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Preconditioners for higher order finite element discretizations of $\boldsymbol{H}(\mathrm{div})$-elliptic problem

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Abstract. In this paper, we are concerned with the fast solvers for higher order finite element discretizations of $\boldsymbol{H}(\mathrm{div})$-elliptic problem. We present the preconditioners for the first family and second family of higher order divergence conforming element equations, respectively. By combining the stable decompositions of two kinds of finite element spaces with the abstract theory of auxiliary space preconditioning, we prove that the corresponding condition numbers of our preconditioners are uniformly bounded on quasi-uniform grids.

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Key words: preconditioner, higher order finite element, stable decomposition, $\boldsymbol{H}(\mathrm{div})$-elliptic problem.

1 Introduction

Let Ω be a simply connected polyhedron in $\mathbb{R}^3$ with boundary Γ and unit outward normal $\boldsymbol{\nu}$. We define the Hilbert spaces $\boldsymbol{H}_0(\mathrm{div};\Omega)$ as follows

$$\boldsymbol{H}_0(\mathrm{div};\Omega) = \left\{\boldsymbol{u} \in (L^2(\Omega))^3 \;\middle|\; \nabla\cdot\boldsymbol{u} \in L^2(\Omega), \;\; \boldsymbol{\nu}\cdot\boldsymbol{u} = 0 \text{ on } \Gamma\right\}$$

with the inner product

$$(\boldsymbol{u},\boldsymbol{v})_{\mathrm{div}} = (\boldsymbol{u},\boldsymbol{v}) + (\nabla\cdot\boldsymbol{u}, \nabla\cdot\boldsymbol{v}),$$

where $(\cdot,\cdot)$ denotes the inner product in $(L^2(\Omega))^3$ or $L^2(\Omega)$.

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In this paper, we consider the following variational problem: Find $\boldsymbol{u} \in \boldsymbol{H}_0(\mathrm{div};\Omega)$ such that

$$a(\boldsymbol{u},\boldsymbol{v}) = (\boldsymbol{f},\boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div};\Omega), \tag{1}$$

where $\boldsymbol{f} \in \boldsymbol{H}_0(\mathrm{div};\Omega)'$ is a given data and

$$a(\boldsymbol{u},\boldsymbol{v}) = (\nabla\cdot\boldsymbol{u},\nabla\cdot\boldsymbol{v}) + \tau(\boldsymbol{u},\boldsymbol{v}), \tag{2}$$

with the constant $\tau > 0$.

The bilinear form $a(\cdot,\cdot)$ induces the energy norm

$$\|\boldsymbol{v}\|_A^2 = a(\boldsymbol{v},\boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div};\Omega). \tag{3}$$

Variational problem of the form (1) arises in numerous problems of practical import. Typical examples include the mixed method for second order elliptic problems, the least squares method of the form discussed in [3], and the sequential regularization method for the time dependent Navier-Stokes equation discussed in [6]. For a more detailed discussion of applications, we refer to [1].

To avoid the repeated use of generic but unspecified constants, following [9], we will use the following short notation: $x \lesssim y$ means $x \leq Cy$, $x \gtrsim y$ means $x \geq cy$, and $x \approx y$ means $cx \leq y \leq Cy$, where c and C are generic positive constants independent of the variables that appear in the inequalities and especially the mesh parameters.

Outline. The remainder of this article is organized as follows. In the next section, we introduce two kinds of higher order finite element equations, and present the corresponding frame of constructing preconditioner. We construct the preconditioners for two kinds of higher order divergence conforming element equations, and prove that their corresponding condition number is uniformly bounded in Section 3 and Section 4, respectively.

2 Finite element equations and framework of preconditioner

Let $\mathcal{T}_h$ be a shape regular tetrahedron meshes of Ω, where h is the maximum diameter of the tetrahedra in $\mathcal{T}_h$. Now, we present two families of divergence conforming finite elements spaces (see [7])

$$\boldsymbol{W}_h^{k,1} = \left\{ \boldsymbol{v}_h^{k,1} \in \boldsymbol{H}_0(\text{div};\Omega) \,\middle|\, \boldsymbol{v}_h^{k,1}|_K \in (\mathcal{P}_{k-1})^3 \oplus \tilde{\mathcal{P}}_{k-1}\boldsymbol{x}, \forall K \in \mathcal{T}_h \right\},$$

$$\boldsymbol{W}_h^{k,2} = \left\{ \boldsymbol{v}_h^{k,2} \in \boldsymbol{H}_0(\text{div};\Omega) \,\middle|\, \boldsymbol{v}_h^{k,2}|_K \in (\mathcal{P}_k)^3, \forall K \in \mathcal{T}_h \right\},$$

where $\mathcal{P}_k$ denote the standard space of polynomials of total degree less than or equal to k, and $\tilde{\mathcal{P}}_k$ denote the space of homogeneous polynomials of order k.

We consider the solution of systems of linear algebraic equations which arise from the finite element discretization of variational problems (1): Find $\boldsymbol{u}_h^{k,l} \in \boldsymbol{W}_h^{k,l} (k \geq 1, l = 1, 2)$ such that

$$a(\boldsymbol{u}_h^{k,l}, \boldsymbol{v}_h^{k,l}) = (\boldsymbol{f}, \boldsymbol{v}_h^{k,l}) \quad \forall \boldsymbol{v}_h^{k,l} \in \boldsymbol{W}_h^{k,l}. \tag{4}$$

Their algebraic systems can be described as

$$A_h^{k,l} U_h^{k,l} = F_h^{k,l}. \tag{5}$$

Since $A_h^{k,l}$ is symmetric positive definite, we use precondition conjugate gradient (PCG) methods to solve algebraic systems (5). In this paper, we will construct the preconditioners for the cases of higher order finite equations, and present some estimates of the corresponding condition numbers.

For this purpose, we need to introduce some auxiliary spaces and corresponding operators.

Let $V = \boldsymbol{W}_h^{k,l}$ with inner product $a(\cdot,\cdot)$ given by (2).

Let $\bar{V}_1, \cdots, \bar{V}_J$, $J \in \mathbb{N}$, be Hilbert spaces endowed with inner products $\bar{a}_j(\cdot,\cdot)$, $j = 1, \cdots, J$. The operators $\bar{A}_j : \bar{V}_j \mapsto \bar{V}_j'$ are isomorphisms induced by $\bar{a}_j(\cdot,\cdot)$, namely

$$\bar{a}_j(\bar{u}_j, \bar{v}_j) = < \bar{A}_j \bar{u}_j, \bar{v}_j > \quad \forall \bar{u}_j, \bar{v}_j \in \bar{V}_j,$$

here we tag dual spaces by $'$ and use angle brackets for duality pairings. For each $\bar{V}_j$, there exist continuous transfer operators $\Pi_j : \bar{V}_j \mapsto V$. Then we can construct the preconditioner for operator $A_h^{k,l}$ as follows:

$$B = \sum_{j=1}^{J} \Pi_j \bar{B}_j \Pi_j^*, \tag{6}$$

where $\bar{B}_j : \bar{V}_j' \mapsto \bar{V}_j$ are given preconditioners for $\bar{A}_j$, and Π_j^* are adjoint operators of Π_j.

Now, we present the following theorem of an estimate for the spectral condition number of the preconditioner given by (6).

Theorem 2.1. *Assume that there exist constants c_j, such that*

$$\|\Pi_j \bar{u}_j\|_A \leq c_j \|\bar{u}_j\|_{\bar{A}_j}, \qquad \forall\, \bar{u}_j \in \bar{V}_j,\ 1 \leq j \leq J, \tag{7}$$

and for $\forall\, u \in V$, there exist $\bar{u}_j \in \bar{V}_j$ such that $u = \sum_{j=1}^{J} \Pi_j \bar{u}_j$ and

$$\left(\sum_{j=1}^{J} \|\bar{u}_j\|_{\bar{A}_j}^2 \right)^{1/2} \leq c_0 \|u\|_A, \tag{8}$$

then for the preconditioner B given by (6), we have the following estimate for the spectral condition number

$$\kappa(B A_h^{k,l}) \leq \max_{1 \leq j \leq J} \kappa(\bar{B}_j \bar{A}_j) c_0^2 \sum_{j=1}^{J} c_j^2. \tag{9}$$

Proof. We define the space

$$\bar{V} = \bar{V}_1 \times \bar{V}_2 \times \cdots \times \bar{V}_J$$

with the inner product

$$(\bar{u}, \bar{u})_{\bar{A}} = \sum_{j=1}^{J} (\bar{u}_j, \bar{u}_j)_{\bar{A}_j}, \bar{u} = (\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_J)^t, \bar{u}_i \in \bar{V}_j,$$

and the following two operators

$$\begin{aligned} \Pi = (\Pi_1, \Pi_2, \cdots, \Pi_J) : \bar{V} \mapsto V, \\ \bar{A} = \operatorname{diag}(\bar{A}_1, \bar{A}_2, \cdots, \bar{A}_J) : \bar{V} \mapsto \bar{V}, \\ \bar{B} = \operatorname{diag}(\bar{B}_1, \bar{B}_2, \cdots, \bar{B}_J) : \bar{V} \mapsto \bar{V}. \end{aligned}$$

Thus we can rewrite the definition of operator B given by (6):

$$B = \Pi \bar{B} \Pi^*.$$

Using the definitions of inner product in $\bar{V}$, operators Π and $\bar{B}$, and conditions (7)-(8), then there exists a constant $\bar{c}_1^2 := \sum_{j=1}^{J} c_j^2$, such that

$$\|\Pi\bar{u}\|_A \leq \bar{c}_1\|\bar{u}\|_{\bar{A}}, \qquad \forall\, \bar{u} \in \bar{V},$$

and for $\forall\, u \in V$, there exists $\bar{u} \in \bar{V}$, such that $u = \Pi\bar{u}$ and

$$\|\bar{u}\|_{\bar{A}} \leq c_0\|u\|_A.$$

From Corollary 2.3 of [5], we immediately get an estimate for the spectral condition number of the preconditioned operator B

$$\kappa(BA_h^{k,l}) \leq \kappa(\bar{B}\bar{A})c_0^2\sum_{j=1}^{J} c_j^2.$$

The desired estimates then follow by combining the above inequality and the following fact

$$\kappa(\bar{B}\bar{A}) \leq \max_{1\leq j\leq J}\kappa(\bar{B}_j\bar{A}_j). \qquad \square$$

The principal challenge confronted in the development of preconditioners by applying Theorem 2.1 is to construct some appropriate spaces and operators which satisfy (7) and (8). In the following two sections, we present the corresponding spaces and operators for two kinds of divergence conforming element spaces, respectively.

3 Preconditioner for finite element equations of first kind

We first introduce Sobolev functional space

$$\boldsymbol{H}_0(\mathbf{curl};\Omega) = \left\{\boldsymbol{u} \in (L^2(\Omega))^3 \,\middle|\, \nabla\times\boldsymbol{u} \in (L^2(\Omega))^3,\ \boldsymbol{\nu}\times\boldsymbol{u} = \mathbf{0} \text{ on } \Gamma\right\}$$

with the norm

$$\|\boldsymbol{u}\|_{\boldsymbol{H}(\mathbf{curl};\Omega)} = \left(\|\boldsymbol{u}\|_0^2 + \|\nabla\times\boldsymbol{u}\|_0^2\right)^{1/2}.$$

There exist two families of edge finite element spaces for the space $\boldsymbol{H}_0(\mathbf{curl};\Omega)$ (see [2, 4, 7]).

1. k order Nédélec element of first kind:

$$\boldsymbol{V}_h^{k,1} = \left\{ \boldsymbol{u}_h^{k,1} \in \boldsymbol{H}_0(\mathbf{curl};\Omega) \middle| \boldsymbol{u}_h^{k,1}|_K \in \mathcal{R}_k, \forall K \in \mathcal{T}_h \right\}, \tag{10}$$

where $\mathcal{R}_k = (\mathcal{P}_{k-1})^3 \oplus \{\boldsymbol{p} \in (\tilde{P}_k)^3 \mid \boldsymbol{p}(\boldsymbol{x}) \cdot \boldsymbol{x} = \boldsymbol{0}\}$.

2. k order Nédélec element of second kind:

$$\boldsymbol{V}_h^{k,2} = \left\{ \boldsymbol{u}_h^{k,2} \in \boldsymbol{H}_0(\mathbf{curl};\Omega) \middle| \boldsymbol{u}_h^{k,2}|_K \in (\mathcal{P}_k)^3, \forall K \in \mathcal{T}_h \right\}. \tag{11}$$

We also need to introduce the following space of piecewise $k-$degree discontinuous scalar elements on $\mathcal{T}_h$:

$$X_h^k = \left\{ q_h^k \in L^2(\Omega) \;\middle|\; q_h^k|_K \in \mathcal{P}_k \text{ for all } K \in \mathcal{T}_h \right\}.$$

The Sobolev spaces $\boldsymbol{H}_0(\text{div};\Omega)$, $\boldsymbol{H}_0(\mathbf{curl};\Omega)$ and the corresponding finite element spaces possess the exceptional exact sequence properties (see [4, 7])

$$\begin{aligned}
\boldsymbol{H}_0(\text{div}\mathbf{0};\Omega) &:= \{\boldsymbol{w} \in \boldsymbol{H}_0(\text{div};\Omega) : \nabla\cdot\boldsymbol{w} = \mathbf{0}\} \\
&= \nabla\times\boldsymbol{H}_0(\mathbf{curl};\Omega), \qquad (12)\\
\boldsymbol{W}_h^{k-1,l}(\text{div}\mathbf{0}) &:= \{\boldsymbol{w}_h^{k-1,l} \in \boldsymbol{W}_h^{k-1,l} : \nabla\cdot\boldsymbol{w}_h^{k-1,l} = \mathbf{0}\} \\
&= \nabla\times\boldsymbol{V}_h^{k,l},\ \ l = 1,2, \qquad (13)\\
\nabla\cdot\boldsymbol{W}_h^{k,l} &\subset X_h^{k-1},\ \ l = 1,2. \qquad (14)
\end{aligned}$$

Assuming that $\boldsymbol{u}$ has the necessary smoothness, we can define two kinds of interpolants: $\Pi_{h,\text{div}}^{k,1}$ and Π_h^k, such that $\Pi_{h,\text{div}}^{k,1}\boldsymbol{u} \in \boldsymbol{W}_h^{k,1}$ and $\Pi_h^k\boldsymbol{u} \in X_h^k$ (more details refer to [4, 7]). Especially, the interpolation $\Pi_{h,\text{div}}^{k,1}$ is not defined for a general function in $\boldsymbol{H}_0(\text{div};\Omega)$. Here let us quote a slightly simplified version (see Theorem 5.25 of [7]).

Lemma 3.1. *Suppose that there are constants $\delta > 0$ such that $\boldsymbol{u} \in (H^{1/2+\delta}(K))^3$ for each K in $\mathcal{T}_h$. Then $\Pi_{h,\text{div}}^{k,1}\boldsymbol{u}$ is well-defined, and we have*

$$\|(Id - \Pi_{h,\text{div}}^{k,1})\boldsymbol{u}\|_{0,K} \lesssim h_K^{1/2+\delta}\|\boldsymbol{u}\|_{(H^{1/2+\delta}(K))^3} \tag{15}$$

with a constant only depending on the shape regularity of $\mathcal{T}_h$.

The finite element spaces $\boldsymbol{W}_h^{k,1}$ is equipped with bases $\mathcal{B}(k,1)$ comprising locally supported functions. These bases are L^2 stable in the sense that

$$\boldsymbol{v}_h^{k,1} = \sum_{\mathbf{b}\in\mathcal{B}(k,1)} \boldsymbol{v}_{\mathbf{b}},\ \boldsymbol{v}_{\mathbf{b}} \in \operatorname{span}\{\mathbf{b}\},\ \sum_{\mathbf{b}\in\mathcal{B}(k,1)} \|\boldsymbol{v}_{\mathbf{b}}\|_0^2 \approx \|\boldsymbol{v}_h^{k,1}\|_0^2\ \forall \boldsymbol{v}_h^{k,1} \in \boldsymbol{W}_h^{k,1}, \quad (16)$$

with constant only depending on the shape-regularity of $\mathcal{T}_h$.

Lemma 3.2. *The interpolation operator $\Pi_{h,\mathrm{div}}^{k,1}$ is bounded on $(H_0^1(\Omega))^3$ and satisfies*

$$\|(Id - \Pi_{h,\mathrm{div}}^{k,1})\boldsymbol{\psi}\|_0 \lesssim h\|\boldsymbol{\psi}\|_{(H^1(\Omega))^3} \quad \forall\, \boldsymbol{\psi} \in (H_0^1(\Omega))^3 \quad (17)$$

with a constant only depending on the shape regularity of $\mathcal{T}_h$.

Furthermore, all above operators possess the following commuting diagram property (see [7])

$$\operatorname{div} \Pi_{h,\mathrm{div}}^{k,1} \;=\; \Pi_h^{k-1} \operatorname{div}. \quad (18)$$

We may apply the quasi-interpolation operators for Lagrangian finite element space introduced in [8] to the components of vector fields separately. This gives rise to the projectors $Q_h : (H_0^1(\Omega))^3 \mapsto (S_h^1)^3$, which inherits the continuity

$$\|Q_h\boldsymbol{\Psi}\|_{(H^1(\Omega))^3} \lesssim \|\boldsymbol{\Psi}\|_{(H^1(\Omega))^3} \quad \forall\, \boldsymbol{\Psi} \in (H_0^1(\Omega))^3 \quad (19)$$

and satisfies the local projection error esitmate

$$\|h^{-1}(Id - Q_h)\boldsymbol{\Psi}\|_0 \lesssim \|\boldsymbol{\Psi}\|_{(H^1(\Omega))^3} \quad \forall\, \boldsymbol{\Psi} \in (H_0^1(\Omega))^3. \quad (20)$$

Now, we present the stable decomposition of $\boldsymbol{W}_h^{k,1}$, $k \geq 2$.

Lemma 3.3. *For any $\boldsymbol{u}_h^{k,1} \in \boldsymbol{W}_h^{k,1}$, there exist $\sum_{\mathbf{b}\in\mathcal{B}(k,1)} \boldsymbol{v}_{\mathbf{b}} \in \boldsymbol{W}_h^{k,1}$, $\boldsymbol{v}_{\mathbf{b}} \in$ Span$\{\mathbf{b}\}$, $\boldsymbol{u}_h^{k-1,2} \in \boldsymbol{W}_h^{k-1,2}$, such that*

$$\boldsymbol{u}_h^{k,1} = \sum_{\mathbf{b}\in\mathcal{B}(k,1)} \boldsymbol{v}_{\mathbf{b}} + \boldsymbol{u}_h^{k-1,2}, \quad (21)$$

and

$$\left(\sum_{\mathbf{b}\in\mathcal{B}(k,1)} \|\boldsymbol{v}_{\mathbf{b}}\|_A^2 + \|\boldsymbol{u}_h^{k-1,2}\|_A^2\right)^{1/2} \leq \tilde{c}_0\|\boldsymbol{u}_h^{k,1}\|_A, \quad (22)$$

where the constant $\tilde{c}_0$ only depends on Ω and the shape regularity of $\mathcal{T}_h$.

Proof. For any given $\boldsymbol{u}_h^{k,1} \in \boldsymbol{W}_h^{k,1}$, using the continuous Helmholtz decomposition, there exist $\boldsymbol{\Psi} \in (H_0^1(\Omega))^3$, $\boldsymbol{p} \in \boldsymbol{H}_0(\mathbf{curl}; \Omega)$ such that

$$\boldsymbol{u}_h^{k,1} = \boldsymbol{\Psi} + \nabla \times \boldsymbol{p}, \tag{23}$$

and

$$\|\boldsymbol{\Psi}\|_{(H^1(\Omega))^3} \lesssim \|\nabla \cdot \boldsymbol{u}_h^{k,1}\|_0, \quad \|\nabla \times \boldsymbol{p}\|_0 \lesssim \|\boldsymbol{u}_h^{k,1}\|_{\boldsymbol{H}(\mathrm{div};\Omega)}, \tag{24}$$

with constants only depending on Ω.

Taking the div of both sides of (23) and using (14), we get

$$\nabla \cdot \boldsymbol{\Psi} = \nabla \cdot \boldsymbol{u}_h^{k,1} \in X_h^{k-1}.$$

Owing to Lemma 3.2, $\Pi_{h,\mathrm{div}}^{k,1}\boldsymbol{\Psi}$ is well defined. Furthermore, the commuting diagram property (18) implies

$$\nabla \cdot \Pi_{h,\mathrm{div}}^{k,1}\boldsymbol{\Psi} = \Pi_h^{k-1}\nabla \cdot \boldsymbol{\Psi} = \nabla \cdot \boldsymbol{\Psi} \;\;\Rightarrow\;\; \nabla \cdot (Id - \Pi_{h,\mathrm{div}}^{k,1})\boldsymbol{\Psi} = 0.$$

This confirms that the third term in the splitting

$$\boldsymbol{\Psi} = \Pi_{h,\mathrm{div}}^{k,1}(Id - Q_h)\boldsymbol{\Psi} + \Pi_{h,\mathrm{div}}^{k,1}Q_h\boldsymbol{\Psi} + (Id - \Pi_{h,\mathrm{div}}^{k,1})\boldsymbol{\Psi} \tag{25}$$

actually belongs to the kernel of div. By (12), then there esists $\boldsymbol{q} \in \boldsymbol{H}_0(\mathbf{curl}; \Omega)$ such that

$$(Id - \Pi_{h,\mathrm{div}}^{k,1})\boldsymbol{\Psi} = \nabla \times \boldsymbol{q}. \tag{26}$$

Noting that $Q_h\boldsymbol{\Psi} \in (S_h^1)^3 \subset \boldsymbol{W}_h^{k,1}$, which leads to

$$\Pi_{h,\mathrm{div}}^{k,1}Q_h\boldsymbol{\Psi} = Q_h\boldsymbol{\Psi}. \tag{27}$$

Substituting (25), (26) and (27) into (23), we have

$$\boldsymbol{u}_h^{k,1} = \Pi_{h,\mathrm{div}}^{k,1}(Id - Q_h)\boldsymbol{\Psi} + Q_h\boldsymbol{\Psi} + \nabla \times (\boldsymbol{q} + \boldsymbol{p}). \tag{28}$$

Since $\boldsymbol{u}_h^{k,1}$, $\Pi_{h,\mathrm{div}}^{k,1}(Id - Q_h)\boldsymbol{\Psi}$, $Q_h\boldsymbol{\Psi} \in \boldsymbol{W}_h^{k,1}$, we obtain $\nabla \times (\boldsymbol{q} + \boldsymbol{p}) \in \boldsymbol{W}_h^{k,1}(\mathrm{div}\mathbf{0})$ by using (28), then observing (13), there exists $\boldsymbol{q}_h \in \boldsymbol{V}_h^{k,1}$, such that

$$\nabla \times \boldsymbol{q}_h = \nabla \times (\boldsymbol{q} + \boldsymbol{p}). \tag{29}$$

Let

$$\tilde{\boldsymbol{u}}_h^{k,1} = \Pi_{h,\mathrm{div}}^{k,1}(Id - Q_h)\boldsymbol{\Psi} = \sum_{\mathbf{b}\in\mathcal{B}(k,1)} \boldsymbol{v}_{\mathbf{b}},\ \boldsymbol{v}_{\mathbf{b}} \in \mathrm{Span}\{\mathbf{b}\}, \tag{30}$$

$$\boldsymbol{u}_h^{k-1,2} = Q_h\boldsymbol{\Psi} + \nabla\times\boldsymbol{q}_h. \tag{31}$$

It's easy to obtain $\boldsymbol{u}_h^{k-1,2} \in \boldsymbol{W}_h^{k-1,2}$ by noting that $Q_h\boldsymbol{\Psi} \in (S_h^1)^3 \subset \boldsymbol{W}_h^{k-1,2}$ and $\nabla\times\boldsymbol{q}_h \in \nabla\times\boldsymbol{V}_h^{k,1} \subset \boldsymbol{W}_h^{k-1,2}$. Substituting (29), (30) and (31) into (28), we conclude

$$\boldsymbol{u}_h^{k,1} = \sum_{\mathbf{b}\in\mathcal{B}(k,1)} \boldsymbol{v}_{\mathbf{b}} + \boldsymbol{u}_h^{k-1,2}, \tag{32}$$

which completes the proof of (21).

Using (30), triangular inequality, Lemma 3.2, (20) and (24), we have

$$\begin{aligned}
\|h^{-1}\tilde{\boldsymbol{u}}_h^{k,1}\|_0 &= \|h^{-1}\Pi_{h,\mathrm{div}}^{k,1}(Id - Q_h)\boldsymbol{\Psi}\|_0 \\
&\le \|h^{-1}(Id - \Pi_{h,\mathrm{div}}^{k,1})(Id - Q_h)\boldsymbol{\Psi}\|_0 + \|h^{-1}(Id - Q_h)\boldsymbol{\Psi}\|_0 \\
&\lesssim \|(Id - Q_h)\boldsymbol{\Psi}\|_{(H^1(\Omega))^3} + \|\boldsymbol{\Psi}\|_{(H^1(\Omega))^3} \\
&\lesssim \|\boldsymbol{\Psi}\|_{(H^1(\Omega))^3} \lesssim \|\nabla\cdot\boldsymbol{u}_h^{k,1}\|_0,
\end{aligned}$$

which leads to

$$\|\tilde{\boldsymbol{u}}_h^{k,1}\|_0 \lesssim h\|\nabla\cdot\boldsymbol{u}_h^{k,1}\|_0. \tag{33}$$

It follows readily from inverse estimate and (16) that

$$\begin{aligned}
\sum_{\mathbf{b}\in\mathcal{B}(k,1)} \|\boldsymbol{v}_{\mathbf{b}}\|_A^2 &= \sum_{\mathbf{b}\in\mathcal{B}(k,1)} \left(\|\nabla\cdot\boldsymbol{v}_{\mathbf{b}}\|_0^2 + \tau\|\boldsymbol{v}_{\mathbf{b}}\|_0^2\right) \\
&\lesssim \sum_{\mathbf{b}\in\mathcal{B}(k,1)} \left(\|h^{-1}\boldsymbol{v}_{\mathbf{b}}\|_0^2 + \tau\|\boldsymbol{v}_{\mathbf{b}}\|_0^2\right) \\
&\lesssim \left(h^{-2} + \tau\right)\|\tilde{\boldsymbol{u}}_h^{k,1}\|_0^2.
\end{aligned} \tag{34}$$

Using inverse estimate again yields

$$\|\tilde{\boldsymbol{u}}_h^{k,1}\|_A^2 = \|\nabla\cdot\tilde{\boldsymbol{u}}_h^{k,1}\|_0^2 + \tau\|\tilde{\boldsymbol{u}}_h^{k,1}\|_0^2 \lesssim \left(h^{-2} + \tau\right)\|\tilde{\boldsymbol{u}}_h^{k,1}\|_0^2. \tag{35}$$

By means of (33) and inverse estimate, we get

$$\begin{aligned}\left(h^{-2}+\tau\right)\|\tilde{\boldsymbol{u}}_h^{k,1}\|_0^2 &\lesssim \left(h^{-2}+\tau\right)h^2\|\nabla\cdot\boldsymbol{u}_h^{k,1}\|_0^2\\ &\lesssim \|\nabla\cdot\boldsymbol{u}_h^{k,1}\|_0^2+\tau\|\boldsymbol{u}_h^{k,1}\|_0^2\\ &= \|\boldsymbol{u}_h^{k,1}\|_A^2. \end{aligned}\tag{36}$$

In view of (32), triangular inequality (34), (35) and (36), we have

$$\begin{aligned}\sum_{\mathbf{b}\in\mathcal{B}(k,1)}\|\boldsymbol{v}_{\mathbf{b}}\|_A^2+\|\boldsymbol{u}_h^{k-1,2}\|_A^2 &\leq \sum_{\mathbf{b}\in\mathcal{B}(k,1)}\|\boldsymbol{v}_{\mathbf{b}}\|_A^2+\left(\|\boldsymbol{u}_h^{k,1}\|_A+\|\tilde{\boldsymbol{u}}_h^{k,1}\|_A\right)^2\\ &\lesssim \left(h^{-2}+\tau\right)\|\tilde{\boldsymbol{u}}_h^{k,1}\|_0^2+\|\boldsymbol{u}_h^{k,1}\|_A^2\\ &\lesssim \|\boldsymbol{u}_h^{k,1}\|_A^2,\end{aligned}$$

which completes the proof of (22). □

We rely on the stable decomposition for $V=\boldsymbol{W}_h^{k,1}$ in Lemma 3.3 and apply the abstract theory in Section 2 to define the preconditioner for finite element equations of first kind.

Let $V=\boldsymbol{W}_h^{k,1}$ and choose two auxiliary spaces and the corresponding transfer operators as follows.

1. $\bar{V}_1=\boldsymbol{W}_h^{k,1}$, with inner product $\bar{a}_1(\cdot,\cdot)$ which is defined by

$$\bar{a}_1(\bar{u}_1,\bar{v}_1):=<\bar{A}_1\bar{u}_1,\bar{v}_1>=\sum_{\mathbf{b}\in\mathcal{B}(k,1)}a(\boldsymbol{u}_{\mathbf{b}},\boldsymbol{v}_{\mathbf{b}}),$$

where

$$\bar{u}_1=\sum_{\mathbf{b}\in\mathcal{B}(k,1)}\boldsymbol{u}_{\mathbf{b}},\ \bar{v}_1=\sum_{\mathbf{b}\in\mathcal{B}(k,1)}\boldsymbol{v}_{\mathbf{b}},\ \boldsymbol{u}_{\mathbf{b}},\boldsymbol{v}_{\mathbf{b}}\in\mathrm{span}\{\mathbf{b}\}.$$

The transfer operator is $\Pi_1=Id$.

2. $\bar{V}_2=\boldsymbol{W}_h^{k-1,2}$ with inner product $\bar{a}_2(\cdot,\cdot)=a(\cdot,\cdot)$ in the sense that

$$\bar{a}_2(\bar{u}_2,\bar{v}_2):=<\bar{A}_2\bar{u}_2,\bar{v}_2>=a(\bar{u}_2,\bar{v}_2)\quad\forall\bar{u}_2,\bar{v}_2\in\bar{V}_2,$$

which concludes that $\bar{A}_2=A_h^{k-1,2}$. The transfer operator is $\Pi_2=Id$.

Making use of (6), the auxiliary space preconditioner for $A_h^{k,1}$ reads

$$B_h^{k,1} = \bar{B}_1 + B_h^{k-1,2}, \tag{37}$$

where $B_h^{k-1,2}$ is the preconditioner of $A_h^{k-1,2}$, $\bar{B}_1$ is the preconditioners of $\bar{A}_1$.

Noting that $\bar{A}_1$ denotes the diagonal matrix of $A_h^{k,1}$, in the practical application, we will take $\bar{B}_1$ as the Jacobi (or Gauss-Seidel) smoothing operator for $A_h^{k,1}$. Obviously, this special choose satisfies

$$\kappa(\bar{B}_1\bar{A}_1) \leq \tilde{C}_1, \tag{38}$$

where the constant $\tilde{C}_1$ is independent of the mesh parameters.

First, we prove that the above transfer operators satisfy the condition (7).

Due to the definitions of inner product and transfer operator in space $\bar{V}_1$, for any given $\bar{u}_1 = \sum_{\mathbf{b}\in\mathcal{B}(k,1)} \alpha_{\mathbf{b}}\mathbf{b} \in \bar{V}_1$, where $\alpha_{\mathbf{b}} \in \mathbb{R}$, we have

$$\begin{aligned} \|\Pi_1\bar{u}_1\|_A^2 &= \|\bar{u}_1\|_A^2 = \Bigg\|\sum_{\mathbf{b}\in\mathcal{B}(k,1)} \alpha_{\mathbf{b}}\mathbf{b}\Bigg\|_A^2 = \sum_{K\in\mathcal{T}_h} \Bigg\|\sum_{j=1}^{M} \alpha_{\mathbf{b}}\mathbf{b}\Bigg\|_{A,K}^2 \\ &\leq M \sum_{K\in\mathcal{T}_h} \sum_{\mathbf{b}\in\mathcal{B}(k,1)} \|\alpha_{\mathbf{b}}\mathbf{b}\|_{A,K}^2 = M\|\bar{u}_1\|_{\bar{A}_1}^2, \end{aligned} \tag{39}$$

where the constant M bounds the number of basis functions whose support overlaps with a single element K.

For any given $\bar{u}_2 \in \bar{V}_2$, it's easy to obain

$$\|\Pi_2\bar{u}_2\|_A = \|\bar{u}_2\|_A = \|\bar{u}_2\|_{\bar{A}_2}. \tag{40}$$

Combining (39) with (40), we conclude that (7) holds with the constants $c_1 = M$ and $c_2 = 1$.

Secondly, the above spaces and operators satisfy the condition (8) by using the Lemma 3.3.

Summing up, we obtain the following theorem by using Theorem 2.1.

Theorem 3.4. *For $B_h^{k,1}$ given by* (37)*, and $\bar{B}_1$ satisfies the condition of* (38)*, then we have*

$$\kappa(B_h^{k,1}A_h^{k,1}) \lesssim \kappa(B_h^{k-1,2}A_h^{k-1,2}), \tag{41}$$

with a constant only depending on the constants $\tilde{c}_0$, $\tilde{C}_1$ and the shape regularity of $\mathcal{T}_h$.

4 Preconditioner for finite element equations of second kind

Now, we present the another stable decomposition of $\boldsymbol{W}_h^{k-1,2}$ with $k \geq 2$.

Lemma 4.1. *For any $\boldsymbol{u}_h^{k-1,2} \in \boldsymbol{W}_h^{k-1,2}$, there are $\boldsymbol{u}_h^{k-1,1} \in \boldsymbol{W}_h^{k-1,1}$ and $\boldsymbol{\varphi}_h \in \boldsymbol{V}_h^{k,2}$ such that*

$$\boldsymbol{u}_h^{k-1,2} = \boldsymbol{u}_h^{k-1,1} + \nabla \times \boldsymbol{\varphi}_h, \tag{42}$$

and

$$\left(\|\boldsymbol{u}_h^{k-1,1}\|_A^2 + \|\nabla \times \boldsymbol{\varphi}_h\|_A^2\right)^{1/2} \leq c_0 \|\boldsymbol{u}_h^{k-1,2}\|_A, \tag{43}$$

where the constant c_0 only depends on Ω and the shape regularity of $\mathcal{T}_h$.

Proof. For any $\boldsymbol{u}_h^{k-1,2} \in \boldsymbol{W}_h^{k-1,2}$, we can interpolate $\boldsymbol{u}_h^{k-1,2}$ by Lemma 3.1. Thus, using (18), we have

$$\nabla \cdot \Pi_{h,\text{div}}^{k-1,1} \boldsymbol{u}_h^{k-1,2} = \Pi_h^{k-2} \nabla \cdot \boldsymbol{u}_h^{k-1,2}. \tag{44}$$

In view of (14), we have

$$\nabla \cdot \boldsymbol{u}_h^{k-1,2} \in X_h^{k-2}. \tag{45}$$

Making use of (45) and noting that $\Pi_h^{k-2}|_{X_h^{k-2}} = Id$ in (44), we get

$$\nabla \cdot \Pi_{h,\text{div}}^{k-1,1} \boldsymbol{u}_h^{k-1,2} = \nabla \cdot \boldsymbol{u}_h^{k-1,2},$$

namely

$$\nabla \cdot \left(\boldsymbol{u}_h^{k-1,2} - \Pi_{h,\text{div}}^{k-1,1} \boldsymbol{u}_h^{k-1,2}\right) = 0. \tag{46}$$

Noting that $\boldsymbol{u}_h^{k-1,2} - \Pi_{h,\text{div}}^{k-1,1} \boldsymbol{u}_h^{k-1,2} \in \boldsymbol{W}_h^{k-1,2}$, then by (46) and (13), there exists $\boldsymbol{\varphi}_h \in \boldsymbol{V}_h^{k,2}$, such that

$$\boldsymbol{u}_h^{k-1,2} = \boldsymbol{u}_h^{k-1,1} + \nabla \times \boldsymbol{\varphi}_h, \tag{47}$$

where $\boldsymbol{u}_h^{k-1,1} = \Pi_{h,\text{div}}^{k-1,1} \boldsymbol{u}_h^{k-1,2}$, which completes the proof of (42).

Using (47), (15) with $\delta = 1/2$, and the inverse estimate, we obtain

$$\begin{aligned} \|\nabla \times \boldsymbol{\varphi}_h\|_{0,K} &= \|\boldsymbol{u}_h^{k-1,2} - \Pi_{h,\mathrm{div}}^{k-1,1} \boldsymbol{u}_h^{k-1,2}\|_{0,K} \\ &\lesssim h\|\boldsymbol{u}_h^{k-1,2}\|_{(H^1(K))^3} \lesssim \|\boldsymbol{u}_h^{k-1,2}\|_{0,K}. \end{aligned}$$

Squaring and summing over all the elements, we get

$$\begin{aligned} \|\nabla \times \boldsymbol{\varphi}_h\|_0^2 &= \sum_{K\in\mathcal{T}_h} \|\nabla \times \boldsymbol{\varphi}_h\|_{0,K}^2 \\ &\lesssim \sum_{K\in\mathcal{T}_h} \|\boldsymbol{u}_h^{k-1,2}\|_{0,K}^2 = \|\boldsymbol{u}_h^{k-1,2}\|_0^2. \end{aligned} \tag{48}$$

In view of (3) and (48), we find

$$\|\nabla \times \boldsymbol{\varphi}_h\|_A^2 = \tau\|\nabla \times \boldsymbol{\varphi}_h\|_0^2 \lesssim \tau\|\boldsymbol{u}_h^{k-1,2}\|_0^2 \leq \|\boldsymbol{u}_h^{k-1,2}\|_A^2. \tag{49}$$

Making use of (47), triangular inequality and (48), we have

$$\|\boldsymbol{u}_h^{k-1,1}\|_0 \leq \|\boldsymbol{u}_h^{k-1,2}\|_0 + \|\nabla \times \boldsymbol{\varphi}_h\|_0 \lesssim \|\boldsymbol{u}_h^{k-1,2}\|_0^2. \tag{50}$$

A direct manipulation of (47) gives that

$$\|\nabla \cdot \boldsymbol{u}_h^{k-1,1}\|_0 = \|\nabla \cdot \boldsymbol{u}_h^{k-1,2}\|_0. \tag{51}$$

A combination of (49), (50) and (51) concludes (43). □

In this case, let $V = \boldsymbol{W}_h^{k-1,2}$. We choose the following two auxiliary spaces and the corresponding transfer operator.

1. $\bar{V}_1 = \boldsymbol{W}_h^{k-1,1}$ with inner product $\bar{a}_1(\cdot,\cdot) = a(\cdot,\cdot)$ in the sense that

$$\bar{a}_1(\bar{u}_1, \bar{v}_1) :=< \bar{A}_1\bar{u}_1, \bar{v}_1 >= a(\bar{u}_1, \bar{v}_1) \quad \forall \bar{u}_1, \bar{v}_1 \in \bar{V}_1,$$

   which concludes that $\bar{A}_1 = A_h^{k-1,1}$. The corresponding transfer operator is $\Pi_1 = Id$.

2. $\bar{V}_2 = \boldsymbol{V}_h^{k,2}$ with inner product

$$\bar{a}_2(\bar{u}_2, \bar{v}_2) :=< \bar{A}_2\bar{u}_2, \bar{v}_2 >= \tau(\nabla \times \bar{u}_2, \nabla \times \bar{v}_2) \;\; \forall \bar{u}_2, \bar{v}_2 \in \bar{V}_2. \tag{52}$$

   The corresponding transfer operator is $\Pi_2 = \textbf{curl}$.

Then by using (6), we obtain the auxiliary space preconditioner for $A_h^{k-1,2}$ as follows

$$B_h^{k-1,2} = B_h^{k-1,1} + \mathbf{curl}\bar{B}_2\mathbf{curl}^*, \tag{53}$$

where $B_h^{k-1,1}$ is the preconditioner of $A_h^{k-1,1}$, and $\bar{B}_2$ is the preconditioners of $\bar{A}_2$ given by (52).

Especially, we adopt the preconditioner $\bar{B}_2$ in [10], this choice satisfy

$$\kappa(\bar{B}_2\bar{A}_2) \leq C_1, \tag{54}$$

where the constant C_1 is independent of the mesh parameters.

It is easy to prove that the above transfer operators satisfy the conditions (7). In fact, using the definitions of inner products and transfer operators in spaces $\bar{V}_l(l = 1, 2)$, we have

$$\|\Pi_1\bar{v}_1\|_A = \|\bar{v}_1\|_A = \|\bar{v}_1\|_{\bar{A}_1},\ \forall\, \bar{v}_1 \in \bar{V}_1, \tag{55}$$

$$\|\Pi_2\bar{v}_2\|_A^2 = \|\nabla\times\bar{v}_2\|_A^2 = \tau\|\nabla\times\bar{v}_2\|_0^2 = \|\bar{v}_2\|_{\bar{A}_2}^2, \forall\,\bar{v}_2 \in \bar{V}_2, \tag{56}$$

namely, the conditions (7) of Theorem 2.1 hold with the constants $c_1 = c_2 = 1$.

Applying Theorem 2.1 and using Lemma 4.1, we have the following Theorem.

Theorem 4.2. *For $B_h^{k-1,2}$ given by* (53)*, and $\bar{B}_2$ satisfies the condition of* (54)*, then we have*

$$\kappa(B_h^{k-1,2}A_h^{k-1,2}) \lesssim \kappa(B_h^{k-1,1}A_h^{k-1,1}), \tag{57}$$

with a constant only depending on the constants c_0 and C_1 and the shape regularity of $\mathcal{T}_h$.

Combining Theorem 3.4 and Theorem 4.2, by using a Jacobi (or Gauss-Seidel) smoothing, we can translate the construction of preconditioner for $A_h^{k,1}$ into the one of $A_h^{k-1,2}$. Furthermore, by using the preconditioner of $\boldsymbol{H}(\mathbf{curl};\Omega)$-elliptic problem, we can translate the preconditioner for $A_h^{k-1,2}$ into the one for $A_h^{k-1,1}$. Since Hiptmair and Xu [5] have constructed an efficient preconditioner $B_h^{1,1}$ for $A_h^{1,1}$, we construct the efficient precondtioners for $A_h^{k,l}(k = 1,\ l = 2 \text{ or } k \geq 2, l = 1, 2)$ and prove the corresponding spectral condition numbers are uniformly bounded and independent of mesh size h and the parameter τ by this recursive form.

5 Implementation of algorithm and numerical experiments

For simplicity, we only give the description of the preconditioning algorithm defined by (53) when $k = 2$.

Note that when $k = 2$, (53) turn to

$$B_h^{1,2} = B_h^{1,1} + \mathbf{curl}\bar{B}_2\mathbf{curl}^*. \tag{58}$$

In the following, we first discuss the description of algorithm about the preconditioner $B_h^{1,1}$. For this purpose, we introduce the following operators

$$P_d^c : \boldsymbol{W}^{1,1} \longrightarrow \nabla \times \boldsymbol{V}^{1,1},$$

$$P_d^s : \boldsymbol{W}^{1,1} \longrightarrow (S_h^1)^3,$$

$$P_c^s : \boldsymbol{V}^{1,1} \longrightarrow (S_h^1)^3,$$

and

$$A_c^{1,1} = P_d^c A_h^{1,1} (P_d^c)^T,$$

$$A_d^s = P_d^s A_h^{1,1} (P_d^s)^T,$$

$$A_c^s = P_c^s A_c^{1,1} (P_c^s)^T,$$

then, the algorithm about the operator $B_h^{1,1}$ can be described by (see [5] for more details)

Algorithm 5.1. *For a given $g \in \boldsymbol{W}_h^{1,1}$, then $u_g = B_h^{1,1} g \in \boldsymbol{W}_h^{1,1}$ can be obtained as follows:*

Step 1: *Applying m_1 times symmetric Gauss_Seidel iterations in variational problem*

$$a(\tilde{u}_1, \boldsymbol{v}_h^{1,1}) = (\boldsymbol{f}, \boldsymbol{v}_h^{1,1}) \quad \forall \boldsymbol{v}_h^{1,1} \in \boldsymbol{W}_h^{1,1}$$

with a zero initial guess to get $\tilde{u}_1$, where $\boldsymbol{f} = g$.

Step 2: *Computing $\tilde{u}_2 \in (S_h^1)^3$ by*

$$(A_d^s \tilde{u}_2, v_2) = (g, v_2), \;\; \forall v_2 \in (S_h^1)^3.$$

Step 3: *Computing $\tilde{u}_3 \in \boldsymbol{V}_h^{1,1}$ by*

$$(A_c^{1,1}\tilde{u}_3, \tilde{v}_3) = (g, \nabla \times \tilde{v}_3), \quad \forall \tilde{v}_3 \in \boldsymbol{V}_h^{1,1}, \tag{59}$$

which can be obtained by

1. *Applying m_2 times symmetric Gauss_Seidel iterations in* (59) *with a zero initial guess to get $\tilde{u}_4$.*
2. *Computing $\tilde{u}_5 \in (S_h^1)^3$ by*

$$(A_c^s\tilde{u}_5, v_5) = (g, v_5), \quad \forall v_5 \in (S_h^1)^3. \tag{60}$$

3. *Set $\tilde{u}_3 = \tilde{u}_4 + (P_c^s)^T\tilde{u}_5$.*

Step 4: *Set $u_g = \tilde{u}_1 + (P_d^s)^T\tilde{u}_2 + (P_d^c)^T\tilde{u}_3$.*

By [5], the preconditioner $B_h^{1,1}$ defined by Algorithm 5.1 satisfy

$$\kappa(B_h^{1,1}A_h^{1,1}) \leq C_1,$$

where the constant C_1 is independent of the mesh size h and parameter τ.

Next, we give the description of algorithm for the operator $\mathbf{curl}\bar{B}_2\mathbf{curl}^*$. Firstly, let

$$n = dim(\boldsymbol{V}_h^{2,1}), \quad m = dim(\boldsymbol{W}_h^{1,2}),$$

and

$$\boldsymbol{V}^{2,1} = span\{\phi_i, i = 1, \cdots, n\}, \quad \boldsymbol{W}^{1,2} = span\{\psi_j, j = 1, \cdots, m\},$$

then we introduce the transfer matrix(or operator) $P_d^{c,2}$

$$\begin{pmatrix} \nabla \times \phi_1 \\ \nabla \times \phi_2 \\ \vdots \\ \nabla \times \phi_n \end{pmatrix} = P_d^{c,2} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_m \end{pmatrix},$$

By using $P_d^{c,2}$, we can define the following matrix(or operator)

$$A_c^{2,1} = P_d^{c,2}A_h^{1,2}(P_d^{c,2})^T.$$

In view of (4.1) in [10], we can construct the preconditioner $\bar{B}_2$ for $A_c^{2,1}$, and its spectral condition number satisfy

$$\kappa(\bar{B}_2 A_c^{2,1}) \leq C_2,$$

where the constant C_2 is independent of the mesh size h and parameter τ.

Noting that the operator $\bar{B}_2$ can be divided into three parts: the first part is to use the Jacobi (or Gauss-Seidel) smoothing for (52) in space $\boldsymbol{V}_h^{2,1}$, the second part is to solve the restriction of (52) in $(S_h^1)^3$, the third part is to solve the restriction of (52) in ∇S_h^2. We can drop the second and third parts by using the fact that the second part is the same as (60) and **curl** $\circ$ **grad** $\equiv 0$. Hence the operator $\mathbf{curl}\bar{B}_2\mathbf{curl}^*$ can be simplified.

Summing up, we can obtain the following algorithm of the preconditioner $B_h^{1,2}$.

Algorithm 5.2. *For $g \in \boldsymbol{W}_h^{1,2}$, the solution $u_g = B_h^{1,2} g \in \boldsymbol{W}_h^{1,2}$ can be gotten as follows:*

Step 1: *Computing $u_1 \in \boldsymbol{W}_h^{1,1}$ by Algorithm 5.1.*

Step 2: *Applying m_3 times symmetric Gauss_Seidel iterations to get $u_2 \in V^{2,1}$ by*

$$(A_c^{2,1} u_2, v_2) = (g, \nabla \times v_2), \quad \forall v_2 \in \boldsymbol{V}^{2,1}.$$

Step 3: *Set*

$$u_g = u_1 + u_2.$$

For variational problem (4), we apply Algorithm 5.2 to the following two examples:

Example 5.1. *The computational domain is $\Omega = [0,1] \times [0,1] \times [0,1]$ and the corresponding structured grids can be seen in Figure 1. For the convenience of computing the exact errors, we construct an exact solution $\boldsymbol{u} = (u_1, u_2, u_3)$ as*

$$\begin{cases} u_1 = xyz(x-1)(y-1)(z-1) \\ u_2 = \sin(\pi x)\sin(\pi y)\sin(\pi z) \\ u_3 = (1-e^x)(1-e^{x-1})(1-e^y)(1-e^{y-1})(1-e^z)(1-e^{z-1}). \end{cases}$$

Example 5.2. *The computational domain is the spheres of radius 1 and the corresponding unstructured grids can be seen in Figure 2, the exact solution* $\boldsymbol{u} = (u_1, u_2, u_3)$ *is*

$$\begin{cases} u_1 = x^2 + y^2 + z^2 - 1 \\ u_2 = x^2 + y^2 + z^2 - 1 \\ u_3 = x^2 + y^2 + z^2 - 1. \end{cases}$$

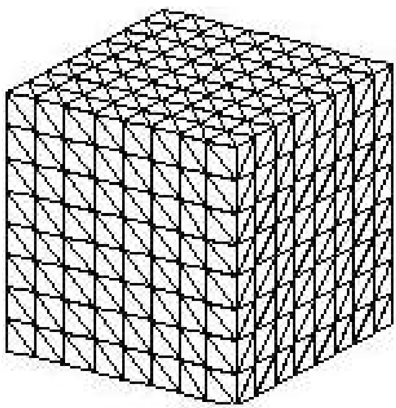

Figure 1

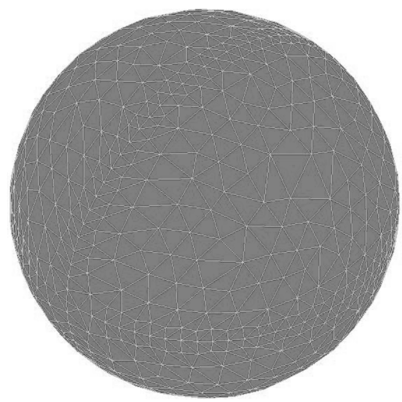

Figure 2

Now, we present some numerical experiments with $m_1 = m_2 = m_3 = 3$.

Table 1 gives the L_2 and $H(div)$ error estimates for Example 5.1 when $\tau = 1$, which shows that $\boldsymbol{u}_h^{1,2}$ is the optimal convergence.

$\mathcal{T}_h$	iter	$\|\boldsymbol{u} - \boldsymbol{u}_h^{1,2}\|_{L^2}$ err	rate	$\|\boldsymbol{u} - \boldsymbol{u}_h^{1,2}\|_{H(div)}$ err	rate
6^3	20	2.051e-2		2.040e-1	
12^3	19	4.685e-3	4.378	1.026e-1	1.988
24^3	19	1.139e-3	4.113	5.141e-2	1.996

Table 1

The condition number estimates and iteration counts for Example 5.1 and Example 5.2 are listed in Tables 2 – 5 for different values of the mesh size h and the scaling parameter τ. By these Tables, we find that the condition number and iteration counts are independent of the mesh size h and weakly dependent on the parameter τ.

		τ				
level	#cells	10^{-5}	10^{-2}	1	10^2	10^5
1	6×6^3	9.577	9.578	10.008	13.282	21.403
2	6×12^3	10.258	10.261	10.254	12.363	19.396
3	6×24^3	10.301	10.291	10.294	11.030	18.098

Table 2 – Unit cube: spectral condition number of $B_h^{1,2} A_h^{1,2}$.

		τ				
level	#cells	10^{-5}	10^{-2}	1	10^2	10^5
1	6×6^3	19	19	20	22	28
2	6×12^3	19	18	19	21	25
3	6×24^3	19	18	19	19	23

Table 3 – Number of PCG-iterations on unit cube.

		τ				
level	#cells	10^{-5}	10^{-2}	1	10^2	10^5
1	2197	11.918	11.920	12.111	17.300	30.293
2	4462	11.745	11.746	11.881	16.783	30.015
3	8865	14.887	14.889	15.051	20.204	34.122
4	17260	16.936	16.937	17.049	22.816	34.089
5	46543	14.876	14.875	14.863	18.830	37.786
6	66402	17.839	17.840	16.524	22.420	43.861

Table 4 – Unit ball: spectral condition number of $B_h^{1,2} A_h^{1,2}$.

		τ				
level	#cells	10^{-5}	10^{-2}	1	10^2	10^5
1	2197	13	17	20	24	30
2	4462	13	17	20	24	30
3	8865	14	17	21	25	31
4	17260	14	17	20	23	29
5	46543	15	17	20	23	28
6	66402	16	17	20	23	27

Table 5 – Number of PCG-iterations on unstructured grids in the unit ball.

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