics looses the detailed balance property, the model still bears some similarity to the equilibrium Potts model, that is also reflected in the overall behaviour of the model in the (q, d) plane. In a certain case, when the range of interactions in our model is increased, we observe that the so-called voter transition [21], that typically occurs for $q = d = 2$, is split into two phase transitions: first a spontaneous symmetry breaking preselects one of the two absorbing states and then a collapse on the preelected absorbing state takes place [22]. Since our model is expressed in terms of spin operators and a Hamiltonian-like function, one can easily construct its various generalizations that would include multi-spin interactions, anisotropies or higher order symmetries ($O(N)$). Such models might exhibit novel properties and also enrich the collection of relations between equilibrium and nonequilibrium systems.

Recently a class of nonequilibrium phase transitions was intensively studied in the context of synchronization [23]. When for example two identical chaotic dynamical systems are sufficiently strongly connected, they can synchronize with one another. Since the system cannot escape from the synchronized state it is actually a certain absorbing state. For spatially extended dynamical systems (e.g., coupled map lattices) the synchronization transition (ST) resembles some absorbing state phase transitions that are typical to statistical mechanics models. Some arguments were given [24] that ST should generically belong to the same universality class as a certain model of a driven interface bounded with a wall (BKPZ). However, for a certain class of dynamical sys-

tems ST has a different critical behaviour, namely it belongs to the DP universality class [25]. Despite some attempts, theoretical understanding of the mechanism that changes the critical behaviour is still missing [26-28]. An interesting related problem is a nature of a multicritical point that possibly joins the critical lines of DP and BKPZ type. There are some indications that these problems might be also related with some particle systems like the recently intensively studied PCPD model [29].

In section II we discuss the Potts model with absorbing states. The synchronization of extended chaotic systems is discussed in section III, and these two sections are essentially independent. Conclusions are presented in section IV.

II   Absorbing-state Potts model

An important step toward the classification of absorbing state phase transitions into universality classes was made by Janssen and Grassberger who predicted that a large group of models falls into the so-called directed percolation universality class (DP) and it was conjectured that all models with a single absorbing state, positive one-component order parameter and short-range dynamics should generically belong to this universality class [30]. By now their conjecture has received numerous and convincing support. It also raised a possibility that the number of absorbing states q might be a relevant parameter that determines the critical behaviour of a given model. And indeed, there are some examples that show that models with $q = 2$ share the same critical behaviour that is called the parity conserving (PC) universality class [31, 32]. However, it turns out that PC criticality appears only for models with symmetrical absorbing states. Even a small asymmetry in the dynamics of the model that would favour one of the absorbing states over the other will drive the system into the DP universality class [33]. Models with $q > 2$ were also examined and using a relation with some particle systems it was predicted that such models are generically in the active phase and only in a limiting case undergo a phase transition related with a certain multispecies branching-annihilating random walk model [32, 34]. As we will discuss in this section, the number q does not fix the critical behaviour because some other details of the dynamics play a role too. For example, we can change the critical behaviour by suppressing certain transition rates or extending the range of interactions. Many factors are therefore responsible for the critical behaviour of a given model.

We should mention that absorbing phase transitions also take place in some particle systems like e.g., branching-annihilating random walk (BARW). For such models the critical behaviour is determined, e.g., through some symmetries of their dynamics, rather than by the number of absorbing states (that is usually the vacuum). For $d = 1$, at the coarse grained level, models with multiple absorbing states can be related with some particle systems. For higher dimensions, such an analogy in general does not hold. There are already some reviews on this subject [17, 18].

II.1 Definition of the model and the Monte Carlo method

Before defining our dynamical model, let us recall some basic properties of the usual equilibrium Potts model [35]. First, we assign at each lattice site i a q-state variable $\sigma_i = 0, 1, ..., q - 1$. Next, we define the energy of this model through the Hamiltonian:

$$H = -\sum_{i,j} \delta_{\sigma_i \sigma_j}, \tag{1}$$

where summation is over pairs of i and j which are usually nearest neighbours and δ is the Kronecker delta function. This equilibrium model was studied using many different analytical and numerical methods and is a rich source of information about phase transitions and critical phenomena [35]. To simulate numerically the equilibrium Potts model defined using the Hamiltonian (1), one introduces a stochastic Markov process with transition rates chosen in such a way that the asymptotic probability distribution is the Boltzmann distribution. One possibility of choosing such rates is the so-called Metropolis algorithm [36]. In this method one looks at the energy difference ΔE between the final and initial configuration and accepts the move with probability $\min\{1, e^{-\Delta E/T}\}$, where T is temperature measured in units of the interaction constant of the Hamiltonian (1), which was set to unity. To obtain a final configuration one selects randomly a site and its state (one out of q in our case). In the above described algorithm for $T > 0$ there is always a positive probability of leaving any given configuration (even when the final configuration has a higher energy). Accordingly, such a model does not have absorbing states for $T > 0$.

A nonequilibrium Potts model having q absorbing states can be obtained by making the following modification in the Metropolis dynamics [19, 20]: when all neighbours of a given site are in the same state as this site, then this site cannot change its state (at least until one of its neighbours is changed). Let us notice that any of the q ground states of the equilibrium Potts model is an absorbing state of the above defined nonequilibrium Potts model. Moreover, the rules of the dynamics of our model depends on the parameter T that for the equilibrium Potts model would be the temperature. Although for our model the thermodynamic temperature cannot be defined, we will refer to T as temperature.

To study the properties of this model we performed standard Monte Carlo simulations. A natural characteristic of models with absorbing states is the steady-state density of active sites ρ. A given site i is active when at least one of its neighbours is in a state different than i. Otherwise the site i is nonactive. In addition to the steady-state density we also looked at its time dependence $\rho(t)$. In the active phase $\rho(t)$ converges to the positive value, while at criticality, $\rho(t)$ has a power-law decay $\rho \sim t^{-\Theta}$. In addition, we used the so-called dynamic Monte Carlo method where one sets the system in the absorbing state, locally initiates activity, and then monitors some stochastic properties of surviving runs [37]. The most frequently used characteristics are the survival probability $P(t)$ that the activity survives at least until time t and the number of active sites $N(t)$ (averaged over all runs). At criticality these quantities are expected to have power-law decay: $P(t) \sim t^{-\delta}$ and $N(t) \sim t^\eta$.

$d = 1$

First, let us consider the case $q = 2$. The simplest possibility is to consider our model on a one-dimensional chain. However, for $q = 2$ and for any temperature T, this model is trivially equivalent to the $T = 0$ temperature Ising model with Metropolis dynamics. Indeed, in this case the allowed moves are only those which do not increase energy and they are always accepted. The same rule governs the dynamics of the $T = 0$ Ising chain. To overcome this difficulty, we studied our model on a ladder-like lattice, where two chains are connected by interchain bonds such that each site has three neighbours.

Monte Carlo simulations of the model show that for large enough T the model remains in the active (disordered) phase. After reducing the temperature below a certain critical value, the model collapses on one of the absorbing states. The evolution to the absorbing state resembles the coarsening process. Measuring the critical exponents at the critical point we found that their values are very close to those of the PC universality class, which is an expected result.

We also did simulations for $q = 3$, 4, and 5. In this case we found that our model remains in the active phase for any $T > 0$ and collapses on one of its absorbing states only at $T = 0$. It was already suggested that an absence of the transition for models with $q > 2$ absorbing states is a generic feature [32, 34]. Such a conclusion can be obtained relating a model with absorbing states with multi-species BARW model that in some case are known to exhibit such a behaviour [38]. However, such a relation is not rigorously established and must be taken with care. And indeed, one can show that in some cases models with $q > 2$ absorbing states behave differently.

In the following we shall describe a modification of our nonequilibrium Potts model that even for $q > 2$ undergoes a transition at positive temperature. To comply with ref. [20], we refer to this modification as a model B. All we have to do is to introduce the following restriction in the dynamics of our model: a flip into a state different than any of its neighbours is forbidden. In other words, we suppress the spontaneous creation of, e.g., domains of type A between domains of type B and C. Here, A, B, and C denote three (out of q) different states. Let us also notice that the above restriction does not break the symmetry and the absorbing states of model B are symmetric with respect to its dynamics. Numerical simulations of such a model for $q > 2$ show that at positive temperature it undergoes a phase transition that belongs to the PC universality class [20]. It was suggested that with the above restriction, the long-time dynamics of model B in the active phase and close to the critical point is dominated by parity conserving processes and that might explain

the origin of PC criticality.

$d > 1$

The nonequilibrium Potts model was also studied on higher dimensional lattices and below we briefly describe the obtained results [19].

(i) $d = 2$, $q = 2$:

Models with a single absorbing state on $d = 2$ lattices typically belong to the DP universality class. It is interesting to ask whether for $d = 2$ models with double absorbing states share the same critical behaviour. Recent numerical calculations show that indeed there is a group of such models that have the same critical behaviour, that was termed the voter universality class [21]. The name of this universality class was given after a voter model that was originally proposed as a model of spreading of an opinion [39]. Later, various generalizations of this model were also studied [40]. In the voter model the order parameter vanishes continuously to 0 upon approaching a critical point but the decay is slower than any power law. In addition, the time decay of the order parameter at criticality is also slower than any power law decay and is in fact logarithmic, as can be shown exactly [41]. Such an unusual behaviour explains the numerical difficulties in studying models of this universality class [19]. The nonequilibrium Potts model for $q = 2$ and on square lattice with nearest neighbour interactions also belongs to the voter universality class [22]. It was suggested that two-dimensional models with double absorbing state should generically belong to the voter universality class [21]. However, as we describe below, there are some exceptions to this rule [22].

An interesting feature of the voter critical point is the fact that at this point actually two phenomena seem to take place. One of them is the symmetry breaking between two competing states of the model, that is similar to the symmetry breaking in the Ising model. The second phenomenon is the phase transition between active and absorbing phases of the model. It turns out that these two phase transitions can be separated and it happens in the $q = 2$ Potts model on the square lattice with interactions (all of the same strength) up to the third nearest neighbour. In this case the behaviour of the model can be thus described as follows. At sufficiently high temperature T the model remains in the disordered phase. Upon reducing the temperature, the model first undergoes the symmetry breaking phase transition. Calculation of the Binder cumulant suggests that this transition belongs to the Ising type universality class. Upon further decrease of T, the model undergoes a second phase transition into an absorbing state. Since at this point the symmetry is already broken and the absorbing state is already preselected, this second transition, as expected, belongs to the DP universality class.

The Ising-type phase transition is just one example of a symmetry breaking. The voter criticality can be regarded as a superposition of this transition with DP transition. One can ask whether other types of symmetries, such as e.g., Z_3 or $U(1)$ can be superposed with DP. Possibly in such a case a new critical behaviour might result.

(ii) $d = 2$, $q = 3$

In this case (nearest-neighbour interactions) there is clear evidence of a discontinuous phase transition. In particular, the steady-state density of active sites ρ has a discontinuous behaviour and the time dependent $\rho(t)$ develops a plateau at a critical point.

(iii) $d = 3$, $q = 2$

To split the voter critical point in the two-dimensional case we had to include further neighbour interactions. Alternatively, increasing the dimensionality up to $d = 3$ also results in two separate phase transitions [22]. It would be interesting to check whether in the three-dimensional case some additional interactions (possibly antiferromagnetic ones) could actually lead to the overlap of these two transitions.

Studying our nonequilibrium Potts model for some values of q and d we were tempted to speculate on the overall behaviour of the phase diagram in the (q, d) plane [19]. Indeed, it seems that the (q, d) plane is divided into three parts with (i) non-mean-field critical behaviour (ii) mean-field critical behaviour, and (iii) discontinuous transitions. Arrangements of these parts suggests that the qualitative behaviour of our model resembles the behaviour of the equilibrium Potts model. If so, it would imply that the modification of the dynamics that we introduced, and that imply the existence of absorbing states, might not change that much the qualitative behaviour of the model (as compared to the equilibrium one).

Since our model is formulated in terms of spin-like variables, we can easily introduce its various modifications that for example will take into account lattice anisotropy, multi-spin interactions, external fields or additional symmetries (gauge, $U(1)$,...). For example some equilibrium homogeneous spin models are known to exhibit glassy behaviour [42]. One of the questions is whether a similar behaviour exists when the dynamics with absorbing states is used.

III   Synchronization of dynamical systems

Recently, synchronization of dynamical, and in particular chaotic, systems has received considerable attention [23]. This is to large extent related with its various experimental realizations in lasers, electronic circuits or chemical reactions [43]. So far, most of the attention has been focused on the behaviour of the low-dimensional systems. More recently, spatially extended, i.e., high-dimensional, systems are also drawing some interest [44]. Since the synchronized state is an attractor of the dynamics, it can be considered as an absorbing state. Consequently, a transition into a synchronized state (ST) for spatially extended systems bears some similarity to absorbing state phase transitions. However, for continuous dynamical systems, like e.g., coupled map lattices (CML) [45], the system cannot reach a perfectly synchronized state in a finite time. This is in contrast with for example some cellular automata that typically

can reach an absorbing state in finite time. In some cases such a difference has probably a negligible effect and ST belongs to the directed percolation universality class [25]. But there are some other arguments suggesting that the critical behaviour at ST typically is different and belongs to the bounded Kardar-Parisi-Zhang universality class [24]. A possible crossover between these two universality classes has recently been intensively studied [26-28].

III.1 Coupled-Map Lattices

To provide a more detailed example we examine a model recently proposed by Ahlers and Pikovsky that consists of two coupled CML's [46],

$$\begin{pmatrix} u_1(x,t+1) \\ u_2(x,t+1) \end{pmatrix} = \begin{pmatrix} 1-\gamma & \gamma \\ \gamma & 1-\gamma \end{pmatrix} \times \begin{pmatrix} (1+\epsilon\Delta)f(u_1(x,t+1)) \\ (1+\epsilon\Delta)f(u_2(x,t+1)) \end{pmatrix}, \tag{2}$$

where Δv is the discrete Laplacian $\Delta v(x) = v(x-1) - 2v(x) + v(x+1)$. Both space ($x$) and time ($t$) are discretized, $x = 1, 2 \ldots, L$ and $t = 0, 1, \ldots$. Periodic boundary conditions are imposed $u_{1,2}(x+L,t) = u_{1,2}(x,t)$ and, similarly to previous studies, we set the intrachain coupling $\epsilon = 1/3$. Varying the interchain coupling γ allows us to study the transition between synchronized (large γ) and chaotic (small γ) phases. Local dynamics is specified through a nonlinear function $f(u)$.

Next, we introduce a synchronization error $w(x,t) = |u_1(x,t) - u_2(x,t)|$ and its spatial average $w(t) = \frac{1}{L}\sum_{x=1}^{L} w(x,t)$. The time average of $w(t)$ in the steady state will be simply denoted as w. In the chaotic phase one has $w > 0$, while in the synchronized phase $w = 0$. Moreover, at criticality, i.e. for $\gamma = \gamma_c$, $w(t)$ is expected to have a power-law decay to zero $w(t) \sim t^{-\Theta}$. In the stationary state, and for γ approaching the critical value γ_c one expects that $w \sim (\gamma_c - \gamma)^\beta$.

For $f(u) = 2u \mod(1)$, i.e., a Bernoulli map, Pikovsky and Ahlers [46] found that ST belongs to the DP universality class. Such a behaviour is in agreement with earlier predictions by Baroni et al. [25] that DP critical behaviour should exist for maps with strong nonlinearities (in the case of the Bernoulli map it is even a discontinuity). Let us notice, however, that DP criticality is typically attributed to models with a single absorbing state. On the other hand, an extended dynamical system as e.g. (2) has infinitely many synchronized states. Recently, we applied a dynamical Monte Carlo method to study model (2). Our results show [47] that exponents η and δ depend on the type of a synchronized state, but their sum $\eta + \delta$ remains constant. Such a situation is known to take place in some other models with infinitely many absorbing states [48].

A different critical behaviour emerges for the symmetric tent map $f(u) = 1 - 2|u - 1/2|$. Since the map is now continuous, as expected, ST belongs to the BKPZ universality class. But happens when the symmetry of the tent map is gradually distorted? In particular, let us examine the following map

$$f(u) = \begin{cases} au & \text{for } 0 \le u < 1/a \\ a(1-u)/(a-1) & \text{for } 1/a \le u \le 1, \end{cases} \tag{3}$$

with $1 < a \le 2$. For $a = 2$ this is the symmetric tent map. Let us notice that in the limit $a \to 1$, the slope of the second part of this map diverges. In such a limit the map has a strong nonlinearity and we expect that ST in this case belongs to the DP universality class. Numerical calculations show, however, that DP criticality sets in already for a in a finite distance from 1 [47]. In such a way model (2) with the map (3) allows us to study the change of the universality class of ST. An interesting possibility is that at a certain $a = a_c > 1$, where DP and BKPZ critical lines intersect, a multicritical behaviour appears. Numerical simulations of model (2) are not yet conclusive enough, but it is possible that additional insight into this problem can be obtained using certain effective models of ST. This problem is discussed in the next subsection.

III.2 Interfacial models of synchronization

An interesting approach to ST was initiated by Pikovsky and Kurths [24]. They have argued that the temporal evolution of the small perturbation of a synchronized state $w(x,t)$ of the system (2) in the continuous limit should obey the following Lagevin-type equation [24, 46]

$$\frac{\partial w(x,t)}{\partial t} = [a + \xi(x,t) - p|w(x,t)|^2]w(x,t) + \epsilon\frac{\partial^2 w(x,t)}{\partial x^2}, \tag{4}$$

where a is a control parameter connected with the transverse Lyapunov exponent $\lambda_\perp$, that describes an exponential growth of $w(x,t)$. The Gaussian stochastic process $\xi(x,t)$ has the properties

$$\langle \xi(x,t) \rangle = 0, \quad \langle \xi(x,t)\xi(x',t') \rangle = 2\sigma^2\delta(x-x')\delta(t-t'). \tag{5}$$

Applying the Hopf-Cole transformation $h = \ln|w|$, Eq. (4) is transformed into a driven interface model

$$\frac{\partial h(x,t)}{\partial t} = a + \xi(x,t) - pe^{2h(x,t)}$$

$$+ \epsilon\frac{\partial^2 h(x,t)}{\partial x^2} + \epsilon[\partial h(x,t)/\partial x]^2, \tag{6}$$

which is the KPZ equation [49] with an additional exponential saturation term. In Eq. (6), synchronization corresponds

to an interface moving towards $-\infty$, and the saturation term prevents the interface from moving towards large positive value. Equation (6) is usually referred to as the bounded KPZ equation.

Critical exponents of model (6) [50] remain in a satisfactory agreement with those obtained for the CML model (2) with the symmetric tent map [46]. However, the relation with model (6) offers little understanding of what changes the universality class to DP for strongly nonlinear local maps $f(u)$. Recently, there were some attempts to understand the emergence of the DP universality class. For example Muñoz and Pastor-Satorras [27], showed numerically that DP might emerge in a more general version of Eq. (4). However, the origin of certain additional terms that appear in their approach is not yet clear.

In another approach [26], we proposed a certain SOS model whose dynamics is motivated by the dynamics of synchronization error in CML model (2). In this SOS model ST transition is essentially mapped onto a certain wetting transition. It turns out that when certain transition rates, that might be related with a binding potential of a wall, decay exponentially fast with the distance from the wall h, the wetting transition belongs to the DP universality class. There is a hope that for another form of these transition rates BKPZ universality class will be recovered but we still did not succeed to confirm it. On the other hand, when these transition rates decay as a power of h, the SOS model exhibits a different critical behaviour. Surprisingly, critical exponents in this case remain in a good agreement with those calculated recently for the bosonic version of the PCPD model [51]. Whether this is just a numerical coincidence or a manifestation of a deeper relation is yet to be seen [52]. Another approach to examine a change of the universality class in ST using an interfacial model was proposed by Ginelli et al [28].

IV   Conclusions

In the present paper we described some of our recent results on models with absorbing states. In particular we examined phase transitions and critical behaviour in the nonequilibrium Potts model with absorbing states. We also described the connections between synchronization of spatially extended chaotic systems and absorbing phase transitions. We related the synchronization problem with some models of a driven or wetting interfaces. It would fulfill our goals if this work would stimulate further research in this field. For example, we showed that when the third nearest neighbour interactions are included, the voter critical point in the Potts model is split. But these further range interactions are of the same strength as nearest neighbour interactions. One can consider a model with varying strength of further range interactions. At a certain amplitude of this interaction splitting should appear. At this point three critical lines will meet: voter, Ising and directed percolation. It would be interesting to examine in more details the nature of such a tricritical point. Synchronization transition and its connection with interfacial models is still intensively studied by several groups and further interesting results are likely to appear.

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