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Ostrowski type inequalities for interval-valued functions using generalized Hukuhara derivative*

Y. CHALCO-CANO, A. FLORES-FRANULIČ and H. ROMÁN-FLORES

Instituto de Alta Investigación, Universidad de Tarapacá, Casilla 7D, Arica-Chile
E-mails: ychalco@uta.cl / aflores@uta.cl / hroman@uta.cl

Abstract. The present paper is devoted to obtaining some Ostrowski type inequalities for interval-valued functions. In this context we use the generalized Hukuhara derivative for interval-valued functions. Also some examples and consequences are presented.

Mathematical subject classification: Primary: 26E25; Secondary: 35A23.

Key words: Ostrowski type inequalities, interval-valued functions, gH-differentiability and integrability of interval-valued functions.

1 Introduction

The importance of the study of set-valued analysis from a theoretical point of view as well as from their application is well known [5, 7]. Many advances in set-valued analysis have been motivated by control theory and dynamical games [6]. Optimal control theory and mathematical programming were a motivating force behind set-valued analysis since the sixties [6]. Interval Analysis is a particular case and it was introduced as an attempt to handle interval uncertainty that appears in many mathematical or computer models of some deterministic real-world phenomena. The first monograph dealing with interval analysis was given by Moore [14]. Moore is recognized to be the first to use intervals in computational

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mathematics, now called numerical analysis. He also extended and implemented the arithmetic of intervals to computers. One of his major achievements was to show that Taylor series methods for solving differential equations not only are more tractable, but also more accurate [15].

The following inequality is known in the literature as Ostrowski's inequality

$$\left|\frac{1}{b-a}\int_a^b f(y)dy - f(x)\right| \leq \left(\frac{1}{4} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right)(b-a)\left\|f'\right\|_\infty, \tag{1}$$

where $f \in C^1([a,b])$, $x \in [a,b]$. Inequality (1) is sharp, see [3]. Since 1938 when A. Ostrowski (see [16]) presented his famous inequality many researchers have been working about and around it, in many different directions and with a lot of applications. In the book edited by Dragomir and Rassias [11] and recently in the book of Anastassiou [1] are given a brief review of state of art about Ostrowski type inequalities and its applications.

Continuing that tradition, in [2] the Ostrowski type inequality has been extended to context of fuzzy-valued functions. In this context has been used the concept of Hukuhara-derivative for fuzzy-valued functions. Note that interval-valued functions are fuzzy-valued functions. Thus, the fuzzy Ostrowski type inequalities obtained in [2] is valid for interval-valued functions. However, the concept of H-derivative for interval-valued functions is very restrictive, see [8, 9]. Generalized Hukuhara differentibility it is the most general differentiability concept for interval-valued functions, see [8, 9, 19].

Motivated by [1, 2, 3, 11] and [8, 9, 13, 19] we extend Ostrowski type inequality (1) for gH-differentiable interval-valued functions.

2 Basic concepts

Let $\mathbb{R}$ be the one-dimensional Euclidean space. Following [10], let $\mathcal{K}_C$ denote the family of all non-empty compact convex subsets of $\mathbb{R}$, that is,

$$\mathcal{K}_C = \{[a,b] \mid a, b \in \mathbb{R} \text{ and } a \leq b\}.$$

The Hausdorff metric H on $\mathcal{K}_C$ is defined by

$$H(A,B) = \max\{d(A,B), d(B,A)\},$$

where

$$d(A, B) = \max_{a \in A} d(a, B) \text{ and } d(a, B) = \min_{b \in B} d(a, b) = \min_{b \in B} |a - b| .$$

It is well known that $(\mathcal{K}_C, H)$ is a complete metric space (see [5, 10]). The Minkowski sum and scalar multiplication are defined by

$$A + B = \{a + b \mid a \in A,\ b \in B\} \text{ and } \lambda A = \{\lambda a \mid a \in A\}. \tag{2}$$

The space $\mathcal{K}_C$ is not a linear space since it does not possess an additive inverse and therefore subtraction is not well defined (see [5, 9, 10, 19]). Actually, $\mathcal{K}_C$ is a quasilinear space [4, 17].

A crucial concept in obtaining a useful working definition of derivative for interval-valued functions is considering a suitable difference between two intervals. Toward this end, one way is to use (2) by requiring

$$A - B = A + (-1)B.$$

However, this definition of difference has the drawback that

$$A - A \neq \{0\} \tag{3}$$

in general (the exception is when we have a zero width interval, $A = [a, a]$, that is, a real number). One of the first attempts to overcome (3) was due to Hukuhara [12] who defined what has become to be known as the Hukuhara difference (H-difference). If $A = B + C$, then the H-difference of A and B, denoted by $A -_H B$, is equal to C. The H-difference of two intervals does not always exists for arbitrary pairs of intervals. It only exists for intervals A and B for which the widths are such that

$$\mu(A) \geq \mu(B),$$

where for $A = [\underline{a}, \overline{a}]$, $\mu(A) = \overline{a} - \underline{a}$ is the lenght of the interval A.

Recently, Stefanini and Bede [19] introduced the concept of generalized Hukuhara difference of two sets $A, B \in \mathcal{K}_C$ (gH-difference for short) and it is defined as follows

$$A \ominus_{gH} B = C \Leftrightarrow \begin{cases} (a) \quad A = B + C \\ \qquad\quad \text{or} \\ (b) \quad B = A + (-1)C. \end{cases} \tag{4}$$

In case (a), the gH-difference is coincident with the H-difference. Thus, the gH-difference is a generalization of the H-difference. On the other hand, gH-difference exists for any two compact intervals $A = [a, b]$, $B = [c, d] \in \mathcal{K}_C$ and

$$A \ominus_{gH} B = [\min\{a - c, b - d\},\ \max\{a - c, b - d\}]\,. \tag{5}$$

For more details and properties of gH-difference see [19, 20].

3 Calculus for interval-valued functions

Henceforth $T = [a, b]$ denotes a closed interval. Let $F : T \to \mathcal{K}_C$ be an interval-valued function. We will denote $F(t) = [\underline{f}(t), \overline{f}(t)]$, where $\underline{f}(t) \leq \overline{f}(t)$, $\forall t \in T$. The functions $\underline{f}$ and $\overline{f}$ are called the lower and the upper (endpoint) functions of F, respectively.

For interval-valued functions it is clear that $F : T \to \mathcal{K}_C$ is continuous at $t_0 \in T$ if

$$\lim_{t \to t_0} F(t) = F(t_0),$$

where the limit is taken in the metric space $(\mathcal{K}_C, H)$. Consequently, F is continuous at $t_0 \in T$ if and only if its endpoint functions $\underline{f}$ and $\overline{f}$ are continuous functions at $t_0 \in T$.

We denote by $C\left([a, b], \mathcal{K}_C\right)$ the family of all continuous interval-valued functions. Then, $C\left([a, b], \mathcal{K}_C\right)$ is a quasilinear spaces, see [4, 17]. On the quasilinear space $C\left([a, b], \mathcal{K}_C\right)$ we can define a quasinorm $\|\cdot\|_\infty$ given by

$$\|F\|_\infty = \sup_{t \in [a,b]} H(F(t), \{0\}).$$

For more details and properties of quasilinear spaces and quasinorms see [4, 17].

Definition 3.1. ([5]) *Let $F : T \to \mathcal{K}_C$ be an interval-valued function. The integral (Aumann integral) of F over T is defined as*

$$\int_{t_1}^{t_2} F(t)dt = \left\{ \int_{t_1}^{t_2} f(t)dt \mid f \in S(F) \right\},$$

where $S(F)$ is the set of all integrable selectors of F, i.e.:

$$S(F) = \{f : T \to \mathbb{R} \mid f \ \ integrable\ and\ f(x) \in F(x)\ a.e.\}\,.$$

If $S(F) \neq \emptyset$, then the integral exists and F is said to be integrable (Aumann integrable).

Note that if F is measurable then has a measurable selector (see [5, 7, 10]) which is integrable and, consequently, $S(F) \neq \emptyset$. More precisely.

Theorem 3.2. ([5]) *Let $F : T \to \mathcal{K}_C$ be a measurable and integrably bounded interval-valued function. Then it is integrable and $\int_a^b F(t)dt \in \mathcal{K}_C$.*

Corollary 3.3. ([5, 10]) *A continuous interval-valued function $F : T \to \mathcal{K}_C$ is integrable.*

The Aumann integral satisfies the following properties.

Proposition 3.4. ([5, 10]) *Let $F, G : T \to \mathcal{K}_C$ be two measurable and integrably bounded interval-valued functions. Then*

(i) $\int_{t_1}^{t_2} (F(t) + G(t))\, dt = \int_{t_1}^{t_2} F(t)dt + \int_{t_1}^{t_2} G(t)dt$

(ii) $\int_{t_1}^{t_2} F(t)dt = \int_{t_1}^{\tau} F(t)dt + \int_{\tau}^{t_2} F(t)dt$, $t_1 < \tau < t_2$.

Theorem 3.5. ([8]) *Let $F : T \to \mathcal{K}_C$ be a measurable and integrably bounded interval-valued function such that $F(t) = [\underline{f}(t), \overline{f}(t)]$. Then $\underline{f}$ and $\overline{f}$ are integrable functions and*

$$\int_{t_1}^{t_2} F(t)dt = \left[\int_{t_1}^{t_2} \underline{f}(t)dt\ ,\ \int_{t_1}^{t_2} \overline{f}(t)dt\right].$$

The H-derivative (differentiability in the sense of Hukuhara) for interval-valued functions was initially introduced in [12] and it is based on the H-difference of intervals.

Definition 3.6. ([12]) *Let $F\colon T \to \mathcal{K}_C$ be interval-valued function. We say that F is differentiable at $t_0 \in T$ if there exists an element $F'(t_0) \in \mathcal{K}_C$ such that the limits*

$$\lim_{h\to 0^+} \frac{F(t_0+h) -_H F(t_0)}{h} \quad \text{and} \quad \lim_{h\to 0^+} \frac{F(t_0) -_H F(t_0-h)}{h}$$

exist and are equal to $F'(t_0)$.

Here the limits are taken in the metric space $(\mathcal{K}_C, H)$. Note that the H-derivative is very restrictive. For example, if we consider the interval-valued function $F(t) = (1-t^3)[-2, 1]$, since $F(0+h) -_H F(0) = (1-h^3)[-2, 1] -_H [-2, 1]$, the H-difference $F(0+h) -_H F(0)$ does not exist as $h \to 0^+$. Therefore, the H-derivative of F does not exist at $t = 0$. In general, if $F(t) = C \cdot g(t)$ where C is an interval and $g : [a, b] \to \mathbb{R}^+$ is a function with $g'(t_0) < 0$, then F is not differentiable at t_0 ([8, 9]). To avoid this difficulty, in [19] the authors have introduced a more general definition of derivative for interval-valued functions. For more details see [9, 19].

Definition 3.7. ([19]) *The gH-derivative of an interval-valued function $F\colon T \to \mathcal{K}_C$ at $t_0 \in T$ is defined as*

$$F'(t_0) = \lim_{h\to 0} \frac{F(t_0+h) \ominus_{gH} F(t_0)}{h}. \tag{6}$$

If $F'(t_0) \in \mathcal{K}_C$ satisfying (6) *exists, we say that F is generalized Hukuhara differentiable (gH-differentiable) at t_0.*

In connection with the endpoint functions of F we have the following result.

Theorem 3.8. ([9]) *Let $F : T \to \mathcal{K}_C$ be an interval-valued function such that $F(t) = [\underline{f}(t), \overline{f}(t)]$. Then, F is gH-differentiable at $t_0 \in T$ if and only if one of the following cases holds*

(a) *$\underline{f}$ and $\overline{f}$ are differentiable at t_0 and*

$$F'(t_0) = \left[\min\left\{(\underline{f})'(t_0), (\overline{f})'(t_0)\right\},\ \max\left\{(\underline{f})'(t_0), (\overline{f})'(t_0)\right\}\right];$$

(b) *$(\underline{f})'_-(t_0)$, $(\underline{f})'_+(t_0)$, $(\overline{f})'_-(t_0)$ and $(\overline{f})'_+(t_0)$ exist and satisfy $(\underline{f})'_-(t_0) = (\overline{f})'_+(t_0)$ and $(\underline{f})'_+(t_0) = (\overline{f})'_-(t_0)$. Moreover*

$$\begin{aligned} F'(t_0) &= \left[\min\left\{(\underline{f})'_-(t_0), (\overline{f})'_-(t_0)\right\},\ \max\left\{(\underline{f})'_-(t_0), (\overline{f})'_-(t_0)\right\}\right] \\ &= \left[\min\left\{(\underline{f})'_+(t_0), (\overline{f})'_+(t_0)\right\},\ \max\left\{(\underline{f})'_+(t_0), (\overline{f})'_+(t_0)\right\}\right] \end{aligned}$$

Example 3.9. Let the interval-valued function $F : \mathbb{R} \to \mathcal{K}_C$ defined by $F(t) = [-|t|, |t|]$. Then F is gH-differentiable in $\mathbb{R}$ but the endpoint functions $\underline{f}$ and $\overline{f}$ are not differentiable at 0. Also, from Theorem 3.8 part (a) we have $F'(t) = \left[(\overline{f})'(t), (\underline{f})'(t)\right] = [-1, 1]$ for all $t \in (-\infty, 0)$ and $F'(t) = \left[(\underline{f})'(t), (\overline{f})'(t)\right] = [-1, 1]$ for all $t \in (0, \infty)$. From part (b) we have $F'(0) = [-1, 1]$.

From Example 3.9 we can see that on the interval $(-\infty, 0)$ the lenght of the interval $F(t)$ (for short, $len(F(t))$) is decreasing while on the interval $(0, \infty)$ the $len(F(t))$ is increasing and $t = 0$ is a switching point for the monotonicity of $len(F(t))$, that is to say, in $t = 0$, $len(F(t))$ change its monotonicity. Thus, we establish that (see [19]):

(I) F is differentiable at $t_0 \in T$ in the first form if $\underline{f}$ and $\overline{f}$ are differentiable at t_0 and
$$F'(t_0) = \left[(\underline{f})'(t_0), (\overline{f})'(t_0)\right];$$

(II) F is differentiable at $t_0 \in T$ in the second form if $\underline{f}$ and $\overline{f}$ are differentiable at t_0 and
$$F'(t_0) = \left[(\overline{f})'(t_0), (\underline{f})'(t_0)\right].$$

Even more, a point $t_0 \in T$ is said to be a switching point for the differentiability of F, if in any neighborhood V of t_0 there exist points $t_1 < t_0 < t_2$ such that

(type I) F is differentiable at t_1 in the first form while it is not differentiable in the second form, and F is differentiable at t_2 in the second form while it is not differentiable in the first form, or

(type II) F is differentiable at t_1 in the second form while it is not differentiable in the first form, and F is differentiable at t_2 in the first form while it is not differentiable in the second form.

Next we give an interval version of the second fundamental theorem of calculus which will be important to obtaining our main results.

Theorem 3.10. ([18]) *Let $F : [a, b] \to \mathcal{K}_C$ be an interval-valued function. If F is gH-differentiable in the first form (or second form) in $[a, b]$ then*

$$\int_a^b F'(t)dt = F(b) \ominus_{gH} F(a).$$

Theorem 3.11. *Let the interval-valued function $F : [a, b] \to \mathcal{K}_C$ gH-differentiable on $[a, b]$ with a finite number of switching points at $a = c_0 < c_1 < c_2 < \cdots < c_n < c_{n+1} = b$ and exactly at these points. Then we have*

$$\int_a^b F'(x)dx = \sum_{i=1}^{n+1} \left[F(c_i) \ominus_{gH} F(c_{i-1})\right].$$

Proof. For simplicity we consider only one switching point, the case of a finite number of switching points follow similarly. Let us suppose that F is differentiable on $[a, c]$ in the first form and F is differentiable on $[c, b]$ in the second form. Then from Proposition 3.4 and Theorem 3.10 we have

$$\begin{aligned} \int_a^b F'(x)dx &= \int_a^c F'(x)dx + \int_c^b F'(x)dx \\ &= (F(c) \ominus_{gH} F(a)) + (F(b) \ominus_{gH} F(c)). \end{aligned}$$

Thus the proof is completed. □

Remark 3.12. In [19] was presented a similar result to Theorem 3.11, but with different arguments used in the proof. Moreover if $c \in [a, b]$ is the only switching point for differentiability of F and $F(c)$ is a singleton not necessarely $\int_a^b F'(x)dx = F(b) \ominus_{gH} F(a)$. For instance, if F is considered as in the Example 3.9, we have $F(0) = 0$ and $\int_{-1}^1 F'(x)dx \neq F(1) \ominus_{gH} F(-1)$. It corrects the Theorem 30 in [19].

Next we present a version of mean value theorem for gH-differentiable interval-valued functions. This result will be also important in the next section.

Theorem 3.13. *Let $F : [a, b] \to \mathcal{K}_C$ be an gH-differentiability interval-value function on $[a, b]$ with a finite number of switching points at $a = c_0 < c_1 < c_2 < \cdots < c_n < c_{n+1} = b$ and exactly at these points. Assume that F' is continuous. Then*

$$H(F(b), F(a)) \leq \left\|F'\right\|_\infty (b - a).$$

Proof. Firstly we suppose that F is gH-differentiable with no switching point in the interval $[a, b]$ then, taking on account the Theorem 3.10, we have

$$H\left(F(b), F(a)\right) = H\left(F(b) \ominus_{gH} F(a), \{0\}\right)$$

$$
\begin{aligned}
&= H\left(\int_a^b F'(t)dt\ ,\ \{0\}\right) \\
&= H\left(\int_a^b F'(t)dt\ ,\ \int_a^b \{0\}dt\right) \\
&\leq \int_a^b H\left(F'(t)\ ,\ \{0\}\right) dt \\
&\leq \left\|F'\right\|_\infty (b-a).
\end{aligned}
$$

Now, we consider only one switching point, the case of a finite number of switching points follow similarly. Let us suppose that F is differentiable on $[a, c]$ in the first form and F is differentiable on $[c, b]$ in the second form. Then

$$
\begin{aligned}
& H\left(F(b), F(a)\right) \\
\leq\ & H\left(F(b), F(c)\right) + H\left(F(c), F(a)\right) \\
\leq\ & (b-c) \sup_{t\in[c,b]} H(F'(t), \{0\}) + (c-a) \sup_{t\in[a,c]} H(F'(t), \{0\}) \\
\leq\ & (b-a) \sup_{t\in[a,b]} H(F'(t), \{0\}) \\
=\ & \left\|F'\right\|_\infty (b-a).
\end{aligned}
$$

So the Theorem is established. □

4 Ostrowski type inequalities

In this Section we present some Ostrowski type inequalities for gH-differentiable interval-valued functions. We want to remark that the concept of gH-differentiability is the more general concept of differentiability than another concept for interval-valued fuctions. For more details see [9, 13, 19].

Theorem 4.1. *Let $F : [a, b] \to \mathcal{K}_C$ be a continuously gH-differentiable interval-valued function on $[a, b]$ with a finite number of switching points at $a = c_0 < c_1 < c_2 < \cdots < c_n < c_{n+1} = b$ and exactly at these points. Then, for $x \in [a, b]$ we have*

$$
H\left(\frac{1}{b-a}\int_a^b F(y)dy, F(x)\right) \leq \left\|F'\right\|_\infty \left(\frac{(x-a)^2 + (b-x)^2}{2(b-a)}\right). \tag{7}
$$

Proof. Taking in account Theorem 3.13 and properties of Hausdorff metric we have

$$
\begin{aligned}
& H\left(\frac{1}{b-a}\int_a^b F(y)dy, F(x)\right) \\
= \; & H\left(\frac{1}{b-a}\int_a^b F(y)dy \, , \, \frac{1}{b-a}\int_a^b F(x)dy\right) \\
\leq \; & \frac{1}{b-a}\int_a^b H(F(y), F(x))dy \\
\leq \; & \frac{1}{b-a}\int_a^b \sup_{y\in[a,b]} H(F'(y), \{0\}) \, |y-x| \, dy \\
= \; & \frac{1}{b-a} \sup_{y\in[a,b]} H(F'(y), \{0\}) \int_a^b |y-x| \, dy \\
= \; & \|F'\|_\infty \left(\frac{(x-a)^2+(b-x)^2}{2(b-a)}\right).
\end{aligned}
$$

And the inequality (7) is proved. □

Proposition 4.2. *Inequality (7) is sharp at $x = a$, in fact attained by $F(y) = (y-a)(b-a)A$, with $A \in \mathcal{K}_C$ being fixed.*

Proof. We denote by $A = [\underline{a}, \overline{a}]$, with $\underline{a} \leq \overline{a}$. Since $(y-a)(b-a) \geq 0$ then $F(y) = (y-a)(b-a)A = [(y-a)(b-a)\underline{a}, (y-a)(b-a)\underline{a}]$. From Theorem 3.8 F is a continuously gH-differentible interval-valued function and $F'(y) = (b-a)A$. Thus, we have that

$$
\begin{aligned}
H\left(\frac{1}{b-a}\int_a^b F(y)dy \, , \, \{0\}\right) &= H\left(\int_a^b ((y-a)A)dy \, , \, \{0\}\right) \\
&= H\left(\left(\int_a^b (y-a)dy\right) A \, , \, \{0\}\right) \\
&= H\left(\frac{(b-a)^2}{2}A, \{0\}\right) \\
&= \frac{(b-a)^2}{2} H\left(A, \{0\}\right) \, ,
\end{aligned}
$$

and

$$
\begin{aligned}
& \left(\sup_{t\in[a,b]} H\left(F'(y),\{0\}\right)\right)\left[\frac{(x-a)^2+(b-x)^2}{2(b-a)}\right] \\
= & \left(\sup_{t\in[a,b]} H((b-a)A,\{0\})\right)\frac{(b-a)^2}{2(b-a)} \\
= & \frac{(b-a)^2}{2}H\left(A,\{0\}\right).
\end{aligned}
$$

So, the equality in (7) is attained. □

Example 4.3. We consider the interval-valued function $F:[0,\pi]\to\mathcal{K}_C$ defined by

$$F(t)=[2,4]\cos(4t),$$

or equivalently

$$
F(t)=\begin{cases}
[2\cos(4t),\ 4\cos(4t)] & \text{if } 0\le t\le \pi/8;\\
[4\cos(4t),\ 2\cos(4t)] & \text{if } \pi/8\le t\le 3\pi/8;\\
[2\cos(4t),\ 4\cos(4t)] & \text{if } 3\pi/8\le t\le 5\pi/8;\\
[4\cos(4t),\ 2\cos(4t)] & \text{if } 5\pi/8\le t\le 7\pi/8;\\
[2\cos(4t),\ 4\cos(4t)] & \text{if } 7\pi/8\le t\le \pi.
\end{cases}
$$

Since $g(t)=\cos(4t)$ is a continuously differentiable function then F is continuously gH-differentiable and $F'(t)=[-16,-8]\sin(4t)$. So, $\left\|F'\right\|_\infty=16$.

On the other hand, F has seven switching points for its gH-differentiability in $(0,\pi)$ which are $\{\pi/8,\pi/4,3\pi/8,\pi/2,5\pi/8,3\pi/4,7\pi/8\}$.

Figure 1 shows the endpoint functions of F, the solid line curve represent the lower function $\underline{f}$ and the dashed one represent the upper function $\overline{f}$.

The left hand of the inequality (7) is given by

$$H\left(\frac{1}{\pi}\int_0^\pi [2,4]\cos(4t)dt\ ,\ F\left(\frac{\pi}{8}\right)\right)=H\left(\frac{1}{\pi}[-2,2],\{0\}\right)=\frac{2}{\pi}$$

while the right hand is

$$16\left(\frac{\left(\frac{\pi}{8}\right)^2+\left(\frac{7\pi}{8}\right)^2}{2\pi}\right)=16\left(\frac{50\pi}{128}\right)=\frac{25\pi}{4}$$

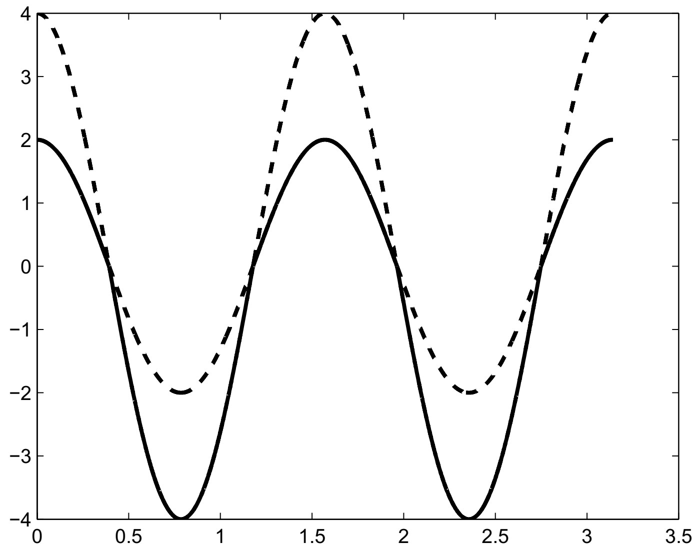

Figure 1 – The interval-valued function F.

So, the inequality (7) is valid for F.

Note that the inequality (7) is valid for any continuously gH-differentiable interval-valued function on $[a, b]$ with a finite number of switching points. From the example above we can see that F is continuously gH-differentiable and (7) is valid however the endpoint functions are not necessarely differentiables. For this special case, when endpoint functions are differentiables we have the following result, where we omitted that F has a finite number of switching points.

Theorem 4.4. *Let $F : [a, b] \to \mathcal{K}_C$ be an interval-valued function such that the endpoint functions $\underline{f}$, $\overline{f}$ are continuously differentiables. Then, F is continuously gH differentiable and for $x \in [a, b]$*

$$H\left(\frac{1}{b-a}\int_a^b F(y)dy, F(x)\right) \leq \|F'\|_\infty \left[\frac{(x-a)^2+(b-x)^2}{2(b-a)}\right]. \quad (8)$$

Proof. Taking in account the Ostrowski inequality (1) we have

$$\begin{aligned} & H\left(\frac{1}{b-a}\int_a^b F(y)dy, F(x)\right) \\ = \; & H\left(\frac{1}{b-a}\int_a^b \left[\underline{f}(y), \overline{f}(y)\right]dy \,,\; \left[\underline{f}(x), \overline{f}(x)\right]\right) \end{aligned}$$

$$
\begin{aligned}
&= H\left(\left[\frac{1}{b-a}\int_a^b \underline{f}(y)dy, \frac{1}{b-a}\int_a^b \overline{f}(y)dy\right], \left[\underline{f}(x), \overline{f}(x)\right]\right) \\
&= \max\left\{\left|\frac{1}{b-a}\int_a^b \underline{f}(y)dy - \underline{f}(x)\right|, \left|\frac{1}{b-a}\int_a^b \overline{f}(y)dy - \overline{f}(x)\right|\right\} \\
&\leq \max\left\{\left\|\left(\underline{f}\right)'\right\|_\infty \left[\frac{(x-a)^2+(b-x)^2}{2(b-a)}\right],\right. \\
&\qquad \left.\left\|\left(\overline{f}\right)'\right\|_\infty \left[\frac{(x-a)^2+(b-x)^2}{2(b-a)}\right]\right\} \\
&= \left[\frac{(x-a)^2+(b-x)^2}{2(b-a)}\right]\max\left\{\left\|\left(\underline{f}\right)'\right\|_\infty, \left\|(\overline{f})'\right\|_\infty\right\} \\
&= \|F'\|_\infty\left[\frac{(x-a)^2+(b-x)^2}{2(b-a)}\right].
\end{aligned}
$$

Thus, the proof is completed. □

Next we present another one generalization of the Ostrowski type inequality (1).

Theorem 4.5. *Let the interval-valued function $F : [a, b] \to \mathcal{K}_c$ gH-differentiable in (a, b) such that the endpoint functions $\underline{f}$, $\overline{f}$ are continuously differentiables. Let $\alpha : [a, b] \to [a, b]$ and $\beta : (a, b] \to [a, b]$, $\alpha(x) \leq x$, $\beta(x) \geq x$. Then, for all $x \in [a, b]$ we have*

$$
\begin{aligned}
&H\left(\int_a^b F(t)dt, (\beta(x)-\alpha(x))F(x) + (b-\beta(x))F(b) + (\alpha(x)-a)F(a)\right) \\
&\qquad \leq \left\|F'\right\|_\infty \left(\frac{1}{2}\left[\left(\frac{b-a}{2}\right)^2 + \left(x - \frac{a+b}{2}\right)^2\right]\right. \\
&\qquad \left. + \left(\alpha(x) - \frac{a+x}{2}\right)^2 + \left(\beta(x) - \frac{b+x}{2}\right)^2\right).
\end{aligned}
$$

Proof. From Theorem 47 in [11] and properties of Hausdorff metric, we have that

$$
\begin{aligned}
&H\left(\int_a^b F(y)dy, (\beta(x)-\alpha(x))F(x) + (b-\beta(x))F(b) + (\alpha(x)-a)F(a)\right) \\
&= H\left(\int_a^b \left[\underline{f}(y), \overline{f}(y)\right]dy, (\beta(x)-\alpha(x))\left[\underline{f}(x), \overline{f}(x)\right]\right.
\end{aligned}
$$

$$
\begin{aligned}
&+ (b-\beta(x))\left[\underline{f}(b), \overline{f}(b)\right] + (\alpha(x)-a)\left[\underline{f}(a), \overline{f}(a)\right]\Big) \\
&= \max\left\{\left|\int_a^b \underline{f}(y)dy - (\beta(x)-\alpha(x))\underline{f}(x) + (b-\beta(x))\underline{f}(b) + (\alpha(x)-a)\underline{f}(a)\right|,\right. \\
&\left.\left|\int_a^b \overline{f}(y)dy - (\beta(x)-\alpha(x))\overline{f}(x) + (b-\beta(x))\overline{f}(b) + (\alpha(x)-a)\overline{f}(a)\right|\right\} \\
&\le \max\left\{\left\|\left(\underline{f}\right)'\right\|_\infty \left(\frac{1}{2}\left[\left(\frac{b-a}{2}\right)^2 + \left(x-\frac{a+b}{2}\right)^2\right]\right.\right. \\
&\quad \left. + \left(\alpha(x)-\frac{a+x}{2}\right)^2 + \left(\beta(x)-\frac{b+x}{2}\right)^2\right), \\
&\quad \left\|\left(\overline{f}\right)'\right\|_\infty \left(\frac{1}{2}\left[\left(\frac{b-a}{2}\right)^2 + \left(x-\frac{a+b}{2}\right)^2\right]\right. \\
&\quad \left.\left. + \left(\alpha(x)-\frac{a+x}{2}\right)^2 + \left(\beta(x)-\frac{b+x}{2}\right)^2\right)\right\} \\
&= \max\left\{\left\|\left(\underline{f}\right)'\right\|_\infty, \left\|\left(\overline{f}\right)'\right\|_\infty\right\} \left(\frac{1}{2}\left[\left(\frac{b-a}{2}\right)^2 + \left(x-\frac{a+b}{2}\right)^2\right]\right. \\
&\quad \left. + \left(\alpha(x)-\frac{a+x}{2}\right)^2 + \left(\beta(x)-\frac{b+x}{2}\right)^2\right) \\
&= \left\|F'\right\|_\infty \left(\frac{1}{2}\left[\left(\frac{b-a}{2}\right)^2 + \left(x-\frac{a+b}{2}\right)^2\right]\right. \\
&\quad \left. + \left(\alpha(x)-\frac{a+x}{2}\right)^2 + \left(\beta(x)-\frac{b+x}{2}\right)^2\right)
\end{aligned}
$$

So, the inequality is established. □

Remark 4.6. *As a consequence of Theorem 4.5 we have the following special inequality: Let the interval-valued function $F : [a, b] \to \mathcal{K}_c$ satisfying the same conditions of Theorem 4.5. Then, if $\alpha(x) = \frac{a+x}{2}$ and $\beta(x) = \frac{b+x}{2}$ we have, for all $x \in [a, b]$,*

$$
\begin{aligned}
H\left(\int_a^b F(t)dt\,,\ \frac{b-a}{2}\left[F(x) + \left(\frac{x-a}{b-a}\right)F(a) + \left(\frac{b-x}{b-a}\right)F(b)\right]\right) \\
\le \frac{1}{2}\left\|F'\right\|_\infty \left[\left(\frac{b-a}{2}\right)^2 + \left(x-\frac{a+b}{2}\right)^2\right]
\end{aligned}
$$

Finally, we stablish that:

a) For this functions α and β we get the best bound for any $x \in [a, b]$ because the inequality in Theorem 4.5 contains a sum of squares and the minimun of this expresion occurs when each quadratic terms are zero.

b) If $x = \frac{a+b}{2}$ (the midpoint of $[a, b]$) we obtain an even more accurate formula from Remark 4.6. In fact,

$$H\left(\int_a^b F(t)dt\ ,\ \frac{b-a}{2}\left[F\left(\frac{a+b}{2}\right) + \frac{F(a)+F(b)}{2}\right]\right) \le \frac{1}{2}\left\|F'\right\|_\infty \left(\frac{b-a}{2}\right)^2$$

REFERENCES

[1] G.A. Anastassiou, *Advanced Inequalities*. World Scientific, New Jersey (2011).

[2] G.A. Anastassiou, *Fuzzy Ostrowski type inequalities*. Computational and Applied Mathematics, **22** (2003), 279–292.

[3] G.A. Anastassiou, *Ostrowski type inequalities*. Proc. AMS, **123** (1995), 3775–3791.

[4] S.M. Assev, *Quasilinear operators and their application in the theory of multivalued mappings*. Proc. Steklov Inst. Math., **2** (1986), 23–52.

[5] J.P. Aubin and A. Cellina, *Differential Inclusions*. Springer-Verleg, New York (1984).

[6] J.P. Aubin and H. Franskowska, *Introduction: Set-valued analysis in control theory*. Set-Valued Analysis, **8** (2000), 1–9.

[7] J.P. Aubin and H. Franskowska, *Set-Valued Analysis*. Birkhäuser, Boston (1990).

[8] B. Bede and S.G. Gal, *Generalizations of the differentiability of fuzzy number valued functions with applications to fuzzy differential equation*. Fuzzy Sets and Systems, **151** (2005), 581–599.

[9] Y. Chalco-Cano, H. Román-Flores and M.D. Jiménez-Gamero, *Generalized derivative and π-derivative for set-valued functions*. Information Science, **181** (2011), 2177–2188.

[10] P. Diamond and P. Kloeden, *Metric Space of Fuzzy Sets: Theory and Application*. World Scientific, Singapore (1994).

[11] S.S. Dragomir and T.M. Rassias, *Ostrowski Type Inequalities and Applications in Numerical Integration*. Kluwer Academic Publishers, London (2002).

[12] M. Hukuhara, *Integration des applications mesurables dont la valeur est un compact convexe*. Funkcialaj Ekvacioj, **10** (1967), 205–223.

[13] S. Markov, *Calculus for interval functions of a real variable*. Computing, **22** (1979), 325–377.

[14] R.E. Moore, *Interval Analysis*. Prince-Hall, Englewood Cliffs, NJ (1966).

[15] R.E. Moore, *Computational Functional Analysis*. Ellis Horwood Limited, England (1985).

[16] A. Ostrowski, *Über die adsolutabweichung einer differentiebaren funktion von ihrem integralmittelwert*. Comment. Math. Helv., **10** (1938), 226–227.

[17] M.A. Rojas-Medar, M.D. Jiménez-Gamero, Y.Chalco-Cano and A.J. Viera-Brandão, *Fuzzy quasilinear spaces and applications*. Fuzzy Sets and Systems, **152** (2005), 173–190.

[18] A. Rufián-Lizana, Y. Chalco-Cano, M.D. Jiménez-Gamero and H. Román-Flores, *Calculus for interval-valued functions using generalized Hukuhara derivative and applications*, Submitted.

[19] L. Stefanini and B. Bede, *Generalized Hukuhara differentiability of interval-valued functions and interval differential equations*. Nonlinear Analysis, **71** (2009), 1311–1328.

[20] L. Stefanini, *A generalization of Hukuhara difference and division for interval and fuzzy arithmetic*. Fuzzy Sets and Systems, **161** (2010), 1564–1584.