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On the generalized nonlinear ultra-hyperbolic heat equation related to the spectrum

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Abstract. In this paper, we study the nonlinear equation of the form

$$\frac{\partial}{\partial t} u(x,t) - c^2 \square^k u(x,t) = f(x,t,u(x,t))$$

where $\square^k$ is *the ultra-hyperbolic operator* iterated k-times, defined by

$$\square^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k,$$

$p + q = n$ is the dimension of the Euclidean space $\mathbb{R}^n$, $(x,t) = (x_1, x_2, \ldots, x_n, t) \in \mathbb{R}^n \times (0,\infty)$, k is a positive integer and c is a positive constant.

On the suitable conditions for f, u and for the spectrum of the heat kernel, we can find the unique solution in the compact subset of $\mathbb{R}^n \times (0,\infty)$. Moreover, if we put $k = 1$ and $q = 0$ we obtain the solution of nonlinear equation related to the heat equation.

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1 Introduction

It is well known that for the heat equation

$$\frac{\partial}{\partial t} u(x,t) = c^2 \triangle u(x,t) \tag{1.1}$$

with the initial condition

$$u(x,0) = f(x)$$

where $\triangle = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator and $(x,t) = (x_1, x_2, \ldots, x_n, t) \in \mathbb{R}^n \times (0,\infty)$, and f is a continuous function, we obtain the solution

$$u(x,t) = \frac{1}{(4c^2\pi t)^{n/2}} \int_{\mathbb{R}^n} \exp\left[-\frac{|x-y|^2}{4c^2 t}\right] f(y)dy \tag{1.2}$$

as the solution of (1.1).

Now, (1.2) can be written as $u(x,t) = E(x,t) * f(x)$ where

$$E(x,t) = \frac{1}{(4c^2\pi t)^{n/2}} \exp\left[-\frac{|x|^2}{4c^2 t}\right]. \tag{1.3}$$

$E(x,t)$ is called *the heat kernel*, where $|x|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$ and $t > 0$, see [1, p. 208–209].

Moreover, we obtain $E(x,t) \to \delta$ as $t \to 0$, where δ is the Dirac-delta distribution. We also have extended (1.1) to be the equation

$$\frac{\partial}{\partial t} u(x,t) = c^2 \Box u(x,t) \tag{1.4}$$

where $\Box$ is *the ultra-hyperbolic operator*, defined by

$$\Box = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \right).$$

We obtain *the ultra-hyperbolic heat kernel*

$$E(x,t) = \frac{(i)^q}{(4c^2\pi t)^{n/2}} \exp\left[\frac{\sum_{i=1}^{p} x_i^2 - \sum_{j=p+1}^{p+q} x_j^2}{4c^2 t}\right]$$

where $p + q = n$ is the dimension of the Euclidean space $\mathbb{R}^n$ and $i = \sqrt{-1}$. For finding the kernel $E(x,t)$ see [4].

In this paper, we extend (1.4) to be the general of the nonlinear form

$$\frac{\partial}{\partial t} u(x,t) - c^2 \square^k u(x,t) = f(x,t,u(x,t)) \tag{1.5}$$

for $(x,t) \in \mathbb{R}^n \times (0,\infty)$ and with the following conditions on u and f as follows,

(1) $u(x,t) \in C^{(2k)}(\mathbb{R}^n)$ for any $t > 0$ where $C^{(2k)}(\mathbb{R}^n)$ is the space of continuous function with $2k$-derivatives.

(2) f satisfies the Lipchitz condition, that is

$$|f(x,t,u) - f(x,t,w)| \leq A|u - w|$$

where A is constant and $0 < A < 1$.

(3)

$$\int_0^\infty \int_{\mathbb{R}^n} |f(x,t,u(x,t))|\,dx\,dt < \infty$$

for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $t \in (0,\infty)$ and $u(x,t)$ is continuous function on $\mathbb{R}^n \times (0,\infty)$.

Under such conditions of f, u and for the spectrum of $E(x,t)$, we obtain the convolution

$$u(x,t) = E(x,t) * f(x,t,u(x,t))$$

as a unique solution in the compact subset of $\mathbb{R}^n \times (0,\infty)$ and $E(x,t)$ is an elementary solution defined by (2.5).

2 Preliminaries

Definition 2.1. *Let $f(x) \in \mathbb{L}_1(\mathbb{R}^n)$-the space of integrable function in $\mathbb{R}^n$. The Fourier transform of $f(x)$ is defined by*

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i(\xi,x)} f(x)\,dx \tag{2.1}$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$, $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $(\xi, x) = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n$ is the usual inner product in $\mathbb{R}^n$ and $dx = dx_1\,dx_2 \ldots dx_n$.

Also, the inverse of Fourier transform is defined by

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi,x)} \widehat{f}(\xi)\,d\xi. \tag{2.2}$$

Definition 2.2. *The spectrum of the kernel $E(x,t)$ defined by* (2.5) *is the bounded support of the Fourier transform $\widehat{E(\xi,t)}$ for any fixed $t > 0$.*

Definition 2.3. *Let $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ be a point in $\mathbb{R}^n$ and we write*

$$u = \xi_1^2 + \xi_2^2 + \ldots + \xi_p^2 - \xi_{p+1}^2 - \xi_{p+2}^2 - \ldots - \xi_{p+q}^2, \quad p + q = n.$$

Denote by

$$\Gamma_+ = \left\{\xi \in \mathbb{R}^n : \xi_1 > 0 \text{ and } u > 0\right\}$$

the set of an interior of the forward cone, and $\overline{\Gamma}_+$ denotes the closure of Γ_+.

Let Ω be spectrum of $E(x,t)$ defined by Definition 2.2 *for any fixed $t > 0$ and $\Omega \subset \overline{\Gamma}_+$. Let $\widehat{E(\xi,t)}$ be the Fourier transform of $E(x,t)$ and define*

$$\widehat{E(\xi,t)} = \begin{cases} \frac{1}{(2\pi)^{n/2}} \exp\left[c^2 t \left(\sum_{j=p+1}^{p+q} \xi_j^2 - \sum_{i=1}^{p} \xi_i^2\right)^k\right] & \text{for } \xi \in \Gamma_+, \\ 0 & \text{for } \xi \notin \Gamma_+. \end{cases} \tag{2.3}$$

Lemma 2.1. *Let* L *be the operator defined by*

$$\mathrm{L} = \frac{\partial}{\partial t} - c^2 \Box^k \tag{2.4}$$

where $\Box^k$ is the ultra-hyperbolic operator iterated k-times defined by

$$\Box^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}\right)^k,$$

$p + q = n$ is the dimension of $\mathbb{R}^n$, $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $t \in (0, \infty)$, k is a positive integer and c is a positive constant. Then we obtain

$$E(x,t) = \frac{1}{(2\pi)^n} \int_\Omega \exp\left[c^2 t \left(\sum_{j=p+1}^{p+q} \xi_j^2 - \sum_{i=1}^{p} \xi_i^2\right)^k + i(\xi, x)\right] d\xi \tag{2.5}$$

as a elementary solution of (2.4) *in the spectrum $\Omega \subset \mathbb{R}^n$ for $t > 0$.*

Proof. Let L$E(x,t) = \delta(x,t)$ where $E(x,t)$ is the kernel or the elementary solution of operator L and δ is the Dirac-delta distribution. Thus

$$\frac{\partial}{\partial t}E(x,t) - c^2\Box^k E(x,t) = \delta(x)\delta(t).$$

Take the Fourier transform defined by (2.1) to both sides of the equation, we obtain

$$\frac{\partial}{\partial t}\widehat{E(\xi,t)} - c^2\left(\sum_{j=p+1}^{p+q}\xi_j^2 - \sum_{i=1}^{p}\xi_i^2\right)^k \widehat{E(\xi,t)} = \frac{1}{(2\pi)^{n/2}}\delta(t).$$

Thus

$$\widehat{E(\xi,t)} = \frac{H(t)}{(2\pi)^{n/2}}\exp\left[c^2 t\left(\sum_{j=p+1}^{p+q}\xi_j^2 - \sum_{i=1}^{p}\xi_i^2\right)^k\right]$$

where $H(t)$ is the Heaviside function. Since $H(t) = 1$ for $t > 0$. Therefore,

$$\widehat{E(\xi,t)} = \frac{1}{(2\pi)^{n/2}}\exp\left[c^2 t\left(\sum_{j=p+1}^{p+q}\xi_j^2 - \sum_{i=1}^{p}\xi_i^2\right)^k\right]$$

which has been already defined by (2.3). Thus

$$E(x,t) = \frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{i(\xi,x)}\widehat{E(\xi,t)}\,d\xi = \frac{1}{(2\pi)^{n/2}}\int_{\Omega} e^{i(\xi,x)}\widehat{E(\xi,t)}\,d\xi$$

where Ω is the spectrum of $E(x,t)$. Thus from (2.3)

$$E(x,t) = \frac{1}{(2\pi)^n}\int_{\Omega}\exp\left[c^2 t\left(\sum_{j=p+1}^{p+q}\xi_j^2 - \sum_{i=1}^{p}\xi_i^2\right)^k + i(\xi,x)\right]d\xi \quad \text{for } t > 0.$$

$\square$

Definition 2.4. *Let us extend $E(x,t)$ to $\mathbb{R}^n \times \mathbb{R}$ by setting*

$$E(x,t) = \begin{cases} \frac{1}{(2\pi)^n}\int_{\Omega}\exp\left[c^2 t\left(\sum_{j=p+1}^{p+q}\xi_j^2 - \sum_{i=1}^{p}\xi_i^2\right)^k + i(\xi,x)\right]d\xi & \text{for } t > 0, \\ 0 & \text{for } t \le 0, \end{cases}$$

3 Main Results

Theorem 3.1. *The kernel $E(x,t)$ defined by* (2.5) *have the following properties:*

(1) $E(x,t) \in C^\infty$*-the space infinitely differentiable.*

(2) $\left(\frac{\partial}{\partial t} - c^2\Box^k\right) E(x,t) = 0$ *for* $t > 0$.

(3)
$$|E(x,t)| \leq \frac{2^{2-n}}{\pi^{n/2}} \frac{M(t)}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)}, \quad \textit{for } t > 0,$$

where $M(t)$ is a function of t in the spectrum Ω and Γ denote the Gamma function. Thus $E(x,t)$ is bounded for any fixed $t > 0$.

(4) $\lim_{t\to 0} E(x,t) = \delta$.

Proof.

(1) From (2.5), since

$$\frac{\partial^n}{\partial x^n} E(x,t) = \frac{1}{(2\pi)^n} \int_\Omega \frac{\partial^n}{\partial x^n} \exp\left[c^2 t \left(\sum_{j=p+1}^{p+q} \xi_j^2 - \sum_{i=1}^{p} \xi_i^2\right)^k + i(\xi, x)\right] d\xi.$$

Thus $E(x,t) \in C^\infty$ for $x \in \mathbb{R}^n$, $t > 0$.

(2) By computing directly, we obtain

$$\left(\frac{\partial}{\partial t} - c^2\Box^k\right) E(x,t) = 0.$$

(3) We have

$$E(x,t) = \frac{1}{(2\pi)^n} \int_\Omega \exp\left[c^2 t \left(\sum_{j=p+1}^{p+q} \xi_j^2 - \sum_{i=1}^{p} \xi_i^2\right)^k + i(\xi, x)\right] d\xi.$$

$$|E(x,t)| \leq \frac{1}{(2\pi)^n} \int_\Omega \exp\left[c^2 t \left(\sum_{j=p+1}^{p+q} \xi_j^2 - \sum_{i=1}^{p} \xi_i^2\right)^k\right] d\xi.$$

By changing to bipolar coordinates

$$\xi_1 = r\omega_1, \xi_2 = r\omega_2, \ldots, \xi_p = r\omega_p \quad \text{and}$$
$$\xi_{p+1} = s\omega_{p+1}, \xi_{p+2} = s\omega_{p+2}, \ldots, \xi_{p+q} = s\omega_{p+q}$$

where $\sum_{i=1}^{p} \omega_i^2 = 1$ and $\sum_{j=p+1}^{p+q} \omega_j^2 = 1$. Thus

$$|E(x,t)| \leq \frac{1}{(2\pi)^n} \int_\Omega \exp\left[c^2 t \left(s^2 - r^2\right)^k\right] r^{p-1} s^{q-1}\, dr\, ds\, d\Omega_p\, d\Omega_q$$

where $d\xi = r^{p-1}s^{q-1}\, dr\, ds\, d\Omega_p\, d\Omega_q$, $d\Omega_p$ and Ω_q are the elements of surface area of the unit sphere in $\mathbb{R}^p$ and $\mathbb{R}^q$ respectively. Since $\Omega \subset \mathbb{R}^n$ is the spectrum of $E(x,t)$ and we suppose $0 \leq r \leq R$ and $0 \leq s \leq L$ where R and L are constants. Thus we obtain

$$\begin{aligned} |E(x,t)| &\leq \frac{\Omega_p\, \Omega_q}{(2\pi)^n} \int_0^R \int_0^L \exp\left[c^2 t \left(s^2 - r^2\right)^k\right] r^{p-1} s^{q-1}\, ds\, dr \\ &= \frac{\Omega_p\, \Omega_q}{(2\pi)^n} M(t) \quad \text{for any fixed } t > 0 \text{ in the spectrum } \Omega \\ &= \frac{2^{2-n}}{\pi^{n/2}} \frac{M(t)}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})} \end{aligned} \tag{3.1}$$

where

$$M(t) = \int_0^R \int_0^L \exp\left[c^2 t \left(s^2 - r^2\right)^k\right] r^{p-1} s^{q-1}\, ds\, dr \tag{3.2}$$

is a function of

$$t > 0, \quad \Omega_p = \frac{2\pi^{p/2}}{\Gamma\left(\frac{p}{2}\right)} \quad \text{and} \quad \Omega_q = \frac{2\pi^{p/2}}{\Gamma\left(\frac{q}{2}\right)}.$$

Thus, for any fixed $t > 0$, $E(x,t)$ is bounded.

(4) By (2.5), we have

$$E(x,t) = \frac{1}{(2\pi)^n} \int_\Omega \exp\left[c^2 t \left(\sum_{j=p+1}^{p+q} \xi_j^2 - \sum_{i=1}^{p} \xi_i^2\right)^k + i(\xi, x)\right] d\xi.$$

Since $E(x,t)$ exists, then

$$\begin{aligned}\lim_{t\to 0} E(x,t) &= \frac{1}{(2\pi)^n}\int_{\Omega} e^{i(\xi,x)}\,d\xi\\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i(\xi,x)}\,d\xi\\ &= \delta(x), \qquad \text{for} \quad x\in\mathbb{R}^n.\end{aligned}$$

See [3, p. 396, Eq. (10.2.19b)]. □

Theorem 3.2. *Given the nonlinear equation*

$$\frac{\partial}{\partial t}u(x,t) - c^2\Box^k u(x,t) = f(x,t,u(x,t)) \tag{3.3}$$

for $(x,t)\in\mathbb{R}^n\times(0,\infty)$*, k is positive number and with the following conditions on u and f as follows,*

(1) $u(x,t)\in C^{(2k)}(\mathbb{R}^n)$ *for any* $t>0$ *where* $C^{(2k)}(\mathbb{R}^n)$ *is the space of continuous function with* $2k$*-derivatives.*

(2) f *satisfies the Lipchitz condition, that is*

$$|f(x,t,u)-f(x,t,w)|\le A|u-w|$$

where A is constant and $0<A<1$.

(3)

$$\int_0^\infty\int_{\mathbb{R}^n}|f(x,t,u(x,t))|\,dx\,dt<\infty$$

for $x=(x_1,x_2,\ldots,x_n)\in\mathbb{R}^n$*,* $t\in(0,\infty)$ *and* $u(x,t)$ *is continuous function on* $\mathbb{R}^n\times(0,\infty)$.

Then, for the spectrum of $E(x,t)$ *we obtain the convolution*

$$u(x,t) = E(x,t) * f(x,t,u(x,t)) \tag{3.4}$$

as a unique solution of (3.3) *for* $x\in\Omega_0$ *where* Ω_0 *is an compact subset of* $\mathbb{R}^n$*,* $0\le t\le T$ *with T is constant and* $E(x,t)$ *is an elementary solution defined by* (2.5) *and also* $u(x,t)$ *is bounded.*

In particular, if we put $k=1$ *and* $q=0$ *in* (3.3) *then* (3.3) *reduces to the nonlinear heat equation.*

Proof. Convolving both sides of (3.3) with $E(x, t)$ and then we obtain the solution

$$u(x, t) = E(x, t) * f(x, t, u(x, t))$$

or

$$u(x, t) = \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} E(r, s) f(x - r, t - s, u(x - r, t - s))\, dr\, ds$$

where $E(r, s)$ is given by Definition 2.4.

We next show that $u(x, t)$ is bounded on $\mathbb{R}^n \times (0, \infty)$. We have

$$\begin{aligned} |u(x, t)| &\le \int_{-\infty}^{\infty} \int_{\mathbb{R}^n} |E(r, s)|\, |f(x - r, t - s, u(x - r, t - s))|\, dr\, ds \\ &\le \frac{2^{2-n}}{\pi^{n/2}} \frac{N.M(t)}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})} \end{aligned}$$

by the condition (3) and (3.1) where

$$N = \int_{0}^{\infty} \int_{\mathbb{R}^n} |f(x, t, u(x, t))|\, dx\, dt.$$

Thus $u(x, t)$ is bounded on $\mathbb{R}^n \times (0, \infty)$.

To show that $u(x, t)$ is unique, suppose there is another solution $w(x, t)$ of equation (3.3). Let the operator

$$\mathrm{L} = \frac{\partial}{\partial t} - c^2 \Box^k$$

then (3.3) can be written in the form

$$\mathrm{L}\, u(x, t) = f(x, t, u(x, t)).$$

Thus

$$\mathrm{L}\, u(x, t) - \mathrm{L}\, w(x, t) = f(x, t, u(x, t)) - f(x, t, w(x, t)).$$

By the condition (2) of the Theorem,

$$|\mathrm{L}\, u(x, t) - \mathrm{L}\, w(x, t)| \le A |u(x, t) - w(x, t)|. \tag{3.5}$$

Let $\Omega_0 \times (0, T]$ be compact subset of $\mathbb{R}^n \times (0, \infty)$ and $\mathrm{L}\colon C^{(2k)}(\Omega_0) \longrightarrow C^{(2k)}(\Omega_0)$ for $0 \le t \le T$.

Now $\left(C^{(2k)}(\Omega_0), \|\cdot\|\right)$ is a Banach space where $u(x,t) \in C^{(2k)}(\Omega_0)$ for $0 \leq t \leq T$, $\|\cdot\|$ given by

$$\|u(x,t)\| = \sup_{x\in\Omega_0} |u(x,t)|.$$

Then, from (3.5) with $0 < A < 1$, the operator L is a contraction mapping on $C^{(2k)}(\Omega_0)$. Since $\left(C^{(2k)}(\Omega_0), \|\cdot\|\right)$ is a Banach space and $\mathrm{L}\colon C^{(2k)}(\Omega_0) \longrightarrow C^{(2k)}(\Omega_0)$ is a contraction mapping on $C^{(2k)}(\Omega_0)$, by Contraction Theorem, see [3, p. 300], we obtain the operator L has a fixed point and has uniqueness property. Thus $u(x,t) = w(x,t)$. It follows that the solution $u(x,t)$ of (3.3) is unique for $(x,t) \in \Omega_0 \times (0, T]$ where $u(x,t)$ is defined by (3.4).

In particular, if we put $k = 1$ and $q = 0$ in (3.3) then (3.3) reduces to the nonlinear heat equation

$$\frac{\partial}{\partial t} u(x,t) - c^2 \triangle u(x,t) = f(x,t,u(x,t))$$

which has solution

$$u(x,t) = E(x,t) * f(x,t,u(x,t))$$

where $E(x,t)$ is defined by (2.5) with $k = 1$ and $q = 0$. That is complete of proof. $\square$

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