NONLINEAR H_∞ CONTROL FOR UNDERACTUATED MANIPULATORS WITH ROBUSTNESS TESTS

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ABSTRACT

In this paper, the model-based robotic control problem with disturbance attenuation (or robotic H_∞ control problem), presented in Chen *et all* (1994), is extended to underactuated manipulators. The dynamic coupling between the links is used to control all manipulator's free joints. A global explicit solution is found solving a minimax Bellman-Isaacs equation, generated via differential game theory. Experimental results, obtained from UArm II manipulator, considering fully actuated and underactuated configurations, are presented.

KEYWORDS: Robust control, nonlinear H_∞ control, underactuated manipulators.

RESUMO

Neste artigo, o problema de controle robótico baseado em modelo com atenuação de distúrbios (ou problema de controle H_∞ robótico), apresentado em Chen *et all* (1994), é estendido para robôs manipuladores subatuados. O acoplamento dinâmico entre os elos é usado para controlar todas as juntas livres do manipulador. Uma solução explícita global é encontrada resolvendo um problema minimax definido através de uma equação de Bellman-Isaacs gerada pela teoria dos jogos. Resultados experimentais, obtidos com o manipulador UArm II, considerando as configurações totalmente atuada e subatuada, são apresentados.

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PALAVRAS-CHAVE: Controle robusto, controle H_∞ não linear, manipuladores subatuados.

1 INTRODUCTION

Motion control of manipulators has been the objective of a great number of researches (Chen *et all*, 1994; Chen and Chang, 1997; Johansson, 1990; Postlethwaite and Bartoszewicz, 1998). Underactuated manipulators, with less actuator than degrees of freedom, are also of interest for many researchers (Arai, 1997; Arai and Tachi, 1991; Bergerman, 1996). The controllability for these mechanical systems and a control strategy were first established in Arai and Tachi (1991). First, all passive joints (without actuator) are controlled to theirs set-point, using the dynamic coupling. Then, with the passive joints locked, the active ones (with actuator) are controlled by themselves. In Bergerman (1996), three possibilities of selecting the joints to be controlled at every control phase are derived. One can select only passive joints, passive and active joints or only active joints.

The effort to control the generalized coordinates of the manipulator (fully actuated or underactuated) to follow a desired trajectory can be a hard task if parameters uncertainties and exogenous disturbances are present. Robust control with a nonlinear H_∞ criterion (Chen *et all*, 1994; Chen and Chang, 1997) and adaptive control strategy (Johansson, 1990) have been proposed to eliminate the effects of these perturbations. In Johansson (1990), the adaptive strategy is based on optimal motion control with minimization of the applied torques.

H_∞ control for nonlinear time-invariant systems has been widely discussed since the past decade (Isidori, 1992; van der Sachft, 1991; van der Schaft, 1992; Ball *et all*, 1991; Lu and Doyle, 1991). The nonlinear H_∞ control problem means that we need to find an L_2 induced norm between input and output signals limited by a level γ. In Lu (1996), these results are extended to time-varying systems with finite-time horizon.

A robotic H_∞ control problem, or a model-based robotic control problem with desired disturbance attenuation, is proposed in Chen *et all* (1994). A global explicit solution for this problem, formulated as a minimax (leader-following) game, is developed using differential game theory (Basar and Oslder, 1982; Basar and Berhard, 1990). From this theory, one needs to solve a minimax Bellman-Isaacs equation, which after some rearrange it is redefined as a Hamilton-Jacobi equation found in Lu (1996). The result of Chen *et all* (1994) is a kind of feedback linearization with a nonlinear term introduced in the control acceleration. Some experimental results were obtained in Postlethwaite and Bartoszewicz (1998) using a similar approach.

The formulation presented in Chen *et all* (1994) is resumed here as a background to the main results of this paper: the extension of the robotic H_∞ control problem for underactuated manipulators and the application of this methodology in the experimental robot manipulator UArm II, a three link planar manipulator with revolute joints, that can be configured as fully actuated or underactuated.

This paper is organized as follows. In Section 2, the robotic H_∞ control problem is formulated based on Chen *et all* (1994). In Section 3, this problem is extended to the underactuated case. In Section 4, the solution presented in Chen *et all* (1994), for this H_∞ control problem, is described. In Section 5, experimental results obtained from the UArm II are presented. In Section 6, the conclusions are presented.

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2 ROBOTIC H$_\infty$ CONTROL PROBLEM

In this section, the H_∞ control problem is formulated for a manipulator where the disturbances are derived from parametric uncertainties and exogenous inputs, following the line defined in Chen *et all* (1994).

The dynamic equations of a manipulator can be found by the Lagrange theory as

$$\tau = M(q)\ddot{q} + C(q,\dot{q})\dot{q} + F(\dot{q}) + G(q) \tag{1}$$

where $q \in \Re^n$ are the joint positions, $M(q) \in \Re^{n\times n}$ is the symmetric positive definite inertia matrix, $C(q,\dot{q}) \in \Re^{n\times n}$ is the Coriolis and centripetal matrix, $F_0(\dot{q}) \in \Re^n$ are the frictional torques, $G(q) \in \Re^n$ are the gravitational torques and $\tau \in \Re^n$ are the applied torques. The parametric uncertainties can be introduced by dividing the parameter matrices $M(q)$, $C(q,\dot{q})$, $F(\dot{q})$, and $G(q)$ into a nominal and a perturbed part

$$\begin{array}{l} M(q) = M_0(q) + \Delta M(q) \\ C(q,\dot{q}) = C_0(q,\dot{q}) + \Delta C(q,\dot{q}) \\ F(\dot{q}) = F_0(\dot{q}) + \Delta F(\dot{q}) \\ G(q) = G_0(q) + \Delta G(q) \end{array}$$

where $M_0(q)$, $C_0(q,\dot{q})$, $F_0(\dot{q})$, and $G_0(q)$ are the nominal matrices and $\Delta M(q)$, $\Delta C(q,\dot{q})$, $\Delta F(\dot{q})$, and $\Delta G(q)$ are the parametric uncertainties. Exogenous inputs, w, can also be introduced, and (1) can be rewritten as

$$\tau + \delta = M_0(q)\ddot{q} + C_0(q,\dot{q})\dot{q} + F_0(\dot{q}) + G_0(q) \tag{2}$$

with

$$\delta = -\left(\Delta M(q)\ddot{q} + \Delta C(q,\dot{q})\dot{q} + \Delta F(\dot{q}) + \Delta G(q) - w\right).$$

The state tracking error is defined as

$$\tilde{x} = \begin{bmatrix} \dot{q} - \dot{q}^d \\ q - q^d \end{bmatrix} = \begin{bmatrix} \dot{\tilde{q}} \\ \tilde{q} \end{bmatrix} \tag{3}$$

where q^d and $\dot{q}^d \in \Re^n$ are the desired reference trajectory and the corresponding velocity, respectively. The variables q^d, $\dot{q}^d$, and $\ddot{q}^d$ (desired acceleration), are assumed to be within the physical and kinematics limits of the control object. The dynamic equation for the state tracking error is given from (2) and (3) as

$$\dot{\tilde{x}} = A(q,\dot{q})\tilde{x} + B_0\left(q,\dot{q},\ddot{q}^d,\dot{q}^d\right) + BM_0^{-1}(q)\tau + BM_0^{-1}(q)\delta \tag{4}$$

where

$$A(q,\dot{q}) = \begin{bmatrix} -M_0^{-1}(q)C_0(q,\dot{q}) & 0 \\ I_n & 0 \end{bmatrix}$$

$$B_0(q,\dot{q},\ddot{q}^d,\dot{q}^d) = \begin{bmatrix} -\ddot{q}^d - M_0^{-1}(q)(F_0(\dot{q}t) + G_0(q) + C_0(q,\dot{q})\dot{q}^d) \\ 0 \end{bmatrix}$$

$$B = \begin{bmatrix} I_n \\ 0 \end{bmatrix}.$$

In order to represent this equation in a canonical form, a control input variable u should be defined. Using the following state-space transformation of $\tilde{x}$ (Chen *et all*, 1994; Johansson, 1990)

$$\tilde{z} = \begin{bmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{bmatrix} = T_0 \tilde{x} = \begin{bmatrix} T_{11} & T_{12} \\ 0 & I \end{bmatrix} \begin{bmatrix} \dot{\tilde{q}} \\ \tilde{q} \end{bmatrix} \tag{5}$$

and selecting the control input as

$$u = \begin{bmatrix} M_0(q) & C_0(q, \dot{q}) \end{bmatrix} \begin{bmatrix} \dot{\tilde{z}}_1 \\ \tilde{z}_1 \end{bmatrix} = M_0(q) T_1 \dot{\tilde{x}} + C_0(q, \dot{q}) T_1 \tilde{x} \tag{6}$$

where T_{11}, $T_{12} \in \Re^{n \times n}$ are constant matrices to be determined later and $T_1 = [T_{11} \quad T_{12}]$, the dynamic equation of the state tracking error (4) can be rewritten as

$$\dot{\tilde{x}} = A_T(\tilde{x}, t)\, \tilde{x} + B_T(\tilde{x}, t)\, u + B_T(\tilde{x}, t)\, d \tag{7}$$

where

$$A_T(\tilde{x}, t) = T_0^{-1} \begin{bmatrix} -M_0^{-1}(q)\, C_0(q, \dot{q}) & 0 \\ T_{11}^{-1} & -T_{11}^{-1} T_{12} \end{bmatrix} T_0$$

$$B_T(\tilde{x}, t) = T_0^{-1} \begin{bmatrix} M_0^{-1}(q) \\ 0 \end{bmatrix}$$

$$d = M_0(q)\, T_{11} M_0^{-1}(q)\, \delta.$$

The control input (6) is a selective applied torque, since it affects the kinetic energy only. It is not necessary to optimize the gravitation-dependent torques during the motion (Johansson, 1990). The relation between the applied torques and the control input is given by

$$\tau = M_0(q)\, \ddot{q} + C_0(q)\, \dot{q} + F_0(q) + G_0(q) \tag{8}$$

where

$$\ddot{q} = \ddot{q}^d - T_{11}^{-1} T_{12} \dot{\tilde{q}} - T_{11}^{-1} M_0^{-1}(q) \left(C_0(q)\, B^T T_0 \tilde{x} - u \right). \tag{9}$$

Equation (9) shows that a control acceleration, $\ddot{q}^c$, with a nonlinear term, is generated by selecting the control input (6).

The H_∞ control strategy aims to attenuate the effects of disturbance, solving the following performance criterion, with a desired attenuation level γ

$$\min_{u(.) \in L_2} \max_{0 \neq d(.) \in L_2} \frac{\int_0^\infty \left(\frac{1}{2} \tilde{x}^T(t) Q \tilde{x}(t) + \frac{1}{2} u^T(t) R u(t) \right) dt}{\int_0^\infty \left(\frac{1}{2} d^T(t) d(t) \right) dt} \leq \gamma^2 \tag{10}$$

where Q and R are weighting matrices and $\tilde{x}(0) = 0$. This performance criterion is actually the H_∞ optimal disturbance attenuation problem for the model-based robotic control.

Remark (Chen *et all*, 1994): Formally, subject to the tracking error dynamics (7) a (full information) H_∞-control problem is to find a state feedback law such that

$$\max_{0 \neq d(.) \in L_2} \frac{\|z(t)\|_{L_2}}{\|d(t)\|_{L_2}} \leq \gamma^2$$

where

$$z(t) = \begin{bmatrix} Q^{1/2} \tilde{x}(t) \\ R^{1/2} u(t) \end{bmatrix}$$

and $||.||_{L2}$ denotes the induced L_2 norm.

3 THE UNDERACTUATED CASE

Underactuated robot manipulators are mechanical systems with less actuators than degrees of freedom. For this reason, the control of the passive joints (joint without actuator) is made considering the dynamic coupling between them and the active joints (with actuator). Here, we consider that the passive joints have brakes. The control strategy consists in controlling all the passive joints to reach the desired positions, applying torques in the active ones, and then turn on the brakes. After that, all the active joints control themselves.

Consider a manipulator with n joints, of which n_p are passive and n_a are active joints. It is known (Arai and Tachi, 1991) that no more than n_a joints of the manipulator can be controlled at every instant. Using this fact, we group the n_a joints being controlled in the vector $q_c \in \Re^{na}$. The remaining joints are grouped in the vector $q_r \in \Re^{n-na}$. There exist three possibilities of forming the vector q_c (Bergerman, 1996)

1. q_c contains only passive joints: when $n_p \geq n_a$ and all other passive joints, if any, are kept locked.

2. q_c contains passive and active joints: all other passive joints, if any, are kept locked.

3. q_c contains active joints.

The control strategy is defined as follows: first, select q_c following the possibilities 1 or 2 (according to n_p), until all passive joints have reached the desired position; second, select q_c following the possibility 3 and control the active joints to the desired position.

The dynamic equation (2) can be partitioned as

$$\begin{bmatrix}\tau_r\\ \tau_c\end{bmatrix}+\begin{bmatrix}\delta_r\\ \delta_c\end{bmatrix}=\begin{bmatrix}M_{rr} & M_{rc}\\ M_{cr} & M_{cc}\end{bmatrix}\begin{bmatrix}\ddot{q}_r\\ \ddot{q}_c\end{bmatrix}+\begin{bmatrix}C_{rr} & C_{rc}\\ C_{cr} & C_{cc}\end{bmatrix}\begin{bmatrix}\dot{q}_r\\ \dot{q}_c\end{bmatrix}+\begin{bmatrix}F_r\\ F_c\end{bmatrix}+\begin{bmatrix}G_r\\ G_c\end{bmatrix} \quad (11)$$

where τ_r are the torques in the remaining joints and τ_c are the torques in the controlled joints. For simplicity of notation, the index 0 representing the nominal system is eliminated from the equations.

For the control strategy 1, $\tau_c = 0$ because there is no torque in the passive joint. For the control strategy 2, τ_c is defined as $\tau_c = [\tau_{ac} \quad 0]$, where τ_{ac} is the torque in the active joints being controlled. From the second line of (11)

$$\tau_c+\delta_c = M_{cr}\ddot{q}_r + M_{cc}\ddot{q}_c + C_{cr}\dot{q}_r + C_{cc}\dot{q}_c + F_c + G_c$$

we can isolate the controlled joint acceleration

$$\ddot{q}_c = -M_{cc}^{-1}\times \left(M_{cr}\ddot{q}_r + C_{cr}\dot{q}_r + C_{cc}\dot{q}_c + F_c + G_c - \tau_c - \delta_c\right). \quad (12)$$

Introducing a desired reference trajectory to the controlled joints (there is no desired reference to the uncontrolled joints), (12) can be rewritten as

$$\ddot{\tilde{q}}_c = -M_{cc}^{-1}C_{cc}\dot{\tilde{q}}_c - \ddot{q}_c^d - M_{cc}^{-1}M_{cr}\ddot{q}_r - M_{cc}^{-1}C_{cr}\dot{q}_r - M_{cc}^{-1}C_{cc}\dot{q}_c^d - M_{cc}^{-1}F_c - M_{cc}^{-1}G_c + M_{cc}^{-1}\tau_c + M_{cc}^{-1}\delta_c. \quad (13)$$

In the state-space form, selecting the state vector as

$$\tilde{x}_c = \begin{bmatrix}\dot{q}_c - \dot{q}_c^d\\ q_c - q_c^d\end{bmatrix} = \begin{bmatrix}\dot{\tilde{q}}_c\\ \tilde{q}_c\end{bmatrix}, \quad (14)$$

(13) can be defined as

$$\dot{\tilde{x}}_c = A(q,\dot{q})\,\tilde{x}_c + B_0\left(q,\dot{q},\ddot{q}_c^d,\dot{q}_c^d\right) + BM_{cc}^{-1}\tau_c + BM_{cc}^{-1}\delta_c \quad (15)$$

where

$$A(q,\dot{q}) = \begin{bmatrix}-M_{cc}^{-1}C_{cc} & 0\\ I_n & 0\end{bmatrix}$$

$$B_0\left(q,\dot{q},\ddot{q}_c^d,\dot{q}_c^d\right) = \begin{bmatrix}-\ddot{q}_c^d - M_{cc}^{-1}\left(F_{cc}+G_{cc}+C_{cc}\dot{q}_c^d + C_{cr}\dot{q}_r\right) - M_{cc}^{-1}M_{cr}\ddot{q}_r\\ 0\end{bmatrix}$$

$$B = \begin{bmatrix}I_n\\ 0\end{bmatrix}.$$

Using a similar transformation like (5), the control input u is selected as

$$u = \begin{bmatrix} M_{cc} & C_{cc}\end{bmatrix}\begin{bmatrix}\dot{\tilde{z}}_1\\ \tilde{z}_1\end{bmatrix} = M_{cc}T_1\dot{\tilde{x}}_c + C_{cc}T_1\tilde{x}_c \quad (16)$$

and the dynamic equation of the state tracking error (15) is redefined as

$$\dot{\tilde{x}}_c = A_T(\tilde{x}_c,t)\,\tilde{x}_c + B_T(\tilde{x}_c,t)\,u + B_T(\tilde{x}_c,t)\,d \quad (17)$$

where

$$A_T(\tilde{x}_c,t) = T_0^{-1}\begin{bmatrix}-M_{cc}^{-1}C_{cc} & 0\\ T_{11}^{-1} & -T_{11}^{-1}T_{12}\end{bmatrix}T_0$$

$$B_T(\tilde{x}_c,t) = T_0^{-1}\begin{bmatrix}M_{cc}^{-1}\\ 0\end{bmatrix}$$

$$d = M_{cc}(q)\,T_{11}M_{cc}^{-1}(q)\,\delta_c.$$

Based on (16), the control acceleration can be given by

$$\ddot{q}_c = \ddot{q}_c^d - T_{11}^{-1}T_{12}\dot{\tilde{q}}_c - T_{11}^{-1}M_{cc}^{-1}\left(C_{cc}B^T T_0\tilde{x}_c - u\right). \quad (18)$$

Equation (18) gives the necessary acceleration to the controlled joints follow the desired reference trajectory. The torques in the active joints can be computed using this control acceleration. One can use another form of representing the underactuated system, similar to (11), partitioning (2) as in Bergerman (1996)

$$\begin{bmatrix}\tau_a\\ 0\end{bmatrix} = \begin{bmatrix}M_{ar} & M_{ac}\\ M_{ur} & M_{uc}\end{bmatrix}\begin{bmatrix}\ddot{q}_r\\ \ddot{q}_c\end{bmatrix} + \begin{bmatrix}b_a\\ b_u\end{bmatrix} \quad (19)$$

where the indexes a and u represent active and unlocked passive joints, respectively, and $b(q,\dot{q}) = C(q,\dot{q})\dot{q} + F(\dot{q}) + G(q) + \delta$. Factoring out the vector $\ddot{q}_r$ in the second line of (19) and substituting it in the first line, one obtains

$$\tau_a = \left(M_{ac} - M_{ar}M_{ur}^{-1}M_{uc}\right)\ddot{q}_c + b_a - M_{ar}M_{ur}^{-1}b_u.$$

If $n_p < n_a$, the redundant control can also be considered. In this case, the vector of controlled joints contains only the passive joints, $q_c = q_u \in \Re^{n_p}$, and the vector of remaining joints contains the active joints, $q_r = q_a \in \Re^{n_a}$. The partitioned equation is defined as follows

$$\begin{bmatrix}\tau_a\\ 0\end{bmatrix} = \begin{bmatrix}M_{aa} & M_{au}\\ M_{ua} & M_{uu}\end{bmatrix}\begin{bmatrix}\ddot{q}_a\\ \ddot{q}_u\end{bmatrix} + \begin{bmatrix}b_a\\ b_u\end{bmatrix}.$$

Factoring out the vector $\ddot{q}_a$ in the second line, one obtains

$$\ddot{q}_a = -M_{ua}^{\#}\left(M_{uu}\ddot{q}_u + b_u\right) + \left(I - M_{ua}^{\#}M_{ua}\right)z$$

where $M_{ua}^{\#}$ is the pseudo-inverse of the $(n_p \times n_a)$ matrix M_{ua} and z is an arbitrary number. The applied torques in the active joints can be computed as

$$\tau_a = \left(M_{au} - M_{aa}M_{ua}^{\#}M_{uu}\right)\ddot{q}_u + b_a - M_{aa}M_{ur}^{\#}b_u + M_{aa}\left(I - M_{ua}^{\#}M_{ua}\right)z.$$

For the underactuated case, the performance criterion (10) is also used to attenuate disturbances.

4 ROBOTIC H_∞ CONTROL PROBLEM SOLUTION

The solution of the robotic H_∞ control problem (10) can be explicitly found via differential game theory (Basar and Oslder, 1982; Basar and Berhard, 1990) with an appropriated Lyapunov function (Chen *et all*, 1994). In this section, a resume of the approach presented in (Chen *et all* (1994)) to solve this problem is presented.

The performance criterion (10) can be rewritten to define the following minimax problem

$$\min_{u(.)\in L_2}\ \max_{0\neq d(.)\in L_2}$$
$$\int_0^\infty \left(\frac{1}{2}\tilde{x}^T(t)\,Q\tilde{x}(t) + \frac{1}{2}u^T(t)\,Ru(t) - \frac{1}{2}\gamma^2 d^T(t)\,d(t)\right)dt \le 0,$$

with $\tilde{x}(0) = 0$. Defining the cost functional

$$J(\tilde{x}(t), u, d, t) = \int_t^\infty L(\tilde{x}(s), u(s), d(s))\,ds$$

with the Lagrangian

$$L(\tilde{x}, u, d) = \frac{1}{2}\tilde{x}^T(t)\,Q\tilde{x}(t) + \frac{1}{2}u^T(t)\,Ru(t) - \frac{1}{2}\gamma^2 d^T(t)\,d(t)$$

and introducing the Lyapunov function

$$V(\tilde{x}(t), t) = \min_{u(.)}\max_{d(.)} J(\tilde{x}(t), u, d, t)$$

the performance criterion (10) can be defined as

$$V(\tilde{x}(0), 0) = \min_{u(.)}\max_{d(.)} J(\tilde{x}(0), u, d, 0) \le 0, \quad \tilde{x}(0) = 0.$$

According to the differential game theory, the solution of this minimax (or leader-follower) problem is found if there exists a continuously differentiable Lyapunov function $V(.,.)$ that satisfies the following minimax Bellman-Isaacs equation

$$-\frac{\partial V(\tilde{x}, t)}{\partial t} = \min_{u(.)}\max_{d(.)}\left\{L(\tilde{x}, u, d) + \left(\frac{\partial V(\tilde{x}, t)}{\partial \tilde{x}}\right)^T \tilde{x}\right\}$$

with terminal condition $V(\tilde{x}(\infty), \infty) = 0$. Choosing a Lyapunov function of the form

$$V(\tilde{x}, t) = \frac{1}{2}\tilde{x}^T P(\tilde{x}, t)\,\tilde{x}$$

where $P(\tilde{x}, t)$ is a positive definite symmetric matrix for all $\tilde{x}$ and t, the Bellman-Isaacs equation is then changed to the following Riccati equation

$$\dot{P}(\tilde{x}, t) + P(\tilde{x}, t)\,A_T(\tilde{x}, t) + A_T^T(\tilde{x}, t)\,P(\tilde{x}, t) - P(\tilde{x}, t)\,B_T(\tilde{x}, t)\left(R^{-1} - \frac{1}{\gamma^2}I\right)B_T(\tilde{x}, t)\,P(\tilde{x}, t) + Q = 0. \tag{20}$$

The corresponding optimal control and the worst case disturbance are given, respectively, by

$$u^* = R^{-1}B_T^T(\tilde{x}, t)\,P(\tilde{x}, t)\,\tilde{x}$$

and

$$d^* = \frac{1}{\gamma^2}R^{-1}B_T^T(\tilde{x}, t)\,P(\tilde{x}, t)\,\tilde{x}.$$

Selecting $P(\tilde{x}, t)$ properly and using the skew symmetric matrix $N(q, \dot{q}) = C_0(q, \dot{q}) + (1/2)\dot{M}_0(q, \dot{q})$ (Chen *et all*, 1994), the Riccati equation (20) can be simplified to an algebraic matrix equation. The matrix $P(\tilde{x}, t)$ defined by Chen *et all* (1994) is given by

$$P(\tilde{x}, t) = T_0^T\begin{bmatrix} M_0(\tilde{x}, t) & 0 \\ 0 & K \end{bmatrix}T_0$$

where K is a positive definite symmetric constant matrix. The simplified algebraic equation is given by

$$\begin{bmatrix} 0 & K \\ K & 0 \end{bmatrix} - T_0^T B\left(R^{-1} - \frac{1}{\gamma^2}I\right)B^T T_0 + Q = 0. \tag{21}$$

The optimal control and the worst case disturbance can be rewritten, respectively, as

$$u^* = R^{-1}B^T T_0\tilde{x} \tag{22}$$

and

$$d^* = \frac{1}{\gamma^2}R^{-1}B^T T_0\tilde{x}.$$

The terminal condition is satisfied for this matrix $P(.,.)$ (Chen *et all*, 1994). Then, to solve the robotic H_∞ problem, we must find matrices K and T_0 which solve the algebraic equation (21). Let the positive definite symmetric matrix Q be factorized as

$$Q = \begin{bmatrix} Q_1^T Q_1 & Q_{12} \\ Q_{12}^T & Q_2^T Q_2 \end{bmatrix}. \tag{23}$$

The solution of (21) is given by

$$T_0 = \begin{bmatrix} R_1^T Q_1 & R_1^T Q_2 \\ 0 & I \end{bmatrix} \tag{24}$$

and

$$K = \frac{1}{2}\left(Q_1^T Q_2 - Q_2^T Q_1\right) - \frac{1}{2}\left(Q_{12}^T + Q_{12}\right)$$

with the conditions: $K > 0$ and $R < \gamma^2$I. The matrix R_1 is defined via Cholesky factorization

$$R_1^T R_1 = \left(R^{-1} - \frac{1}{\gamma^2} I\right)^{-1}. \tag{25}$$

Finally, the design algorithm can be outlined as follows

Step 1. Select a desired level of attenuation, $\gamma > 0$.

Step 2. Select the weighting matrix $R > 0$ such that $\lambda_{max} < \gamma^2$ and the weighting matrix Q as (23), satisfying $K > 0$.

Step 3. Calculate the Cholesky factorization (25) and T_0 (24).

Step 4. Obtain the optimal control u^* (22) and the optimal applied torque (8).

Considering the underactuated state tracking error (17), $P(\tilde{x}, t)$ can be chosen as

$$P(\tilde{x}, t) = T_0^T \begin{bmatrix} M_{cc}(\tilde{x}, t) & 0 \\ 0 & K \end{bmatrix} T_0$$

since the matrix M_{cc} is symmetric positive definite. Note that $N_{cc}(q, \dot{q}) = C_{cc}(q, \dot{q}) + (1/2)\dot{M}_{cc}(q, \dot{q})$ is also skew symmetric, then the design algorithm used to the totally actuated case can be applied to the underactuated case.

5 EXPERIMENTAL RESULTS

To validate the proposed H_∞ control solution, it is applied in our experimental underactuated manipulator

Figure 1: Underactuated Arm II.

UArm II (Underactuated Arm II), designed and built by H. Ben Brown, Jr. of Pittsburgh, PA, USA (Figure 1). This 3-link manipulator has special-purpose joints containing each one an actuator and a brake, so that they can act as active or passive joints. The manipulator configuration can be changed enabling or not the DC motor of each joint. Optical encoders with quadrature decoding are used to measure the joint positions. Joint velocities are obtained by numerical differentiation and filtering.

For interfacing between computer and manipulator, an input-output interface Servo-To-Go board is used. The board driver is accessed by dynamically linked libraries (*dll*s) compiled in the MatLab workspace by use of C++ program that contain *mex*-functions.

A control environment was developed in a suitable way that all changes of configuration and the robot action can be done in a user friendly way. The UMCE (Underactuated Manipulator Control Environment) is written in Matlab language and it is possible to see the real robot motion reproduced in its graphical interface (Figure 2). Simulation tests can also be done in this environment.

All possible configurations, according to active (A) and passive (P) joints location in the arm, are accepted: AAA, AAP, APA, PAA, APP, PAP, and PPA. For example, the configuration AAP means that joints 1 and 2 are active and joint 3 is passive.

The matrices $M(q)$, $C(q, \dot{q})$, and $G(q)$ of (1) are easily found via Lagrange theory for planar manipulators (Craig, 1989) (See Appendix A). However, the term $F(\dot{q})$ is determined according to the kind of frictional torques acting in the robot. In this work, we select a

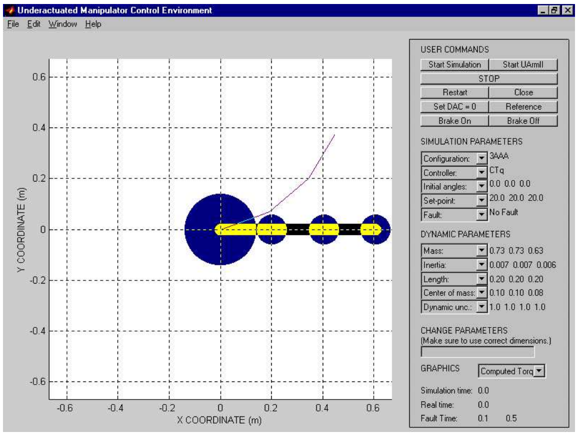

Figure 2: Graphical interface of the UMCE.

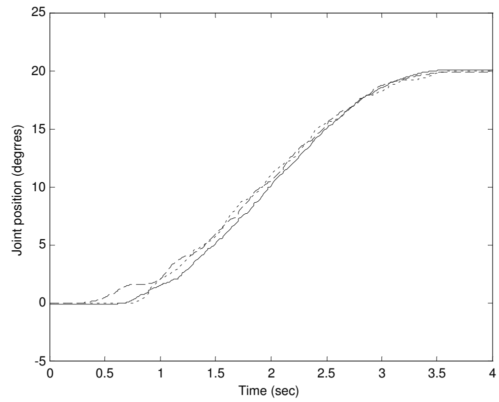

Figure 3: Joint positions, configuration AAA.

velocity-dependent frictional term $F(\dot{q})$ as

$$F(\dot{q}) = \begin{bmatrix} f_1\dot{q}_1 \\ f_2\dot{q}_2 \\ f_3\dot{q}_3 \end{bmatrix}$$

where the values f_1, f_2, and f_3 are selected after empirical tests. The manipulator's kinematic and dynamic nominal parameters, which are used to calculate the nominal matrices $M_0(q)$, $C_0(q,\dot{q})$, $F_0(\dot{q})$, and $G_0(q)$, are shown in Table I.

Table 1: Robot Parameters

Joint	m_i (kg)	I_i (kg.m^2)	l_i (m)	lc_i (m)	f_i (kg.m^2/s)
1	0.850	0.0075	0.203	0.096	0.45
2	0.850	0.0075	0.203	0.096	0.22
3	0.625	0.0060	0.203	0.077	0.22

The initial position for the experiment with configuration AAA is defined as $q(0) = [0,\,0,\,0]^o$ and the set-point defined as $q(T) = [20,\,20,\,20]^o$, where the vector $T = [T_1 \quad T_2 \quad T_3]$ contains the time duration for the reference trajectory for each joint. This vector is adequately selected taking into account the difference between the initial and final positions. The reference trajectory, q^d, is a fifth-degree polynomial trajectory.

The desired level of attenuation selected for the fully actuated case is $\gamma = 3$ with the following weighting matrices

$$Q_1 = I_3, \quad Q_2 = 2 * I_3, \quad Q_{12} = 0, \quad \text{and} \quad R = 5 * I_3.$$

Applying the design algorithm described in Section 4,

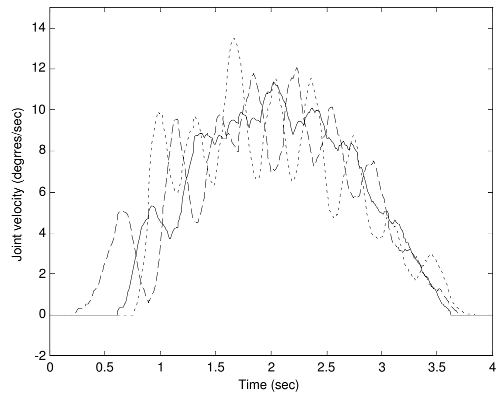

Figure 4: Joint velocities, configuration AAA.

since all conditions are satisfied, one can obtain

$$T_0 = \begin{bmatrix} 3.35 & 0 & 0 & 6.71 & 0 & 0 \\ 0 & 3.35 & 0 & 0 & 6.71 & 0 \\ 0 & 0 & 3.35 & 0 & 0 & 6.71 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The experimental results: joint positions, joint velocities and applied torques, for the configuration AAA, with $T = [4.0\ 4.0\ 4.0]$ sec., are shown in Figures 3, 4, and 5, respectively. In the following graphics the solid line represents the joint 1, the dashdot line represents the joint 2 and the dashed line represents the joint 3.

The configurations APA and PAP were used to validate the extension of the robotic H_∞ control for underactuated manipulators. For the configuration APA, two

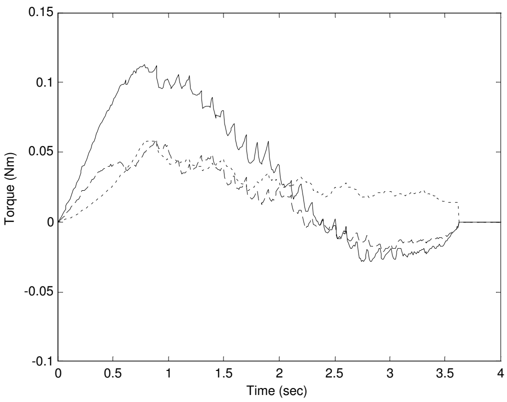

Figure 5: Applied torques, configuration AAA.

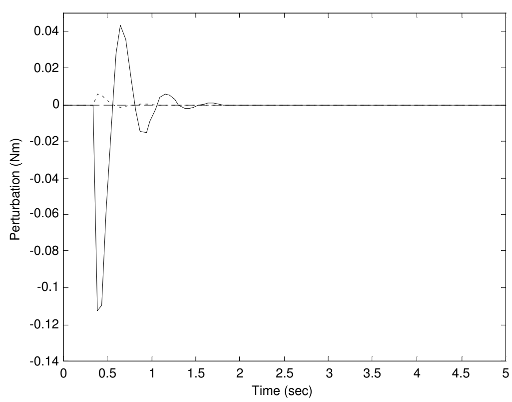

Figure 6: Disturbances in joints 1 and 3.

control phases are necessary to control all joints to the set-point. Since the configuration APA has $n_a = 2$ and $n_p = 1$, we can use two ways to select the controlled joints in the first control phase: 1) the passive joint and one active joint; and 2) only the passive joint, considering the actuation redundant control.

Case 1: In the first control phase, the vector of controlled joints, q_c, is selected as $q_c = [q_2,\ q_3]$, i.e., the passive (2) and the active (3) joints are selected (possibility 2 described in Section 3). In the second control phase, the active joints are selected to form the vector of controlled joints, $q_c = [q_1,\ q_2]$ (possibility 3). In this phase the passive joint 2 is kept locked, since it has already reached the set-point.

The initial position is $q(0) = [0,\ 0,\ 0]^o$, the set-point is $q(T_c, T_a) = [20,\ 20,\ 20]^o$. Two vectors of time duration, $T_c = [T_2\ T_3]$ and T_a= $[T_1\ T_3]$, related with each control phase, have to be constructed to the underactuated case. Exogenous disturbances, starting at t = 0.3 sec, are introduced in the active joints 1 and 3 in the form

$$\begin{aligned} w_1 &= -0.5e^{-4t} sin(4\pi t) \\ w_3 &= 0.5e^{-6t} sin(4\pi t) \end{aligned}$$

respectively. These disturbances are shown in Figure 6, where the solid line represents the disturbance in the joint 1 and the dashed line the disturbance in the joint 3.

The desired level of attenuation is defined as $\gamma = 4$ and the weighting matrices are given by

$$Q_1 = I_2, \quad Q_2 = 4 * I_2, \quad Q_{12} = 0, \quad \text{and} \quad R = 5 * I_2$$

and based on Section 4,

$$T_0 = \begin{bmatrix} 2.6968 & 0 & 10.7872 & 0 \\ 0 & 2.6968 & 0 & 10.7872 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

For the second control phase the desired level of attenuation is $\gamma = 4.5$. The weighting matrices and T_0 are given by

$$Q_1 = I_2, \quad Q_2 = 4 * I_2 \quad , \quad Q_{12} = 0, R = 5 * I_2$$

and

$$T_0 = \begin{bmatrix} 2.5767 & 0 & 10.3068 & 0 \\ 0 & 2.5767 & 0 & 10.3068 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The experimental results: joint positions, joint velocities and applied torques, for the configuration APA, with $T_c = [1.0\ 1.0]$ sec. and $T_a = [4.0\ 4.0]$ sec., are shown in Figures 7, 8, and 9, respectively.

Case 2: In the first control phase, the vector of controlled joints, q_c, is selected as $q_c = q_2$, i.e., only the passive joint (2) is selected. Considering the actuation redundant control, the remaining joints are the two actives ones, $q_r = [q_1,\ q_3]$. Here, the arbitrary number z is set to zero. The second control phase is the same as in case 1.

The desired level of attenuation, for the first control phase, is defined as $\gamma = 4$ and the weighting matrices are given by

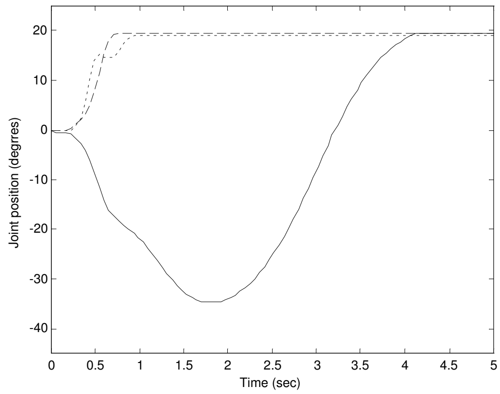

Figure 7: Joint positions, configuration APA.

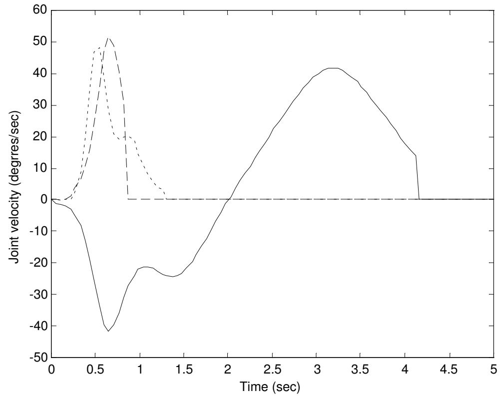

Figure 8: Joint velocities, configuration APA.

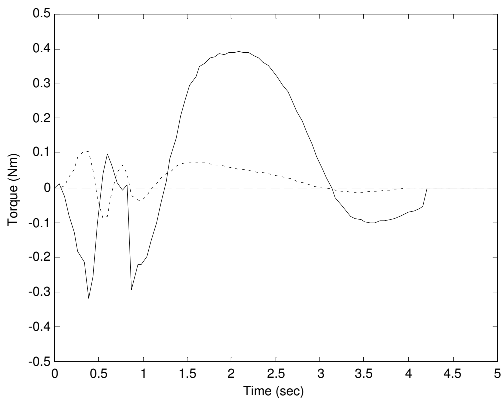

Figure 9: Applied torques, configuration APA.

$$Q_1 = 1, \quad Q_2 = 4, \quad Q_{12} = 0, \quad \text{and} \quad R = 5.$$

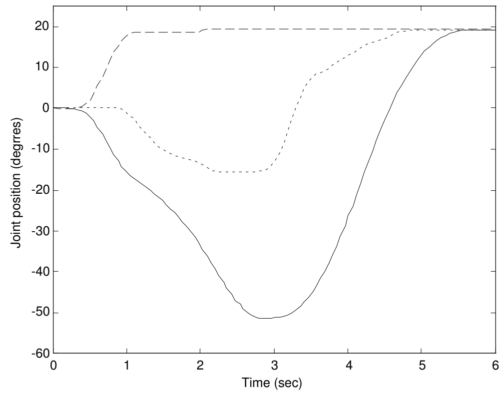

Figure 10: Joint positions, configuration APA with actuation redundant control.

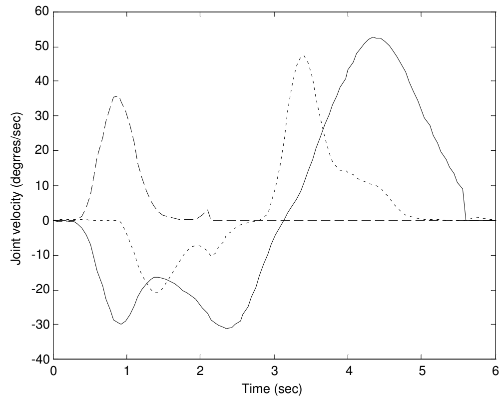

Figure 11: Joint velocities, configuration APA with actuation redundant control.

Applying again the design algorithm described in Section 4, one can obtain

$$T_0 = \begin{bmatrix} 2.6968 & 10.7872 \\ 0 & 1 \end{bmatrix}.$$

The experimental results: joint positions, joint velocities and applied torques, for the configuration APA with actuation redundant control, and with $T_c = [1.0]$ sec. and $T_a = [4.0\ 4.0]$ sec., are shown in Figures 10, 11, and 12, respectively.

For the PAP configuration, three control phases are necessary to control all joints to the set-point. In the first phase, the vector of controlled joints, qc, is selected as qc = q3 (possibility 1 described in Section 3). In the second phase, qc = q1 (possibility 1). And finally, the active joint, qc = q2, is controlled (possibility 3). In

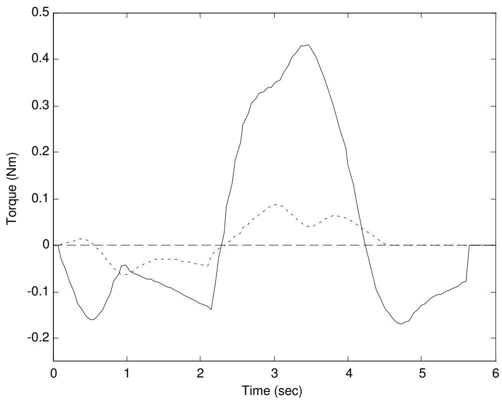

Figure 12: Applied torques, configuration APA with actuation redundant control.

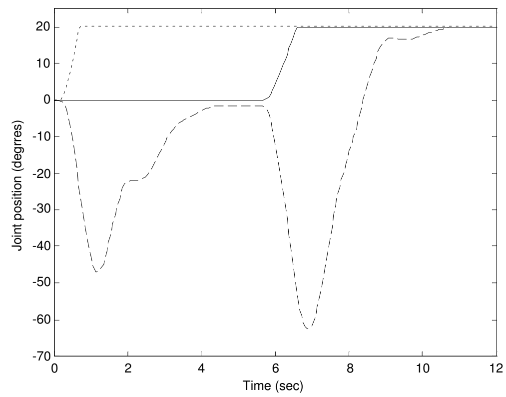

Figure 13: Joint positions, configuration PAP.

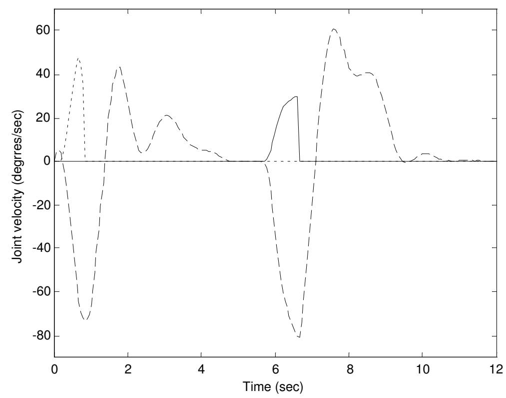

Figure 14: Joint velocities, configuration PAP.

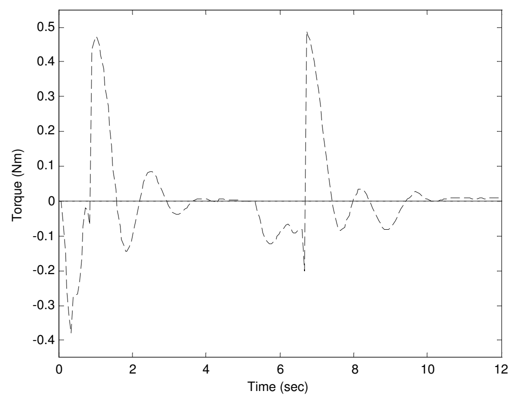

Figure 15: Applied torques, configuration PAP.

the first phase, the joint 1 is kept locked since it is not been controlled. Between the first and the second control phases the joint 2 is repositioned in order to obtain the necessary workspace to control the joint 1. The vector of time duration is defined as T = [T1 Tad T2 T3], where Ti is the time duration of phase i and Tad is the additional time to reposition the joint 1.

For the first control phase $\gamma = 5$,

$$Q_1 = 0.8, \quad Q_2 = 1.5, \quad Q_{12} = 0, \quad R = 5$$

and

$$T_0 = \begin{bmatrix} 2.0 & 3.75 \\ 0 & 1 \end{bmatrix}.$$

For the second control phase $\gamma = 5$,

$$Q_1 = 0.5, \quad Q_2 = 3, \quad Q_{12} = 0, \quad R = 5$$

and

$$T_0 = \begin{bmatrix} 1.25 & 7.5 \\ 0 & 1 \end{bmatrix}.$$

Finally, for the third control phase $\gamma = 3$,

$$Q_1 = 1, \quad Q_2 = 2.2, \quad Q_{12} = 0, \quad R = 5$$

and

$$T_0 = \begin{bmatrix} 3.35 & 7.38 \\ 0 & 1 \end{bmatrix}.$$

The experimental results: joint positions, joint velocities and applied torques, for the configuration PAP, with $T = [1.0\ 4.0\ 0.7\ 3.0]$ sec., are shown in Figures 13, 14, and 15, respectively.

6 CONCLUSION

It was presented in this paper the directives to solve the robotic H_∞ control problem for underactuated manipulators. Since the $M_{cc}(.)$ and $C_{cc}(.,.)$ matrices from (16) are formed by components of $M(.)$ and $C(.,.)$ matrices, respectively, keeping their proprieties, the solution for the underactuated problem is equivalent to the fully actuated case. The experimental results presented in this paper validate the proposed H_∞ controller for fully actuated and underactuated manipulators. The application of linear parameter varying (LPV) techniques (Huang and Jadbabaie, 1998) to solve the robotic H_∞ control problem is of author's interest for further works. The LPV methodology is used to solve nonlinear matrix inequalities (NLMI) generated by convex characterization of the nonlinear H_∞ control (Lu and Doyle, 1995).

APPENDIX A

The matrices $M(q)$, $C(q,\dot{q})$, and $G(q)$ for a 3-link planar manipulator with revolute joints, are given by

$$
\begin{aligned}
&M(1,1) = m_1 lc_1^2 + m_2\left(l_1^2 + lc_2^2 + 2l_1 lc_2\cos_2\right) + \\
&m_3\left(l_1^2 + l_2^2 + lc_3^2 + 2l_1 l_2\cos_2 + 2l_1 lc_3\cos_{23} + 2l_2 lc_3\cos_3\right) + \\
&I_1 + I_2 + I_3 \\
&M(1,2) = M(2,1) = m_2\left(lc_2^2 + l_1 lc_2\cos_2\right) + \\
&m_3\left(l_2^2 + lc_3^2 + l_1 l_2\cos_2 + l_1 lc_3\cos_{23} + 2l_2 lc_3\cos_3\right) + I_2 + I_3 \\
&M(1,3) = M(3,1) = m_3\left(lc_3^2 + l_1 lc_3\cos_{23} + l_2 lc_3\cos_3\right) + I_3 \\
&M(2,2) = m_2 lc_2^2 + m_3\left(l_2^2 + lc_3^2 + 2l_2 lc_3\cos_3\right) + I_2 + I_3 \\
&M(2,3) = M(3,2) = m_3\left(lc_3^2 + l_2 lc_3\cos_3\right) + I_3 \\
&M(3,3) = m_3 lc_3^2 + I_3
\end{aligned}
$$

$$
\begin{aligned}
&C(1,1) = -\left(m_2 l_1 lc_2 sin_2 + m_3 l_1 l_2 sin_2 + m_3 l_1 lc_3 sin_{23}\right)\dot{q}_2 - \\
&\left(m_3 l_1 lc_3 sin_{23} + m_3 l_2 lc_3 sin_3\right)\dot{q}_3 \\
&C(1,2) = -\left(m_2 l_1 lc_2 sin_2 + m_3 l_1 l_2 sin_2 + m_3 l_1 lc_3 sin_{23}\right)\times \\
&\left(\dot{q}_1 + \dot{q}_2\right) - \left(m_3 l_1 lc_3 sin_{23} + m_3 l_2 lc_3 sin_3\right)\dot{q}_3 \\
&C(1,3) = -\left(m_3 l_1 lc_3 sin_{23} + m_3 l_2 lc_3 sin_3\right)\left(\dot{q}_1 + \dot{q}_2 + \dot{q}_3\right) \\
&C(2,1) = \left(m_2 l_1 lc_2 sin_2 + m_3 l_1 l_2 sin_2 + m_3 l_1 lc_3 sin_{23}\right)\dot{q}_1 - \\
&\qquad m_3 l_2 lc_3 sin_3\dot{q}_3 \\
&C(2,2) = -m_3 l_2 lc_3 sin_3\dot{q}_3 \\
&C(2,3) = -m_3 l_2 lc_3 sin_3\dot{q}_3\left(\dot{q}_1 + \dot{q}_2 + \dot{q}_3\right) \\
&C(3,1) = \left(m_3 l_1 lc_3 sin_{23} + m_3 l_2 lc_3 sin_3\right)\dot{q}_1 + m_3 l_2 lc_3 sin_3\dot{q}_3 \\
&C(3,2) = m_3 l_2 lc_3 sin_3\dot{q}_3\left(\dot{q}_1 + \dot{q}_2\right) \\
&C(3,3) = 0
\end{aligned}
$$

and

$$
\begin{aligned}
G(1) &= m_1 g lc_1\cos_1 + m_2 g\left(l_1\cos_1 + lc_2\cos_{12}\right) + \\
&\qquad m_3 g\left(l_1\cos_1 + l_2\cos_{12} + lc_3\cos_{123}\right) \\
G(2) &= m_2 g lc_2\cos_{12} + m_3 g\left(l_2\cos_{12} + lc_3\cos_{123}\right) \\
G(3) &= m_3 g lc_3\cos_{123}
\end{aligned}
$$

where m_i, l_i, lc_i, and I_i are the mass, length, center of mass and inertia of the i-th link and $sin_i = sin(q_i)$, $sin_{ij} = sin(q_i + q_j)$, $cos_i = \cos(q_i)$, $cos_{ij} = cos(q_i + q_j)$, and $cos_{ijk} = cos(q_i + q_j + q_k)$.

REFERENCES

Arai, H. (1997). Feedback control of a 3-DOF planar underactuated manipulator. *Proceedings of the 1997 IEEE International Conference on Robotics and Automation*, Albuquerque, New Mexico, USA, pp. 703-709.

Arai, H. and S. Tachi (1991). Position control of a manipulator with passive joints using dynamic coupling. *IEEE Transactions on Robotics and Automation*, Vol. 7, no. 4, pp. 528-534.

Ball, J. A., J. W. Helton and M. L. Walker (1991). H_∞ control for nonlinear systems with output feedback. *IEEE Transactions on Automatic Control*, Vol. 38, no. 4, pp. 546-559, 1991.

Basar, T. and G. J. Olsder (1982). *Dynamic Noncooperative Game Theory*. Academic Press, New York, USA.

Basar, T. and P. Berhard (1990). *H_∞-Optimal Control and Related Minimax Problems*. Birkhäuser, Berlin, Germany.

Bergerman, M. (1996). Dynamics and control of underactuated manipulators. Ph.D. Thesis, Carnegie Mellon University, Pittsburgh, PA, USA.

Craig, J. J. (1989). *Introduction to Robots: Mechanics and Control*. Addison-Wesley, Reading, MA.

Chen, B. S., T. S. Lee and J. H. Feng (1994). A nonlinear H_∞ control design in robotic systems under parameter perturbation and external disturbance. *International Journal of Control*, Vol. 59, no. 2, pp. 439-461.

Chen, B. S. and Y. C. Chang (1997). Nonlinear mixed H_2/H_∞ control for robust tracking design of robotic systems. *International Journal of Control*, Vol. 67, no. 6, pp. 837-857.

Huang, Y. and A. Jadbabaie (1998). Nonlinear H_∞ control: an enhanced quasi-LPV approach. *Workshop in nonlinear H_∞ control by J. C. Doyle, Caltech, IEEE Conference on Decision and Control,* Tampa, Florida, USA.

Isidori, A. and A. Astolfi (1992). Disturbance attenuation and H_∞-control via measurement feedback in nonlinear systems. *IEEE Transactions on Automatic Control*, Vol. 37, no. 9, pp. 1283-1293.

Johansson, R. (1990). Quadratic optimization of motion coordination and control. *IEEE Transactions on Automatic Control*, Vol. 35, no. 11, pp. 1197-1208.

Lu, W. M. (1996). H_∞ control of nonlinear time-varying systems with finite-time horizon. *International Journal of Control*, Vol. 64, no. 2, pp. 241-262.

Lu, W. M. and J. C. Doyle (1991). H_∞ control of nonlinear systems via output feedback: controller parametrization. *IEEE Transactions on Automatic Control*, Vol. 39, no. 12, pp. 2517 - 2521.

Lu, W. M. and J. C. Doyle (1995). H_∞ control of nonlinear systems: a convex characterization. *IEEE Transactions on Automatic Control*, Vol. 40, no. 9, pp. 1668 – 1675.

Postlethwaite, I. and A. Bartoszewicz (1998). Application of non-linear H_∞ control to the Tetrabot robot manipulator. *Proceedings of the Institution of Mechanical Engineers-Part I - Journal of Systems and Control Engineering*, Vol. 212, no. 16, pp 459-465.

van der Schaft, A. J. (1991). On a state space approach to nonlinear H_∞ control. *Systems and Control Letters*, Vol. 16, pp. 1-8.

van der Schaft, A. J. (1992). L_2-gain analysis of nonlinear systems and nonlinear state feedback H_∞ control. *IEEE Transactions on Automatic Control*, Vol. 37, no. 6, pp. 770-784.