Volume 25, N. 2-3, pp. 331–351, 2006
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ISSN 0101-8205
www.scielo.br/cam

Identification of the collision kernel in the linear Boltzmann equation by a finite number of measurements on the boundary

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Abstract. In this paper we consider the inverse problem of recovering the collision kernel for the time dependent linear Boltzmann equation via a finite number of boundary measurements. We prove that this kernel can be uniquely determined by at most k measurements, provided that it belongs to a finite k-dimensional vector space.

Mathematical subject classification: 35R30, 83D75.

Key words: inverse problem, linear Boltzmann equation, albedo operator, boundary measurements.

1 Introduction

In this paper we consider an inverse problem for the linear Boltzmann equation

$$\partial_t u + \omega \cdot \nabla_x u + qu = qK_\kappa[u] \text{ in } (0,T) \times \mathbb{S} \times \Omega, \tag{1.1}$$

where $T > 0$, Ω is a smooth bounded convex domain of $\mathbb{R}^N$, $N \geq 2$, $\mathbb{S}$ denotes the unit sphere of $\mathbb{R}^N$, $q \in L^\infty(\Omega)$ and K_κ is the integral operator with kernel $\kappa(x, \omega', \omega)$ defined by

$$K_\kappa[u](t,\omega,x) = \int_{\mathbb{S}} \kappa(x,\omega',\omega) u(t,\omega',x)\, d\omega'. \tag{1.2}$$

#691/06. Received: 01/IX/06. Accepted: 01/X/06.

In applications, the equation (1.1) describes the dynamics of a monokinetic flow of particles in a body Ω under the assumption that the interaction between them is negligible (which allows us to discard nonlinear terms). For instance, in the case of a low-density flux of neutrons (see [7], [10]), $q \geq 0$ is the total extinction coefficient and the collision kernel κ is given by

$$\kappa(x, \omega', \omega) = c(x)h(x, \omega' \cdot \omega),$$

where c corresponds to the within-group scattering probability and h describes the anisotropy of the scattering process. In this model, $q(x)u(t, \omega, x)$ describes the loss of particles at x in the direction ω at time t due to absorption or scattering and $q(x)K_\kappa[u](t, \omega, x)$ represents the production of particles at x in the direction ω from those coming from directions ω'.

Our focus here is the inverse problem of recovery the coefficients in (1.1) via boundary measurements. More precisely, we are interested to recover q and κ by giving the incoming flux of particles on the boundary and measuring the outgoing one. Since these operations are described mathematically by the *albedo* operator

$$\mathcal{A}_{q,\kappa} : L^1\big(0, T; L^1(\Sigma^-; d\xi)\big) \longrightarrow L^1\big(0, T; L^1(\Sigma^+; d\xi)\big)$$

(the spaces will be precised below), a general mathematical question concerning this inverse problem is to know if the knowledge of $\mathcal{A}_{q,\kappa}$ uniquely determines q, κ, i.e., if the map $(q, \kappa) \mapsto \mathcal{A}_{q,\kappa}$ is invertible.

Taking into account the applications, we have to precise this question. A first one is to know if the knowledge of $\mathcal{A}_{q,\kappa}[f]$ for all f determines (q, κ) (*infinitely many measurements*); a second one is to know if the knowledge of $\mathcal{A}_{q,\kappa}[f_j]$, for $j = 1, 2, \ldots, k$, determines (q, κ) (*finite number of measurements*).

There is a wide bibliography devoted to the first problem. We specially mention the general results obtained by Choulli and Stefanov [4]: they show that q and κ are uniquely determined by the *albedo* operator (see also [9]). We also mention the stability results obtained by Cipolatti, Motta and Roberty (see [5] and the references therein).

There is also a lot of papers concerning the stationary case (see for instance those by V.G. Romanov [11], [12], P. Stefanov and G. Uhlmann [13], Tamasan [14], J.N. Wang [15], and also the references therein).

In this work we focus on the second question, concerning the recovery by a finite number of measurements. This may be interesting from the numerical point of view (finite element methods, for instance). Assuming that $\kappa(t, \omega', \omega) = c(x)h(\omega', \omega)$, we prove that c can be uniquely determined by at most k measurements, provided that c belongs to a finite k-dimensional vector space of $C(\overline{\Omega})$. More precisely:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded convex domain of class C^1, $T > \operatorname{diam}(\Omega)$ and $\mathcal{X} := \operatorname{span}\{\rho_1, \rho_2, \dots, \rho_k\}$, where $\{\rho_1, \rho_2, \dots, \rho_k\}$ is a linearly independent subset of $C(\overline{\Omega})$. We assume that $c \in \mathcal{X}$ and $\kappa(x, \omega', \omega) = c(x)h(\omega', \omega)$, where $h \in C(\mathbb{S} \times \mathbb{S})$ satisfies $h(\omega, \omega) \neq 0$ for every $\omega \in \mathbb{S}$. Then, there exist $f_1, \dots, f_k \in C_0\big((0, T) \times \Sigma^-\big)$ and $\widetilde{\omega}_1, \dots, \widetilde{\omega}_k \in \mathbb{S}$ that determine κ uniquely.*

The proof of Theorem 1.1 is based on the construction of highly oscillatory solutions (*à la* Calderón [1]) introduced in [5] and some arguments already used by the author in [6]. In fact, we consider solutions of the form

$$u_j(t, \omega, x) = \chi_s(\widetilde{\omega}_j, \omega)\phi_j(x - t\omega)\mathrm{e}^{-\int_0^t \tilde{q}(x-\tau\omega)d\tau}\mathrm{e}^{\mathrm{i}\lambda(t - x\cdot\omega)} + R_{\lambda,s}(t, \omega, x),$$

where χ_s converges (as $s \to 1$) to $\delta_{\tilde{\omega}_j}$, the spherical atomic measure concentrated on $\widetilde{\omega}_j$ and $R_{\lambda,s}$ vanishes as $\lambda \to \infty$. Therefore, by choosing $\widetilde{\omega}_j$ and ϕ_j conveniently, we obtain the result.

We organize the paper as follows: in Section 2 we recall the standard functional framework in which the Cauchy problem for (1.1) is well posed in the sense of the semigroup theory and the albedo operator is defined; in Section 3, we introduce the highly oscillatory functions that will be used, in Section 4, to prove Theorem 1.1.

2 Notation and functional framework

In this section we introduce the notation and we recall some well known results on the Transport Operator and the semigroup it generates in the *Neutronic Function Spaces* (see [5] and the references therein for the proofs).

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be a convex and bounded domain of class C^1 and $\mathbb{S}$ the unit sphere of $\mathbb{R}^N$. We denote by $Q := \mathbb{S} \times \Omega$ and Σ its boundary, i.e.,

$\Sigma := \mathbb{S} \times \partial\Omega$. For $p \in [1, +\infty)$ we consider the space $L^p(Q)$ with the usual norm

$$\|u\|_{L^p(Q)} := \left(\int_Q |u(\omega, x)|^p \, dx \, d\omega \right)^{1/p},$$

where $d\omega$ denotes the surface measure on $\mathbb{S}$ associated to the Lebesgue measure in $\mathbb{R}^{N-1}$.

For each $u \in L^p(Q)$ we define $A_0 u$ by

$$(A_0 u)(\omega, x) := \omega \cdot \nabla_x u(\omega, x) = \sum_{k=1}^{N} \omega_k \frac{\partial u}{\partial x_k}(\omega, x), \quad \omega = (\omega_1, \ldots, \omega_N)$$

where the derivatives are taken in the sense of distributions in Ω.

One checks easily that setting $\mathcal{W}_p := \{u \in L^p(Q)\,;\ A_0 u \in L^p(Q)\}$, the operator $\left(A_0, \mathcal{W}_p\right)$ is a closed densely defined operator and $\mathcal{W}_p$ with the graph norm is a Banach space.

For every $\sigma \in \partial\Omega$, we denote $\nu(\sigma)$ the unit outward normal at $\sigma \in \partial\Omega$ and we consider the sets (respectively, the incoming and outgoing boundaries)

$$\Sigma^{\pm} := \left\{(\omega, \sigma) \in \mathbb{S} \times \partial\Omega\,;\ \pm\, \omega \cdot \nu(\sigma) > 0\right\}.$$

In order to well define the albedo operator as a trace operator on the outgoing boundary, we consider $L^p(\Sigma^{\pm}; d\xi)$, where $d\xi := |\omega \cdot \nu(\sigma)| d\sigma d\omega$, and we introduce the spaces

$$\widetilde{\mathcal{W}}_p^{\pm} := \left\{u \in \mathcal{W}_p\,;\ u_{|_{\Sigma^{\pm}}} \in L^p(\Sigma^{\pm}; \xi)\right\},$$

which are Banach spaces if equipped with the norms

$$\|u\|_{\widetilde{\mathcal{W}}_p^{\pm}} := \left(\|u\|_{\mathcal{W}_p}^p + \int_{\Sigma^{\pm}} |\omega \cdot \nu(\sigma)| |u(\omega, \sigma)|^p \, d\sigma d\omega \right)^{1/p}.$$

The next two lemmas concern the continuity and surjectivity of the trace operators (see [2], [3] and [5]):

$$\gamma_{\pm} : \widetilde{\mathcal{W}}_p^{\pm} \to L^p(\Sigma^{\mp}; d\xi), \quad \gamma_{\pm}(u) := u_{|_{\Sigma^{\mp}}}. \tag{2.1}$$

Lemma 2.2. *Let $1 \leq p < +\infty$. Then there exists $C > 0$ (depending only on p) such that*

$$\int_{\Sigma^{\mp}} |\omega \cdot \nu(\sigma)||u(\omega,\sigma)|^p \, d\sigma d\omega \leq C\|u\|^p_{\widetilde{\mathcal{W}}^{\pm}_p}, \quad \forall u \in \widetilde{\mathcal{W}}^{\pm}_p. \tag{2.2}$$

Moreover, if $p > 1$ and $1/p + 1/p' = 1$, we have the Gauss identity

$$\int_Q \mathrm{div}_x(uv\omega)\, dx d\omega = \int_\Sigma \omega \cdot \nu(\sigma) u(\omega,\sigma) v(\omega,\sigma)\, d\sigma d\omega, \tag{2.3}$$

for all $u \in \widetilde{\mathcal{W}}^{\pm}_p$ and $v \in \widetilde{\mathcal{W}}^{\pm}_{p'}$.

As an immediate consequence of Lemma 2.1, we can introduce the space

$$\widetilde{\mathcal{W}}_p :\,= \left\{ f \in \mathcal{W}_p\,;\; \int_\Sigma |\omega \cdot \nu(\sigma)||f(\omega,\sigma)|^p \, d\omega d\sigma < +\infty \right\}$$

an we have that $\widetilde{\mathcal{W}}^+_p = \widetilde{\mathcal{W}}^-_p = \widetilde{\mathcal{W}}_p$ with equivalent norms.

Lemma 2.2. *The trace operators $\gamma_\pm$ are surjective from $\widetilde{\mathcal{W}}^{\pm}_p$ onto $L^p(\Sigma^{\mp}; d\xi)$. More precisely, for each $f \in L^p(\Sigma^{\mp}; d\xi)$, there exists $h \in \widetilde{\mathcal{W}}^{\pm}_p$ such that $\gamma_\pm(h) = f$ and*

$$\|h\|_{\widetilde{\mathcal{W}}^{\pm}_p} \leq C\|f\|_{L^p(\Sigma^{\mp}, d\xi)},$$

where $C > 0$ is independent of f.

We consider the operator $A : D(A) \to L^p(Q)$, defined by $(Au)(\omega, x) :\,= \omega \cdot \nabla u(\omega, x)$, with $D(A) :\,= \{u \in \widetilde{\mathcal{W}}_p\,;\; \gamma_-(u) = 0\}$.

Theorem 2.3. *The operator A is m-accretive in $L^p(Q)$, for $p \in [1, +\infty)$.*

Corollary 2.4. *Let $f \in L^p(Q)$, $p \in [1, +\infty)$ and assume that $u \in D(A)$ is a solution of $u + Au = f$. If $f \geq 0$ a.e. in Q, then $u \geq 0$ a.e. in Q. In particular, it follows that*

$$\|u\|_{L^1(Q)} \leq \|f\|_{L^1(Q)}.$$

It follows from Theorem 2.3 and Corollary 2.4 that the operator A generates a positive semigroup $\{U_0(t)\}_{t\geq 0}$ of contractions acting on $L^p(Q)$.

Let $q \in L^\infty(\Omega)$ and $\kappa : \Omega \times \mathbb{S} \times \mathbb{S} \to \mathbb{R}$ be a real measurable function satisfying

$$\begin{cases} \displaystyle\int_{\mathbb{S}} |\kappa(x, \omega', \omega)|\, d\omega' \leq M_1 \text{ a.e. } \Omega \times \mathbb{S}, \\ \displaystyle\int_{\mathbb{S}} |\kappa(x, \omega', \omega)|\, d\omega \leq M_2 \text{ a.e. } \Omega \times \mathbb{S}. \end{cases} \tag{2.4}$$

Associated to these functions, we define the following continous operators:

1) $B \in \mathcal{L}(L^p(Q), L^p(Q))$ defined by $B[u](\omega, x) :\, = q(x)u(\omega, x)$,

2) $K_\kappa[u](\omega, x) :\, = \int_{\mathbb{S}} \kappa(x, \omega', \omega)u(\omega', x)\, d\omega'$.

It follows from (2.4) that $K_\kappa \in \mathcal{L}(L^p(Q), L^p(Q))$ $\forall p \in [1, +\infty)$ and (see [7])

$$\|K_\kappa[u]\|_{L^p(Q)} \leq M_1^{1/p'} M_2^{1/p} \|u\|_{L^p(Q)} \leq \max\{M_1, M_2\}\|u\|_{L^p(Q)}. \tag{2.5}$$

The operator $A+B-K_\kappa : D(A) \to L^p(Q)$ generates a c_0-semigroup $\{U(t)\}_{t\geq 0}$ on $L^p(Q)$ satisfying

$$\|U(t)\|_{\mathcal{L}} \leq \mathrm{e}^{Ct}, \quad C :\, = \|q^-\|_\infty + M_2. \tag{2.6}$$

We consider the initial-boundary value problem for the linear Boltzmann equation

$$\begin{cases} \partial_t u(t, \omega, x) + \omega \cdot \nabla u(t, \omega, x) + q(x)u(t, \omega, x) = qK_\kappa[u](t, \omega, x) \\ u(0, \omega, x) = 0, \quad (\omega, x) \in \mathbb{S} \times \Omega \\ u(t, \omega, \sigma) = f(t, \omega, \sigma), \quad (\omega, \sigma) \in \Sigma^-, \ \ t \in (0, T), \end{cases} \tag{2.7}$$

where $q \in L^\infty(\Omega)$, $K_\kappa[u]$ is defined by (1.2) with κ satisfying (2.4).

By the previous results, it follows that, for $f \in L^p\big(0, T; L^p(\Sigma^-, d\xi)\big)$, $p \in [1, +\infty)$, there exists a unique solution $u \in C\big([0, T]; \widetilde{W}_p\big) \cap C^1\big([0, T]; L^p(Q)\big)$ of (2.7). This solution u allows us to define the *albedo* operator

$$\mathcal{A}_{q,\kappa} : L^p\big(0, T; L^p(\Sigma^-, d\xi)\big) \to L^p\big(0, T; L^p(\Sigma^+, d\xi)\big)$$
$$\mathcal{A}_{q,\kappa}[f](t, \omega, \sigma) :\, = u(t, \omega, \sigma), \quad (\omega, \sigma) \in \Sigma^+.$$

As a consequence of Lemmas 2.1 and 2.2, $\mathcal{A}_{q,\kappa}$ is a linear and bounded operator.

We also consider the following backward-boundary value problem, called the *adjoint problem* of (2.7):

$$\begin{cases} \partial_t u^*(t,\omega,x) + \omega \cdot \nabla u^*(t,\omega,x) - q(x)u^*(t,\omega,x) = -qK_\kappa^*[u^*](t,\omega,x) \\ u^*(T,\omega,x) = 0, \quad (\omega,x) \in \mathbb{S} \times \Omega \\ u^*(t,\omega,\sigma) = f^*(t,\omega,\sigma), \quad (\omega,\sigma) \in \Sigma^+, t \in (0,T), \end{cases} \tag{2.8}$$

where $f^* \in L^{p'}\big(0,T; L^{p'}(\Sigma^+, d\xi)\big)$, $p' \in [1,+\infty)$,

$$K_\kappa^*[u^*](t,\omega',x) := \int_{\mathbb{S}} \kappa(x,\omega',\omega)u^*(t,\omega,x)\, d\omega$$

with the corresponding albedo operator $\mathcal{A}_{q,\kappa}^*$

$$\begin{gathered} \mathcal{A}_{q,\kappa}^* : L^{p'}\big(0,T; L^{p'}(\Sigma^+, d\xi)\big) \to L^{p'}\big(0,T; L^{p'}(\Sigma^-, d\xi)\big) \\ \mathcal{A}_{q,\kappa}^*[f^*](t,\omega,\sigma) := u^*(t,\omega,\sigma), \quad (\omega,\sigma) \in \Sigma^-. \end{gathered}$$

The operators $\mathcal{A}_{q,\kappa}$ and $\mathcal{A}_{q,\kappa}^*$ satisfy the following property:

Lemma 2.5. *Let*

$$f \in L^p\big(0,T\,;\, L^p(\Sigma^-; d\xi)\big) \quad \textit{and} \quad f^* \in L^{p'}\big(0,T\,;\, L^{p'}(\Sigma^+; d\xi)\big),$$

where $p,\, p' \in (1,+\infty)$ *are such that* $1/p + 1/p' = 1$. *Then, we have*

$$\begin{aligned} &\int_0^T \int_{\Sigma^-} (\omega \cdot \nu(\sigma))\, f(t,\omega,\sigma)\mathcal{A}_{q,\kappa}^*[f^*](t,\omega,\sigma)\, d\sigma d\omega dt \\ &\qquad = -\int_0^T \int_{\Sigma^+} (\omega \cdot \nu(\sigma))\, f^*(t,\omega,\sigma)\mathcal{A}_{q,\kappa}[f](t,\omega,\sigma)\, d\sigma d\omega dt. \end{aligned}$$

Proof. It is a direct consequence of Lemma 2.1. Let $u(t,\omega,x)$ the solution of (2.7) with boundary condition f and $u^*(t,\omega,x)$ the solution of (2.8) with boundary f^*. We obtain the result by using (2.3), once the equation in (2.7) is multiplied by u^* and integrated over $(0,T) \times Q$. □

As a direct consequence of Lemma 2.5, we have:

Lemma 2.6. *Let $T > 0$, $q_1, q_2 \in L^\infty(\Omega)$ and κ_1, κ_2 satisfying* (2.4)*. Assume that u_1 is the solution of* (2.7) *with coefficients q_1, κ_1 and satisfying the boundary condition $f \in L^p\big(0, T; L^p(\Sigma^-, d\xi)\big)$, $p \in (1, +\infty)$ and that u_2^* is the solution of* (2.8)*, with q_2, κ_2 and boundary condition $f^* \in L^{p'}\big(0, T; L^{p'}(\Sigma^+, d\xi)\big)$, $1/p + 1/p' = 1$. Then we have*

$$\begin{aligned}
&\int_0^T\int_Q \big(q_2(x) - q_1(x)\big)u_1(t,\omega,x)u_2^*(t,\omega,x)\,dxd\omega dt \\
&\qquad - \int_0^T\int_Q \big(q_2(x)K_{\kappa_2}[u_1](t,\omega,x) - q_1(x)K_{\kappa_1}[u_1](t,\omega,x)\big)u_2^*(t,\omega,x)\,dxd\omega dt \\
&\quad = \int_0^T\int_{\Sigma^+} (\omega\cdot\nu(\sigma))\big[\mathcal{A}_{q_1,\kappa_1}[f] - \mathcal{A}_{q_2,\kappa_2}[f]\big](t,\omega,\sigma)\,f^*(t,\omega,\sigma)\,d\sigma d\omega dt.
\end{aligned}$$

3 Highly oscillatory solutions

In this section we prove some technical results related to special solutions of (2.7) and (2.8) that will be useful in the proof of Theorem 1.1. We denote by $\tilde q$ the zero extension of q in the exterior of Ω.

Proposition 3.1. *Let $T > 0$, $q_1, q_2 \in L^\infty(\Omega)$, and κ_1, κ_2 satisfying* (2.4)*. We consider $\psi_1, \psi_2 \in C\big(\mathbb{S}, C_0^\infty(\mathbb{R}^N)\big)$ such that*

$$\operatorname{supp}\psi_1(\omega,\cdot)\cap\overline{\Omega} = (\operatorname{supp}\psi_2(\omega,\cdot) + T\omega)\cap\overline{\Omega} = \emptyset, \quad \forall\omega\in\mathbb{S}. \tag{3.1}$$

Then, there exists $C_0 > 0$ such that, for each $\lambda > 0$, there exist $R_{1,\lambda} \in C\big([0,T]; \widetilde{W}_2\big)$ and $R_{2,\lambda}^* \in C\big([0,T]; \widetilde{W}_2\big)$ satisfying

$$\|R_{1,\lambda}\|_{C([0,T];L^2(Q))} \le C_0, \quad \|R_{2,\lambda}^*\|_{C([0,T];L^2(Q))} \le C_0, \tag{3.2}$$

for which the functions u_1, u_2^* defined by

$$\begin{cases} u_1(t,\omega,x) := \psi_1(\omega, x - t\omega)e^{-\int_0^t \tilde q_1(x - s\omega)\,ds}e^{\mathrm{i}\lambda(t-\omega\cdot x)} + R_{1,\lambda}(t,\omega,x) \\ u_2^*(t,\omega,x) := \psi_2(\omega, x - t\omega)e^{\int_0^t \tilde q_2(x - s\omega)\,ds}e^{-\mathrm{i}\lambda(t-\omega\cdot x)} + R_{2,\lambda}^*(t,\omega,x) \end{cases} \tag{3.3}$$

are solutions of (2.7) *with $(q,\kappa) = (q_1,\kappa_1)$ and* (2.8) *with $(q,\kappa) = (q_2,\kappa_2)$ respectively. Moreover, if $\kappa_j \in L^\infty\big(\Omega; L^2(\mathbb{S}\times\mathbb{S})\big)$, then we have*

$$\lim_{\lambda\to+\infty}\|R_{1,\lambda}\|_{C([0,T];L^2(Q))} = \lim_{\lambda\to+\infty}\|R_{2,\lambda}^*\|_{C([0,T]L^2(Q))} = 0. \tag{3.4}$$

Proof. Let u be the function

$$u(t,\omega,x):=\psi_1(\omega,x-t\omega)\mathrm{e}^{-\int_0^t \tilde{q}_1(x-s\omega)\,ds}\,\mathrm{e}^{\mathrm{i}\lambda(t-\omega\cdot x)}+R(t,\omega,x). \qquad (3.5)$$

By direct calculations, we easily verify that

$$\begin{aligned}&\partial_t u+\omega\cdot\nabla u+q_1u-q_1K_{\kappa_1}[u]\\ &=\partial_t R+\omega\cdot\nabla R+q_1R-q_1K_{\kappa_1}[R]-\mathrm{e}^{\mathrm{i}\lambda t}q_1Z_{1,\lambda},\end{aligned} \qquad (3.6)$$

where

$$Z_{1,\lambda}(t,\omega,x):=\int_{\mathbb{S}}\kappa_1(x,\omega',\omega)\psi_1(\omega',x-t\omega')\mathrm{e}^{-\int_0^t \tilde{q}_1(x-s\omega')\,ds}\mathrm{e}^{-\mathrm{i}\lambda\omega'\cdot x}d\omega'. \quad (3.7)$$

From (2..6), there exists $R_{1,\lambda}\in C^1\big([0,T];L^2(Q)\big)\cap C\big([0,T];D(A)\big)$ a unique solution of

$$\begin{cases}\partial_t R+\omega\cdot\nabla R+q_1R=q_1K_{\kappa_1}[R]+\mathrm{e}^{\mathrm{i}\lambda t}q_1Z_{1,\lambda},\\ R(0,\omega,x)=0, \qquad (\omega,x)\in S\times\Omega,\\ R(t,\omega,\sigma)=0, \qquad (\omega,\sigma)\in\Sigma^-,\end{cases} \qquad (3.8)$$

and it follows from (3.1) that the function u defined by (3.5) satisfies (2.7) with boundary condition

$$f_\lambda(t,\omega,\sigma):=\psi_1(\omega,\sigma-t\omega)\mathrm{e}^{-\int_0^t \tilde{q}_1(\sigma-s\omega)\,ds}\mathrm{e}^{\mathrm{i}\lambda(t-\omega\cdot\sigma)}, \quad (\omega,\sigma)\in\Sigma^-.$$

Multiplying both sides of the equation in (3.7) by the complex conjugate of R, integrating it over Q and taking its real part, we get, from Lemma 2.1,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_Q|R(t)|^2d\omega dx+\frac{1}{2}\int_{\Sigma^+}\omega\cdot\nu(\sigma)|R(t)|^2d\omega d\sigma+\int_Q q_1|R(t)|^2d\omega dx\\ &-\Re\int_Q q_1K_{\kappa_1}[R](t)\overline{R(t)}d\omega dx=\Re\left[\mathrm{e}^{\mathrm{i}\lambda t}\int_Q q_1Z_{1,\lambda}(t)\overline{R(t)}\,d\omega dx\right].\end{aligned}$$

It follows from the Cauchy-Schwarz inequality and (2.5) that

$$\int_Q|K_{\kappa_1}[R(t)]||R(t)|\,dxd\omega\le C_1\|R(t)\|^2_{L^2(Q)},$$

where $C_1:=\max\{M_1,M_2\}$. Therefore, we obtain

$$\frac{d}{dt}\|R(t)\|^2_{L^2(Q)}\le C_2\|q_1\|_\infty\|R(t)\|^2_{L^2(Q)}+\|q_1\|_\infty\|Z_{1,\lambda}(t)\|^2_{L^2(Q)},$$

where $C_2 := 3+2C_1$. Since $R(0) = 0$, we get, by integrating this last inequality on $[0, t]$,

$$\begin{aligned} \|R(t)\|^2_{L^2(Q)} &\leq \|q_1\|_\infty e^{\|q_1\|_\infty T C_2} \int_0^t \|Z_{1,\lambda}(\tau)\|^2_{L^2(Q)} \, d\tau \\ &\leq \|q_1\|_\infty e^{\|q_1\|_\infty T C_2} \|Z_{1,\lambda}\|^2_{L^2((0,T)\times Q)} \end{aligned}, \forall t \in [0, T]. \quad (3.9)$$

The first inequality in (3.2) follows easily because

$$|Z_{1,\lambda}(t, \omega, x)| \leq \|\psi_1\|_\infty e^{\|q_1\|_\infty T} M_1$$

and, as the same arguments hold for u_2^* and $R^*_{2,\lambda}$, we also obtain the second inequality.

We assume now $\kappa \in L^\infty\big(\Omega; L^2(\mathbb{S} \times \mathbb{S})\big)$. For each $x \in \mathbb{R}^N$, the map $\omega' \mapsto \exp(\mathrm{i}\lambda\omega' \cdot x)$ converges weakly to zero in $L^2(\mathbb{S})$ when $\lambda \to +\infty$ and the integral operator with kernel $\kappa(x, \cdot, \cdot)$ is compact in $L^2(\mathbb{S})$. So, we obtain from (3.6),

$$\lim_{\lambda\to+\infty} \|Z_{1,\lambda}(t, \cdot, x)\|_{L^2(\mathbb{S})} = 0 \quad \text{a.e. in} \quad [0, T] \times \Omega.$$

Moreover, $\|Z_{1,\lambda}(t, \cdot, x)\|_{L^2(\mathbb{S})} \leq C$, where $C > 0$ is a constant that does not depend on λ. The Lebesgue's Dominated Convergence Theorem implies that

$$\lim_{\lambda\to+\infty} \|Z_{1,\lambda}\|_{L^2([0,T]\times Q)} = 0.$$

From (3.9) and (3.8) we obtain (3.4), and our proof is complete. □

Corollary 3.2. *Under the hypothesis of Proposition* 3.1*, if* $q_1, q_2 \in C(\overline{\Omega})$ *and* $\kappa_1, \kappa_2 \in L^\infty\big(\Omega; C(\mathbb{S} \times \mathbb{S})\big)$*, we have, for every* $\omega \in \mathbb{S}$*,*

$$\lim_{\lambda\to+\infty} \|R_{1,\lambda}(\cdot, \omega, \cdot)\|_{C([0,T];L^2(\Omega))} = \lim_{\lambda\to+\infty} \|R^*_{2,\lambda}(\cdot, \omega, \cdot)\|_{C([0,T];L^2(\Omega))} = 0.$$

Proof. By multiplying both sides of the equation in (3.7) by the complex conjugate of $R(t, \omega, x)$, integrating it over Ω, taking its real part and applying the Hölder inequality, we get

$$\begin{aligned} \frac{d}{dt}\|R(t, \omega)\|^2_{L^2(\Omega)} &\leq 4\|q_1\|_\infty \|R(t, \omega)\|^2_{L^2(\Omega)} \\ &+ \|q_1\|_\infty \left(\|K_{\kappa_1}[R](t, \omega)\|^2_{L^2(\Omega)} + \|Z_{1,\lambda}(t, \omega)\|^2_{L^2(\Omega)}\right). \end{aligned} \quad (3.10)$$

Since

$$
\begin{aligned}
|K_{\kappa_1}[R](t,\omega,x)| &\le \int_{\mathbb{S}} |\kappa_1(x,\omega',\omega)||R(t,\omega',x)|\,d\omega' \\
&\le \left(\int_{\mathbb{S}} |\kappa_1(x,\omega',\omega)|\,d\omega'\right)^{1/2} \left(\int_{\mathbb{S}} |\kappa_1(x,\omega',\omega)||R(t,\omega',x)|^2\,d\omega'\right)^{1/2} \\
&\le M_1^{1/2}\|\kappa_1\|_\infty^{1/2} \left(\int_{\mathbb{S}} |R(t,\omega',x)|^2\,d\omega'\right)^{1/2},
\end{aligned}
$$

we obtain

$$
\|K_{\kappa_1}[R](t,\omega)\|^2_{L^2(\Omega)} \le M_1\|\kappa_1\|_\infty \|R(t)\|^2_{L^2(Q)}. \tag{3.11}
$$

From (3.8), (3.10) and (3.11) we have

$$
\begin{aligned}
\frac{d}{dt}\|R(t,\omega)\|^2_{L^2(\Omega)} &\le 4\|q_1\|_\infty \|R(t,\omega)\|^2_{L^2(\Omega)} \\
&+ C\left(\|Z_{1,\lambda}\|^2_{L^2((0,T)\times Q)} + \|Z_{1,\lambda}(t,\omega)\|^2_{L^2(\Omega)}\right).
\end{aligned}
$$

Now, integrating this last inequality on time, we get

$$
\begin{aligned}
\|R(t,\omega)\|^2_{L^2(\Omega)} &\le C\mathrm{e}^{\|q_1\|_\infty T}\left(t\|Z_{1,\lambda}\|^2_{L^2((0,T)\times Q)} + \int_0^t \|Z_{1,\lambda}(\tau,\omega)\|^2_{L^2(\Omega)}\,d\tau\right) \\
&\le C\mathrm{e}^{\|q_1\|_\infty T}\left(T\|Z_{1,\lambda}\|^2_{L^2((0,T)\times Q)} + \|Z_{1,\lambda}(\cdot,\omega,\cdot)\|^2_{L^2((0,T)\times\Omega)}\right).
\end{aligned}
$$

From Proposition 3.1 we know that $\|Z_{1,\lambda}\|_{L^2((0,T)\times Q)} \to 0$ as $\lambda \to +\infty$. On the other hand, as the map $\omega' \mapsto \mathrm{e}^{\mathrm{i}\omega'\cdot x}$ converges weakly to zero in $L^2(\mathbb{S})$, we have from (3.6), for almost $x \in \Omega$,

$$
\lim_{\lambda\to\infty} Z_{1,\lambda}(t,\omega,x) = 0, \quad \forall\omega \in S,\ \forall t \in [0,T]
$$

and the conclusion follows from the Lebesgue's Theorem. □

Lemma 3.3. *We assume that $q \in L^\infty(\Omega)$ and κ satisfies (2.4). Let S^*_λ be the solution of*

$$
\begin{cases}
\partial_t S + \omega\cdot\nabla S - qS = -qK^*_\kappa[S] + qe^{-\mathrm{i}\lambda t}Z, \\
S(T,\omega,x) = 0, \quad (\omega,x) \in \mathbb{S}\times\Omega, \\
S(t,\omega,\sigma) = 0, \quad (\omega,\sigma) \in \Sigma^+,
\end{cases} \tag{3.12}
$$

where $Z \in H^1\big(0,T; L^2(Q)\big)$ *such that* $Z(T)=0$. *Then we have*

$$\|S_\lambda^*\|_{C([0,T];L^2(Q))} \le C_0 \quad \textit{and} \quad \lim_{\lambda\to\infty} \|S_\lambda^*\|_{H^{-1}(0,T;L^2(Q))} = 0, \tag{3.13}$$

where C_0 *is a constant independent of* λ.

Proof. Multiplying both sides of the equation in (3.12) by the complex conjugate of S_λ^*, integrating it over Q and taking its real part, we get

$$\frac{1}{2}\frac{d}{dt}\|S_\lambda^*(t)\|^2_{L^2(Q)} + \frac{1}{2}\int_{\Sigma^-} (\omega\cdot\nu(\sigma))|S_\lambda^*(t,\omega,\sigma)|\, d\omega d\sigma \ge -\|q\|_\infty \|S_\lambda^*(t)\|^2_{L^2(Q)}$$
$$-\|q\|_\infty \|K_\kappa^*[S](t)\|_{L^2(Q)} \|S_\lambda^*(t)\|_{L^2(Q)} - \|q\|_\infty \|Z(t)\|_{L^2(Q)} \|S_\lambda^*(t)\|_{L^2(Q)}$$

Since

$$\|K_\kappa^*[S](t)\|_{L^2(Q)} \le \max\{M_1, M_2\} \|S_\lambda^*(t)\|_{L^2(Q)},$$

we have

$$\frac{d}{dt}\|S_\lambda^*(t)\|^2_{L^2(Q)} \ge -C_2 \|S_\lambda^*(t)\|^2_{L^2(Q)} - \|q\|_\infty \|Z(t)\|^2_{L^2(Q)},$$

where $C_2 : = (3 + 2\max\{M_1, M_2\})\|q\|_\infty$. Integrating this last inequality on $[t,T]$ and taking into account that $S_\lambda^*(T) = 0$, we obtain

$$\begin{aligned} \|S_\lambda^*(t)\|^2_{L^2(Q)} &\le \|q\|_\infty e^{C_2 T} \int_t^T \|Z(\tau)\|^2_{L^2(Q)}\, d\tau \\ &\le \|q\|_\infty e^{C_2 T} \|Z\|_{L^2(0,T;L^2(Q))} \end{aligned} \tag{3.14}$$

and the inequality in (3.13) follows easily.

We consider now

$$\begin{aligned} w_\lambda(t,\omega,x) &: = \int_t^T S_\lambda^*(\tau,\omega,x)\, d\tau, \\ h(t,\omega,x) &: = \int_t^T e^{-i\lambda\tau} Z(\tau,\omega,x)\, d\tau. \end{aligned} \tag{3.15}$$

Then, it is easy to check that w_λ satisfies

$$\begin{cases} \partial_t w + \omega\cdot\nabla w - qw = -qK_\kappa^*[w] + qh, \\ w(T,\omega,x) = 0, \quad (\omega,x) \in \mathbb{S}\times\Omega, \\ w(t,\omega,\sigma) = 0, \quad (\omega,\sigma) \in \Sigma^+, \end{cases} \tag{3.16}$$

Multiplying both sides of the equation in (3.16) by the complex conjugate of w_λ, integrating it over Q, taking its real part and applying the Cauchy-Schwarz inequality, we get as before,

$$\begin{aligned}\|w_\lambda(t)\|^2_{L^2(Q)} &\leq \|q\|_\infty e^{C_2T}\|h\|^2_{L^2(0,T;L^2(Q))} \\ &\leq \|q\|_\infty T^2 e^{C_2T}\|Z\|^2_{L^2(0,T;L^2(Q))}.\end{aligned} \tag{3.17}$$

As $S^*_\lambda = -\partial_t w_\lambda$, it follows from (3.14) and (3.17) that the set $\{w_\lambda\}$ is bounded in $C^1\big([0,T]; L^2(Q)\big)$ and, in particular, is relatively compact in $C\big([0,T]; L^2(Q)\big)$.

On the other hand, by integrating by parts the second integral in (3.15), it is easy to check that there exists $C > 0$ (depending only on T) such that

$$\|h\|_{L^2(0,T:L^2(Q))} \leq \frac{C}{|\lambda|}\|Z\|_{H^1(0,T;L^2(Q))}. \tag{3.18}$$

Hence, by (3.17), it follows that $\|w_\lambda\|_{C([0,T];L^2(Q))} \to 0$ as $\lambda \to \infty$. Since the partial derivative in t, $\partial_t : C\big([0,T]; L^2(Q)\big) \to H^{-1}\big(0,T; L^2(Q)\big)$, is a continuous operator, there exists a constant $C_3 > 0$ such that

$$\|S^*_\lambda\|_{H^{-1}(0,T;L^2(Q))} = \|\partial_t w_\lambda\|_{H^{-1}(0,T;L^2(Q))} \leq C_2\|w_\lambda\|_{C(0,T;L^2(Q))}$$

and we have the conclusion. □

4 Recovery by a finite number of boundary measurements

In this section we assume that $\{\rho_1, \rho_2, \ldots, \rho_k\}$ is a given linearly independent set of functions of $C(\overline{\Omega})$ and we denote $\mathcal{X} := \operatorname{span}\{\rho_1, \rho_2, \ldots, \rho_k\}$. For each $\widetilde{\omega} \in \mathbb{S}$ we consider $P_{\tilde{\omega}}[\rho_i]$ the X-ray transform of ρ_i in the direction $\widetilde{\omega}$, i.e.,

$$P_{\tilde{\omega}}[\rho_i](x) := \int_{-\infty}^{\infty} \rho_i(x + t\widetilde{\omega})\, dt$$

and, for each $\varepsilon > 0$, $\Omega_\varepsilon := \big\{x \in \mathbb{R}^N \setminus \overline{\Omega};\ \operatorname{dist}(x,\Omega) < \varepsilon\big\}$.

The following Lemma, which the proof is given in [6], will be essential for the proof of Theorem 1.1:

Lemma 4.1. *For all $\varepsilon > 0$, there exist $\widetilde{\omega}_j \in \mathbb{S}$ and $\phi_j \in C_0^\infty(\Omega_\varepsilon)$, $j = 1, \ldots, k$, such that the matrix $A = (a_{ij})$, with entries defined by*

$$a_{ij} :\, = \int_{\mathbb{R}^N} P_{\tilde{\omega}_j}[\rho_i](x)\phi_j^2(x)\, dx, \tag{4.1}$$

is invertible.

In order to prove Theorem 1.1, we define, for $0 < r < 1$, the function $\chi_r : \mathbb{S} \times \mathbb{S} \to \mathbb{R}$ as $\chi_r(\widetilde{\omega}, \omega) :\, = P(r\widetilde{\omega}, \omega)$, where P is the Poisson kernel for $B_1(0)$, i.e.,

$$P(x, y) :\, = \frac{1 - |x|^2}{\alpha_N |x - y|^N}.$$

From the well known properties of P (see [8]), we have

$$\begin{aligned} &\int_{\mathbb{S}} \chi_r(\widetilde{\omega}, \omega)\, d\omega = 1, \quad \forall\, r \in (0, 1), \quad \forall \widetilde{\omega} \in \mathbb{S}, \\ &\lim_{r\to 1} \int_{\mathbb{S}} \chi_r(\widetilde{\omega}, \omega)\psi(\omega)\, d\omega = \psi(\widetilde{\omega}), \end{aligned} \tag{4.2}$$

where the above limit is taken in the topology of $L^p(\mathbb{S})$, $p \in [1, +\infty)$ and uniformly on $\mathbb{S}$ if $\psi \in C(\mathbb{S})$. We are now in position to prove our main result.

Proof of Theorem 1.1. Let $\varepsilon :\, = (T - \mathrm{diam}(\Omega))/2$. We assume that $q_1 = q_2 = q$ and $\kappa_i(x, \omega', \omega) = c_i(x)h(\omega', \omega)$, where $c_1, c_2 \in \mathcal{X}$. For $\widetilde{\omega} \in \mathbb{S}$, we define $\psi_1(\omega, x) = \chi_s(\widetilde{\omega}, \omega)\phi(x)$ and $\psi_2(\omega, x) = \chi_r(\widetilde{\omega}, \omega)\phi(x)$, where $0 < r, s < 1$ and $\phi \in C_0^\infty(\Omega_\varepsilon)$. Then ψ_1 and ψ_2 satisfy the condition (3.1) and we may consider the solutions u_1 and u_2^* defined by (3.3), i.e.,

$$\begin{aligned} u_1(t, \omega, x) &:= \chi_s(\widetilde{\omega}, \omega)\phi(x - t\omega)\mathrm{e}^{-\int_0^t \tilde{q}(x-\tau\omega)d\tau}\mathrm{e}^{\mathrm{i}\lambda(t-x\cdot\omega)} + R_{1,\lambda,s}(t, \omega, x), \\ u_2^*(t, \omega, x) &:= \chi_r(\widetilde{\omega}, \omega)\phi(x - t\omega)\mathrm{e}^{\int_0^t \tilde{q}(x-\tau\omega)d\tau}\mathrm{e}^{-\mathrm{i}\lambda(t-x\cdot\omega)} + R^*_{2,\lambda,r}(t, \omega, x), \end{aligned}$$

where $\lambda > 0$ will be chosen a posteriori. We shall write

$$\begin{aligned} \Phi_\lambda(t, \omega, x) &:= \phi(x - t\omega)\mathrm{e}^{-\int_0^t \tilde{q}(x-\tau\omega)d\tau}\mathrm{e}^{\mathrm{i}\lambda(t-x\cdot\omega)} \\ \Psi_\lambda(t, \omega, x) &:= \phi(x - t\omega)\mathrm{e}^{\int_0^t \tilde{q}(x-\tau\omega)d\tau}\mathrm{e}^{-\mathrm{i}\lambda(t-x\cdot\omega)} \end{aligned}$$

in such a way that

$$
\begin{aligned}
u_1(t,\omega,x) &= \chi_s(\widetilde{\omega},\omega)\Phi_\lambda(t,\omega,x) + R_{1,\lambda,s}(t,\omega,x), \\
u_2^*(t,\omega,x) &= \chi_r(\widetilde{\omega},\omega)\Psi_\lambda(t,\omega,x) + R_{2,\lambda,r}^*(t,\omega,x).
\end{aligned}
\tag{4.3}
$$

Substituting u_1 and u_2^* in the indentity given in Lemma 2.6, we have

$$
J(\lambda,r,s) = L(\lambda,r,s), \tag{4.4}
$$

where

$$
J(\lambda,r,s) := \int_0^T\int_Q q(x)\big(c_1(x)-c_2(x)\big)K_h[u_1](t,\omega,x)u_2^*(t,\omega,x)\,dxd\omega dt,
$$

$$
L(\lambda,r,s) := \int_0^T\int_{\Sigma^+}(\omega\cdot\nu(\sigma))\big(\mathcal{A}_1[f_{\lambda,s}]-\mathcal{A}_2[f_{\lambda,s}]\big)\,f_{\lambda,r}^*\,d\sigma d\omega dt.
$$

In the above formulas, we are denoting $\mathcal{A}_i = \mathcal{A}_{c_i}$, $i = 1, 2$ and

$$
\begin{aligned}
f_{\lambda,s}(t,\omega,\sigma) &:= \chi_s(\widetilde{\omega},\omega)\Phi_\lambda(t,\omega,\sigma), \quad (\omega,\sigma)\in\Sigma^-, \\
f_{\lambda,r}^*(t,\omega,\sigma) &:= \chi_r(\widetilde{\omega},\omega)\Psi_\lambda(t,\omega,\sigma), \quad (\omega,\sigma)\in\Sigma^+.
\end{aligned}
\tag{4.5}
$$

In particular, it follows from the definition of the Albedo Operator and (4.3),

$$
\mathcal{A}_1[f_{\lambda,s}]-\mathcal{A}_2[f_{\lambda,s}] = R_{1,\lambda,s}-R_{2,\lambda,s}, \quad \text{on } (0,T)\times\Sigma^+. \tag{4.6}
$$

By denoting $\eta(x) = \widetilde{q}(x)\big(\widetilde{c}_1(x)-\widetilde{c}_2(x)\big)$ and by considering the special form of u_1 and u_2^*, we may write $J(\lambda,r,s)$ as $J = J_1+J_2+J_3+J_4$, where

$$
\begin{aligned}
J_1(\lambda,r,s) := \int_0^T\int_Q \eta(x)&\left[\int_{\mathbb{S}} h(\omega',\omega)\chi_s(\widetilde{\omega},\omega')\Phi_\lambda(t,\omega',x)d\omega'\right] \\
&\chi_r(\widetilde{\omega},\omega)\Psi_\lambda(t,\omega,x)\,dxd\omega dt,
\end{aligned}
$$

$$
\begin{aligned}
J_2(\lambda,r,s) := \int_0^T\int_Q \eta(x)&\left[\int_{\mathbb{S}} h(\omega',\omega)\chi_s(\widetilde{\omega},\omega')\Phi_\lambda(t,\omega',x)d\omega'\right] \\
&R_{2,\lambda,r}^*(t,\omega,x)\,dxd\omega dt,
\end{aligned}
$$

$$
\begin{aligned}
J_3(\lambda,r,s) := \int_0^T\int_Q \eta(x)&\left[\int_{\mathbb{S}} h(\omega',\omega)R_{1,\lambda,s}(t,\omega',x)d\omega'\right] \\
&\chi_r(\widetilde{\omega},\omega)\Psi_\lambda(t,\omega,x)\,dxd\omega dt,
\end{aligned}
$$

$$J_4(\lambda, r, s) := \int_0^T \int_Q \eta(x) \left[\int_{\mathbb{S}} h(\omega', \omega) R_{1,\lambda,s}(t, \omega', x) d\omega' \right] R^*_{2,\lambda,r}(t, \omega, x)\, dx d\omega dt.$$

Taking the limit as $r \to 1^-$ in the above expressions, we get from the definition of χ_r, $J_i(\lambda, r, s) \to J_i(\lambda, s)$, where

$$J_1(\lambda, s) := \int_0^T \int_\Omega \eta(x) \left[\int_{\mathbb{S}} h(\omega', \widetilde{\omega}) \chi_s(\widetilde{\omega}, \omega') \Phi_\lambda(t, \omega', x) d\omega' \right] \Psi_\lambda(t, \widetilde{\omega}, x)\, dx dt,$$

$$J_2(\lambda, s) := \int_0^T \int_Q \eta(x) \left[\int_{\mathbb{S}} h(\omega', \omega) \chi_s(\widetilde{\omega}, \omega') \Phi_\lambda(t, \omega', x) d\omega' \right] S^*_{2,\lambda}(t, \omega, x)\, dx d\omega dt,$$

$$J_3(\lambda, s) := \int_0^T \int_\Omega \eta(x) \left[\int_{\mathbb{S}} h(\omega', \widetilde{\omega}) R_{1,\lambda,s}(t, \omega', x) d\omega' \right] \Psi_\lambda(t, \widetilde{\omega}, x)\, dx dt,$$

$$J_4(\lambda, s) := \int_0^T \int_Q \eta(x) \left[\int_{\mathbb{S}} h(\omega', \omega) R_{1,\lambda,s}(t, \omega', x) d\omega' \right] S^*_{2,\lambda}(t, \omega, x)\, dx d\omega dt$$

and $S^*_{2,\lambda}$ is the unique solution of

$$\begin{cases} \partial_t S + \omega \cdot \nabla S - qS = -qK^*_{\kappa_2}[S] + \mathrm{e}^{-\mathrm{i}\lambda t} q Z^*_{2,\lambda}, \\ S(T, \omega, x) = 0, \quad (\omega, x) \in \mathbb{S} \times \Omega, \\ S(T, \omega, \sigma) = 0, \quad (\omega, \sigma) \in \Sigma^+, \end{cases} \tag{4.7}$$

Moreover, from (4.5) and (4.2), it follows that $L(\lambda, r, s) \to L(\lambda, s)$, where

$$\begin{aligned} L(\lambda, s) :=& \int_0^T \int_{\partial\Omega} (\widetilde{\omega} \cdot \nu(\sigma))^+ \big(\widetilde{\mathcal{A}}_1[f_{\lambda,s}] - \widetilde{\mathcal{A}}_2[f_{\lambda,s}] \big) \\ & (t, \widetilde{\omega}, \sigma) \Psi_\lambda(t, \widetilde{\omega}, \sigma)\, d\sigma dt \\ =& \int_0^T \int_{\partial\Omega} (\widetilde{\omega} \cdot \nu(\sigma))^+ \big(R_{1,\lambda,s}(t, \widetilde{\omega}, \sigma) - R_{2,\lambda,s}(t, \widetilde{\omega}, \sigma) \big) \\ & \Psi_\lambda(t, \widetilde{\omega}, \sigma)\, d\sigma dt, \end{aligned} \tag{4.8}$$

where $\widetilde{\mathcal{A}}_i[f_{\lambda,s}]$ denotes the zero extension of $\mathcal{A}_i[f_{\lambda,s}]$ on $\partial\Omega$. Therefore, by taking the limit as $r \to 1^-$ in (4.4), we have

$$J_1(\lambda, s) + J_2(\lambda, s) + J_3(\lambda, s) + J_4(\lambda, s) = L(\lambda, s).$$

Now, it is time to take the limit as $s \to 1^-$. For the first two terms of the right hand side of the above identity, we get (for $i = 1, 2$) $J_i(\lambda, s) \to J_i(\lambda)$, where

$$\begin{aligned} J_1(\lambda) \,:=\, & \int_0^T\!\!\int_\Omega \eta(x) h(\widetilde{\omega}, \widetilde{\omega}) \Phi_\lambda(t, \widetilde{\omega}, x) \Psi_\lambda(t, \widetilde{\omega}, x)\, dx dt \\ =\, & h(\widetilde{\omega}, \widetilde{\omega}) \int_0^T\!\!\int_\Omega \eta(x) \phi(x - t\widetilde{\omega})^2 dx dt, \qquad (4.9) \\ J_2(\lambda) \,:=\, & \int_0^T\!\!\int_Q \eta(x) h(\widetilde{\omega}, \omega) \Phi_\lambda(t, \widetilde{\omega}, x) S^*_{2,\lambda}(t, \omega, x)\, dx d\omega dt. \end{aligned}$$

On the other hand, the dependence on s in the other terms is given by $R_{1,\lambda,s}$ and $R_{2,\lambda,s}$, which are the solution of ($j = 1, 2$)

$$\begin{cases} \partial_t R + \omega \cdot \nabla R + qR = qK_{\kappa_j}[R] + \mathrm{e}^{\mathrm{i}\lambda t} q Z_{j,\lambda,s}, \\ R(0, \omega, x) = 0, \quad (\omega, x) \in \mathbb{S} \times \Omega, \\ R(t, \omega, \sigma) = 0, \quad (\omega, \sigma) \in \Sigma^+, \end{cases} \tag{4.10}$$

where

$$Z_{j,\lambda,s}(t, \omega, x) \colon = \int_{\mathbb{S}} \kappa_j(x, \omega', \omega) \chi_s(\widetilde{\omega}, \omega') \Phi_\lambda(t, \omega', x) d\omega'. \tag{4.11}$$

It is an immediate consequence of (4.2) and the Lebesgue's Theorem that, as $s \to 1$, $Z_{j,\lambda,s} \to Z_{j,\lambda}$ in $C\big([0, T]; L^2(Q)\big)$, where

$$Z_{j,\lambda}(t, \omega, x) \colon = \kappa_j(x, \widetilde{\omega}, \omega) \Phi_\lambda(t, \widetilde{\omega}, x). \tag{4.12}$$

Hence,

$$\lim_{s \to 1^-} R_{j,\lambda,s} = S_{j,\lambda} \quad \text{in} \quad C\big([0, T]; L^2(Q)\big),$$

where $S_{j,\lambda}$ is the solution of

$$\begin{cases} \partial_t S + \omega \cdot \nabla S + qS = qK_{\kappa_j}[S] + \mathrm{e}^{\mathrm{i}\lambda t} q Z_{j,\lambda}, \\ S(0, \omega, x) = 0, \quad (\omega, x) \in \mathbb{S} \times \Omega, \\ S(t, \omega, \sigma) = 0, \quad (\omega, \sigma) \in \Sigma^-, \end{cases} \tag{4.13}$$

and

$$Z_{j,\lambda}(t, \omega, x) \colon = c_j(x) h(\widetilde{\omega}, \omega) \Phi_\lambda(t, \widetilde{\omega}, x).$$

Therefore, $J_i(\lambda, s) \to J_i(\lambda)$, $(i = 3, 4)$ and $L(\lambda, s) \to L(\lambda)$, where

$$J_3(\lambda) := \int_0^T \int_\Omega \eta(x) \left[\int_{\mathbb{S}} h(\omega', \widetilde{\omega}) S_{1,\lambda}(t, \omega', x) d\omega' \right] \Psi_\lambda(t, \widetilde{\omega}, x)\, dx dt,$$

$$J_4(\lambda) := \int_0^T \int_Q \eta(x) \left[\int_{\mathbb{S}} h(\omega', \widetilde{\omega}) S_{1,\lambda}(t, \omega', x) d\omega' \right] S^*_{2,\lambda}(t, \omega, x)\, dx d\omega dt. \tag{4.14}$$

$$L(\lambda) := \int_0^T \int_{\partial\Omega} \big(\widetilde{\omega} \cdot \nu(\sigma)\big)^+ \big(S_{1,\lambda}(t, \widetilde{\omega}, \sigma) - S_{2,\lambda}(t, \widetilde{\omega}, \sigma)\big) \Psi_\lambda(t, \widetilde{\omega}, \sigma)\, d\sigma dt$$

and we obtain

$$|J_1(\lambda)| \leq |J_2(\lambda)| + |J_3(\lambda)| + |J_4(\lambda)| + |L(\lambda)|, \tag{4.15}$$

where

$$\begin{aligned}
|J_2(\lambda)| &\leq \|\eta\|_\infty \|h\|_\infty \mathrm{e}^{MT} \int_0^T \int_Q |\phi(x - t\widetilde{\omega}) S^*_{2,\lambda}(t, \omega, x)|\, dx d\omega dt, \\
|J_3(\lambda)| &\leq \|\eta\|_\infty \|\phi\|_\infty \mathrm{e}^{MT} \|K_h[S_{1,\lambda}]\|_{L^2(0,T;L^2(Q))}, \\
|J_4(\lambda)| &\leq \|\eta\|_\infty \|K_h[S_{1,\lambda}]\|_{L^2(0,T;L^2(Q))} \|S^*_{2,\lambda}\|_{L^2(0,T;L^2(Q))}, \\
|L(\lambda)| &\leq \|\phi\|_\infty \mathrm{e}^{MT} \int_0^T \int_{\partial\Omega} \big(\widetilde{\omega} \cdot \nu(\sigma)\big)^+ \\
&\qquad \big| S_{1,\lambda}(t, \widetilde{\omega}, \sigma) - S_{2,\lambda}(t, \widetilde{\omega}, \sigma) \big|\, d\sigma dt.
\end{aligned} \tag{4.16}$$

Since $\phi \in C_0^\infty(\Omega_\varepsilon)$, it follows from the choice of ε that the function $(t, \omega, x) \mapsto \phi(x - t\widetilde{\omega})$ belongs to $H_0^1(0, T; L^2(Q))$ (as a constant function on ω). Hence, we have

$$|J_2(\lambda)| \leq \|\rho\|_\infty \mathrm{e}^{MT} \|\phi\|_{H_0^1(0,T;L^2(Q))} \|S^*_{2,\lambda}\|_{H^{-1}(0,T;L^2(Q))}.$$

On the other hand, from the weak convergence to zero in $L^2\big(0, T; L^2(Q)\big)$ of $S_{1,\lambda}$, it follows that

$$\lim_{\lambda \to +\infty} \|K_h[S_{1,\lambda}]\|_{L^2(0,T;L^2(Q))} = 0. \tag{4.17}$$

Hence, we have from (4.15)–(4.17) and Lemma 3.3,

$$
\begin{aligned}
|J_1(\lambda)| &= |h(\widetilde{\omega},\widetilde{\omega})| \left| \int_0^T\!\!\int_\Omega \eta(x)\phi(x-t\widetilde{\omega})^2\,dxdt \right| \\
&\le C(\lambda)\|\eta\|_\infty + C_2 \int_0^T\!\!\int_{\partial\Omega} \big(\widetilde{\omega}\cdot\nu(\sigma)\big)^+ \\
&\qquad \big|S_{1,\lambda}(t,\widetilde{\omega},\sigma) - S_{2,\lambda}(t,\widetilde{\omega},\sigma)\big|\; d\sigma dt,
\end{aligned}
\tag{4.18}
$$

where $C(\lambda) \to 0$ as $\lambda \to +\infty$.

Since $(\operatorname{supp}\phi + s\widetilde{\omega}) \cap \Omega = \emptyset$ for all $|s| \ge T$, we have

$$
\begin{aligned}
\left| \int_0^T\!\!\int_\Omega \eta(x)\phi(x-t\widetilde{\omega})^2\,dxdt \right| &= \left| \int_{\mathbb{R}^N} \int_0^T \eta(y+s\widetilde{\omega})\phi(y)^2\,dsdy \right| \\
&= \left| \int_{\mathbb{R}^N} \int_{-\infty}^{\infty} \eta(y+s\widetilde{\omega})\phi(y)^2\,dsdy \right| \\
&= \left| \int_{\mathbb{R}^N} P_{\tilde{\omega}}[\eta](y)\phi(y)^2\,dy \right|
\end{aligned}
\tag{4.19}
$$

and we get

$$
\begin{aligned}
|h(\widetilde{\omega},\widetilde{\omega})| \left| \int_{\mathbb{R}^N} P_{\tilde{\omega}}[\eta](y)\phi(y)^2\,dy \right| &\le C(\lambda)\|\eta\|_\infty \\
+ C_2 \int_0^T\!\!\int_{\partial\Omega} \big(\widetilde{\omega}\cdot\nu(\sigma)\big)^+ &\big|S_{1,\lambda}(t,\widetilde{\omega},\sigma) - S_{2,\lambda}(t,\widetilde{\omega},\sigma)\big|\; d\sigma dt
\end{aligned}
$$

We are now in position to conclude the proof. First of all, we consider in the above inequality the directions $\widetilde{\omega}_1,\ldots,\widetilde{\omega}_k$ and the functions $\phi_1,\ldots,\phi_k$ given by Lemma 4.1, in such a way that we can write

$$
\begin{aligned}
C_0\|c_1-c_2\|_\infty &\le C(\lambda)\|c_1-c_2\|_\infty + \\
+ C_2 \sum_{j=1}^k \int_0^T\!\!\int_{\partial\Omega} \big(\widetilde{\omega}_j\cdot\nu(\sigma)\big)^+ &\big|S_{1,\lambda}(t,\widetilde{\omega}_j,\sigma) - S_{2,\lambda}(t,\widetilde{\omega}_j,\sigma)\big|\; d\sigma dt,
\end{aligned}
$$

for some constant $C_0 > 0$. If we denote by

$$
u_{i,j}(t,\omega,\sigma) = \chi_s(\widetilde{\omega}_j,\omega)\Phi_\lambda(t,\omega,x) + R_{i,\lambda,s}(t,\omega,x), \quad i=1,2, \quad j=1,\ldots,k
$$

it follows from (4.2) that, as $s \to 1^-$, $u_{i,j} \to u^{\#}_{i,j}$, where

$$u^{\#}_{i,j} = \delta_{\tilde{\omega}_j} \Phi_\lambda + S_{i,\lambda}, \quad i = 1, 2, \; j = 1, \ldots, k$$

and $\delta_{\tilde{\omega}_j}$ is the spherical atomic measure concentrated on $\widetilde{\omega}_j$.

It is clear from (4.13) that $u^{\#}_{1,j}(t, \omega, \sigma) = u^{\#}_{2,j}(t, \omega, \sigma)$, for $\sigma \in \Sigma^-_\omega$ and $j = 1, \ldots, k$. Moreover, $u^{\#}_{1,j} - u^{\#}_{2,j} = S_{1,\lambda} - S_{2,\lambda}$. Therefore, if $u^{\#}_{1,j}(t, \widetilde{\omega}_j, \sigma) = u^{\#}_{2,j}(t, \widetilde{\omega}_j, \sigma)$ on $\Sigma^+_{\tilde{\omega}_j}$, for $j = 1, \ldots, k$, it follows that

$$C_0 \|c_1 - c_2\|_\infty \leq C(\lambda) \|c_1 - c_2\|_\infty$$

and the conclusion follows easily if we choose $\lambda > 0$ large enough. $\square$

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