

A special class of continuous general linear methods

D.G. YAKUBU^{1*}, A.M. KWAMI and M.L. AHMED

Mathematical Sciences Programme, Abubakar Tafawa Balewa University,
PMB 0248, Bauchi, Nigeria

E-mails: daudagyakubu@yahoo.com / amkwamiatbu@yahoo.com / mukhtarliman@yahoo.com

Abstract. We consider the construction of a class of numerical methods based on the general matrix inverse [14] which provides continuous interpolant for dense approximations (output). Their stability properties are similar to those for Runge-Kutta methods. These methods provide a unifying scope for many families of traditional methods. They are self-starting, to change stepsize during integration is not difficult when using them. We exploited these properties by first obtaining the direct block methods associated with the continuous schemes and then converting the block methods into uniformly A-stable high order general linear methods that are acceptable for solving stiff initial value problems. However, we will limit our formulation only for the step numbers $k = 2, 3, 4$. From our preliminary experiments we present some numerical results of some initial value problems in ordinary differential equations illustrating various features of the new class of methods.

Mathematical subject classification: 65L05.

Key words: block method, continuous scheme, general linear method, matrix inverse.

1 Introduction

General linear methods emerged as a result of the desire to obtain a wider generalization of a large family of traditional numerical methods for ordinary differential equations. They were first introduced by [2] as a unifying theory for studying

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*Corresponding author.

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stability, consistency and convergence for a wide variety of traditional methods. Their formulations include both the multi-stage nature of Runge-Kutta methods as well as the multi-value nature of linear multistep methods which also allows for many generalizations of the traditional methods [4]. General linear methods, however, have not yet gained the popularity they deserve despite they have been in existence for over forty years. Their discovery opened up many possibilities of obtaining essentially new methods that were neither Runge-Kutta nor linear multistep methods which exist practically and have advantages over the traditional methods. For example “Almost Runge-Kutta” methods [5], two step Runge-Kutta methods [9, 12] and Hybrid methods [10] etc. Some of the reasons for their generalizations are:

- Runge-Kutta methods, which are always termed to be the best known one-step methods, have been regarded as expensive because of their multistage structure (multiple function calls in each time step [7]). Runge-Kutta methods use more function evaluations to attain the same accuracy as compared with the linear multistep methods. The implementation costs for implicit Runge-Kutta methods (Gauss, Lobatto and Radau) present obstacle to finding cheap implementation because of the structure of the coefficient matrix A in Butcher’s array, which has a pair of complex conjugate eigenvalue. For both explicit and implicit Runge-Kutta methods it is very difficult to estimate errors for variable stepsize h and order p [7].
- Linear multistep methods, on the other hand suffer the disadvantages of poor stability property as the step number increases with accuracy and requiring additional starting values with constant step size from other one-step methods. For the A-stability which is a desirable property for stiff problems, order is limited by Dahlquist barriers to two.

In this paper we consider some possible generalizations which retains the Runge-Kutta stability with the general nature of linear multistep methods but overcome some of the handicaps involved in the two well known traditional methods with some advantages than the two traditional methods. This could be done by including an “off-step point” midway between the step numbers which yields order $2k + 1$ [3]. In this way the idea of looking for a starter for a particular method is avoided, since the general linear method can now be used in a block form, see [15].

2 General linear methods

The name “general linear methods” applies to a large family of numerical methods for ordinary differential equations. Runge-Kutta methods and linear multi-step methods are examples of these methods. Further, a general linear method used for the numerical solution of system of initial value problem in ordinary differential equations of the form

$$y' = f(x, y), y(x_0) = y_0, \quad a \leq x \leq b, \quad (1)$$

is both multistage as the Runge-Kutta methods and multivalued as the linear multistep methods. In the general linear methods we denote the internal stage values of step number n by

$$Y_1^{[n]}, Y_2^{[n]}, \dots, Y_s^{[n]}$$

and the derivatives evaluated at these steps by

$$f(Y_1^{[n]}), f(Y_2^{[n]}), \dots, f(Y_s^{[n]}).$$

At the start of the step number n , r quantities denoted by

$$y_1^{[n-1]}, y_2^{[n-1]}, \dots, y_r^{[n-1]}$$

are available from approximations computed in step $n - 1$. Corresponding quantities

$$y_1^{[n]}, y_2^{[n]}, \dots, y_r^{[n]}$$

are evaluated in the step number n . Introducing the vectors

$$Y^{[n]}, f(Y^{[n]}), y^{[n-1]} \quad \text{and} \quad y^{[n]}$$

we can write them as follows:

$$Y^{[n]} = \begin{bmatrix} Y_1^{[n]} \\ Y_2^{[n]} \\ \vdots \\ Y_s^{[n]} \end{bmatrix}, \quad f(Y^{[n]}) = \begin{bmatrix} f(Y_1^{[n]}) \\ f(Y_2^{[n]}) \\ \vdots \\ f(Y_s^{[n]}) \end{bmatrix}, \quad y^{[n-1]} = \begin{bmatrix} y_1^{[n-1]} \\ y_2^{[n-1]} \\ \vdots \\ y_r^{[n-1]} \end{bmatrix}, \quad y^{[n]} = \begin{bmatrix} y_1^{[n]} \\ y_2^{[n]} \\ \vdots \\ y_r^{[n]} \end{bmatrix}$$

where r denotes quantities as output from each step and input to the next step and s denotes stage values used in the computation of the step $Y_1^{[n]}, Y_2^{[n]}, \dots, Y_s^{[n]}$.

If the stepsize is h then the quantities imported into and evaluated in step number n are related by the relations

$$\begin{aligned} Y^{[n]} &= h(A \oplus I_m)f(Y^{[n]}) + (U \oplus I_m)y^{[n-1]}, \\ y^{[n]} &= h(B \oplus I_m)f(Y^{[n]}) + (V \oplus I_m)y^{[n-1]}, \end{aligned}$$

where $n = 1, 2, \dots, N$; I is the identity matrix of size equal to the differential equation system to be solved and m is the dimension of the system. Also \oplus is the Kronecker product of two matrices. For simplicity, we write the method as:

$$\begin{aligned} Y^{[n]} &= hAf(Y^{[n]}) + Uy^{[n-1]}, \\ y^{[n]} &= hBf(Y^{[n]}) + Vy^{[n-1]}, \end{aligned} \quad (2)$$

and the coefficients of the method, that is, the elements of A , B , U and V as a partitioned $(s+r) \times (s+r)$ matrix:

$$M = \left[\begin{array}{c|c} A & U \\ \hline B & V \end{array} \right].$$

This formulation of general linear methods was introduced by Burrage and Butcher [1]. The structure of the leading coefficient matrix A which is similar to that of the A matrix in Runge-Kutta methods, determines the implementation cost of these methods. The V matrix determines the stability of these methods. The B matrix gives the weights. The U matrix is simply e . The vector $y^{[n]}$ can have a very general structure. That is to say the quantities could approximate the solution and the derivatives of various previous points, backward difference approximations to the derivatives or approximations to a Nordsieck vector, all of which are common choices in linear multistep methods. In the case of the Runge-Kutta methods, $y^{[n]}$ could be an approximation to y_n or perturbation of y_n using the generalization to effective order.

Definition 2.1. [8] *For a general linear method (A, U, B, V) the ‘stability matrix’ $M(z)$ is defined by*

$$M(z) = V + zB(I - zA)^{-1}U$$

and the characteristic polynomial is given by

$$\Phi(\omega, z) = \det(\omega I - M(z)).$$

Definition 2.2. [8] *If a general linear method (A, U, B, V) has a stability function which takes the special form:*

$$\Phi(\omega, z) = \omega^{r-1}(\omega - R(z))$$

where the rational function $R(z)$ is known as the ‘stability function’ of the method, then the method is said to have Runge-Kutta stability.

Definition 2.3. [8] *A general linear method (A, U, B, V) is A -stable if for all $z \in C^-$, $I - zA$ is non-singular and $M(z)$ is a stability matrix.*

For methods with this property the step size is never restricted by stability on linear constant coefficient problems, regardless of the stiffness.

Definition 2.4. [6] *A general linear method (A, U, B, V) is L -stable if it is A -stable and $\rho(M(\infty)) = 0$ or the stronger condition $M(\infty) = 0$.*

3 Derivation technique

A particularly useful class of discrete methods for the numerical integration of (1) is the class of linear multistep methods of the form

$$y_{n+k} = \sum_{j=0}^k \phi_j y_{n+j} + h \sum_{j=0}^k \psi_j f_{n+j} \quad (3)$$

where $k > 0$ is the step number, ϕ_j , $j = 0, 1, \dots, t-1$; ψ_j , $j = 0, 1, \dots, s-1$ are the coefficients of the discrete scheme, with $y_{n+j} = y(x_{n+j})$, $j = 0, 1, \dots, k-1$, h is assumed (for simplicity of the analysis) to be a constant step-size given by

$$h = x_{n+1} - x_n; n = 0, 1, \dots, N; hN = b - a;$$

and a set of equally spaced points on the integration interval also given by

$$a = x_0 < x_1 < \dots < x_n < x_{n+1} < \dots < x_{n+k} < \dots < x_N = b.$$

To obtain its continuous formulation, in the sense of [14] which was a generalization of [13], we consider a polynomial $\bar{y}(x)$ of degree $p = t + s - 1$, $t > 0$, $s > 0$, of the form

$$\bar{y}(x) = \sum_{j=0}^{t-1} \phi_j(x) y_{n+j} + h \sum_{j=0}^{s-1} \psi_j(x) f(\bar{x}_j, \bar{y}(\bar{x}_j)) \quad (4)$$

defined over the k -steps, $x \in \{x_n, x_{n+k}\}$ such that it satisfies the conditions

$$\bar{y}(x_{n+j}) = y_{n+j}, \quad j \in \{0, 1, \dots, t-1\} \quad (5)$$

$$\bar{y}'(\bar{x}_j) = f(\bar{x}_j, \bar{y}(\bar{x}_j)), \quad j = 0, 1, \dots, s-1, \quad (6)$$

where $\phi_j(x)$ and $\psi_j(x)$ are assumed polynomials of the form:

$$\phi_j(x) = \sum_{i=0}^{t+s-1} \phi_{j,i+1} x^i, \quad j \in \{0, 1, \dots, t-1\}; \quad (7)$$

$$h\psi_j(x) = h \sum_{i=0}^{t+s-1} \psi_{j,i+1} x^i, \quad j = 0, 1, 2, \dots, s-1, \quad (8)$$

x_{n+j} in (5) are t ($t > 0$) arbitrarily chosen interpolation points taken from $\{x_n, x_{n+k}\}$ and the collocation points \bar{x}_j , $j = 0, 1, \dots, s-1$ in (6) also belong to $\{x_n, x_{n+k}\}$. From the interpolation and collocation conditions (5) and (6), and the expression for $\bar{y}(x)$ in (4), the following conditions are imposed on $\phi_j(x)$ and $\psi_j(x)$:

$$\begin{aligned} \phi_j(x_{n+i}) &= \delta_{ij}, \quad j = 0, 1, \dots, t-1; i = 0, 1, \dots, t-1 \\ h\psi_j(x_{n+i}) &= 0, \quad j = 0, 1, \dots, s-1; i = 0, 1, \dots, t-1 \end{aligned} \quad (9)$$

and

$$\begin{aligned} \phi_j'(\bar{x}_i) &= 0, \quad j = 0, 1, \dots, t-1; i = 0, 1, \dots, t-1 \\ h\psi_j'(\bar{x}_i) &= \delta_{ij}, \quad j = 0, 1, \dots, s-1; i = 0, 1, \dots, t-1. \end{aligned} \quad (10)$$

Next we write (9)-(10) in a matrix equation of the form

$$DC = I \quad (11)$$

where I is an identity matrix of appropriate dimension,

$$D = \begin{pmatrix} 1 & x_n & x_n^2 & \cdots & x_n^{t+s-1} \\ 1 & x_{n+1} & x_{n+1}^2 & \cdots & x_{n+1}^{t+s-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n+t-1} & x_{n+t-1}^2 & \cdots & x_{n+t-1}^{t+s-1} \\ 0 & 1 & 2\bar{x}_n & \cdots & (t+s-1)\bar{x}_n^{t+s-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 2\bar{x}_{n+s} & \cdots & (t+s-1)\bar{x}_{n+s}^{t+s-2} \end{pmatrix} \quad (12)$$

and

$$C = \begin{pmatrix} \phi_{0,1} & \phi_{1,1} & \cdots & \phi_{t-1,1} & h\psi_{0,1} & \cdots & h\psi_{s-1,1} \\ \phi_{0,2} & \phi_{1,2} & \cdots & \phi_{t-1,2} & h\psi_{0,2} & \cdots & h\psi_{s-1,2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \phi_{0,t+s} & \phi_{1,t+s} & \cdots & \phi_{t-1,t+s} & h\psi_{0,t+s} & \cdots & h\psi_{s-1,t+s} \end{pmatrix}. \quad (13)$$

The matrices D and C are both of dimensions $(t+s) \times (t+s)$. It follows from (11) that the columns of $C = D^{-1}$ give the continuous coefficients $\phi_j(x)$ and $\psi_j(x)$. We now derive the continuous formulation of the general linear methods following the derivation techniques discussed in Section 3.

4 The class of continuous general linear methods

Here we propose a more elegant and computationally attractive procedure, which leads to a class of stable general linear methods for both non-stiff and stiff systems of initial value problems. Although these methods were formulated in terms of multistep collocation methods, yet they preserve many of the Runge-Kutta properties, such as being self-starting and of permitting easy change of step length during implementation and have more advantages than for the traditional Runge-Kutta methods. In this family for $k = 2$, $\zeta = (x - x_n)$, the matrix D

in (12) becomes

$$D = \begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 \\ 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 \\ 0 & 1 & 2\bar{x}_n & 3\bar{x}_n^2 & 4\bar{x}_n^3 & 5\bar{x}_n^4 \\ 0 & 1 & 2\bar{x}_{n+1} & 3\bar{x}_{n+1}^2 & 4\bar{x}_{n+1}^3 & 5\bar{x}_{n+1}^4 \\ 0 & 1 & 2\bar{x}_{n+2} & 3\bar{x}_{n+2}^2 & 4\bar{x}_{n+2}^3 & 5\bar{x}_{n+2}^4 \end{pmatrix}. \quad (14)$$

Inverting the matrix in (14) once, using computer algebra, for example, Maple or Matlab software package, give rise to the following continuous scheme

$$\begin{aligned} \bar{y}(x) = & \phi_0(x)y_n + \phi_1(x)y_{n+1} + \phi_2(x)y_{n+2} \\ & + [\psi_0(x)f_n + \psi_1(x)f_{n+1} + \psi_2(x)f_{n+2}], \end{aligned} \quad (15)$$

where

$$\phi_0(x) = \left[\frac{3\zeta^5 - 17h\zeta^4 + 33h^2\zeta^3 - 23h^3\zeta^2 + 4h^5}{4h^5} \right],$$

$$\phi_1(x) = \left[\frac{\zeta^4 - 4h\zeta^3 + 4h^2\zeta^2}{h^4} \right],$$

$$\phi_2(x) = \left[\frac{-3\zeta^5 + 13h\zeta^4 - 17h^2\zeta^3 + 7h^3\zeta^2}{4h^5} \right],$$

$$\psi_0(x) = \left[\frac{\zeta^5 - 6h\zeta^4 + 13h^2\zeta^3 - 12h^3\zeta^2 + 4h^4\zeta}{4h^4} \right],$$

$$\psi_1(x) = \left[\frac{\zeta^5 - 5h\zeta^4 + 8h^2\zeta^3 - 4h^3\zeta^2}{h^4} \right],$$

$$\psi_2(x) = \left[\frac{\zeta^5 - 4h\zeta^4 + 5h^2\zeta^3 - 2h^3\zeta^2}{4h^4} \right].$$

Evaluating the continuous scheme in (15), we first obtain the block method associated with the continuous scheme and we converted the block method into uniformly accurate order general linear method:

$$\left[\begin{array}{ccccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{9}{128} & 0 & \frac{-9}{32} & 0 & \frac{-3}{128} & \frac{11}{128} & \frac{9}{16} & \frac{45}{128} \\ \frac{5}{32} & \frac{2}{3} & \frac{1}{8} & 0 & \frac{-1}{96} & \frac{1}{32} & 0 & \frac{31}{32} \\ \frac{3}{128} & 0 & \frac{9}{32} & 0 & \frac{-9}{128} & \frac{45}{128} & \frac{9}{16} & \frac{11}{128} \\ \frac{-1}{93} & 0 & \frac{4}{31} & \frac{64}{93} & \frac{5}{31} & 0 & \frac{32}{31} & \frac{-1}{31} \\ \hline \frac{-1}{93} & 0 & \frac{4}{31} & \frac{64}{93} & \frac{5}{31} & 0 & \frac{32}{31} & \frac{-1}{31} \\ \frac{3}{128} & 0 & \frac{9}{32} & 0 & \frac{-9}{128} & \frac{45}{128} & \frac{9}{16} & \frac{11}{128} \\ \frac{5}{32} & \frac{2}{3} & \frac{1}{8} & 0 & \frac{-1}{96} & \frac{1}{32} & 0 & \frac{31}{32} \end{array} \right] \cdot \quad (16)$$

We plotted the region of absolute stability of the general linear method (16) using the method used in [9] as shown below:

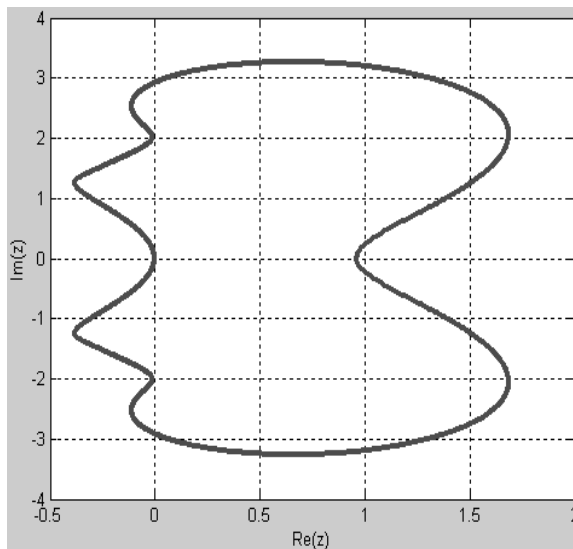


Figure 1 – Region of absolute stability of the general linear method (16).

For $k=3$, the matrix D in (12) takes the following form:

$$D = \begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 \\ 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 \\ 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 & x_{n+2}^7 \\ 1 & x_{n+3} & x_{n+3}^2 & x_{n+3}^3 & x_{n+3}^4 & x_{n+3}^5 & x_{n+3}^6 & x_{n+3}^7 \\ 0 & 1 & 2\bar{x}_n & 3\bar{x}_n^2 & 4\bar{x}_n^3 & 5\bar{x}_n^4 & 6\bar{x}_n^5 & 7\bar{x}_n^6 \\ 0 & 1 & 2\bar{x}_{n+1} & 3\bar{x}_{n+1}^2 & 4\bar{x}_{n+1}^3 & 5\bar{x}_{n+1}^4 & 6\bar{x}_{n+1}^5 & 7\bar{x}_{n+1}^6 \\ 0 & 1 & 2\bar{x}_{n+2} & 3\bar{x}_{n+2}^2 & 4\bar{x}_{n+2}^3 & 5\bar{x}_{n+2}^4 & 6\bar{x}_{n+2}^5 & 7\bar{x}_{n+2}^6 \\ 0 & 1 & 2\bar{x}_{n+3} & 3\bar{x}_{n+3}^2 & 4\bar{x}_{n+3}^3 & 5\bar{x}_{n+3}^4 & 6\bar{x}_{n+3}^5 & 7\bar{x}_{n+3}^6 \end{pmatrix} \quad (17)$$

and we obtain

$$\begin{aligned} \bar{y}(x) = & \phi_0(x)y_n + \phi_1(x)y_{n+1} + \phi_2(x)y_{n+2} + \phi_3(x)y_{n+3} \\ & + [\psi_0(x)f_n + \psi_1(x)f_{n+1} + \psi_2(x)f_{n+2} + \psi_3(x)f_{n+3}]. \end{aligned} \quad (18)$$

as the continuous scheme, where also

$$\phi_0(x) = \left[\frac{11\zeta^7 - 129h\zeta^6 + 602h^2\zeta^5 - 1410h^3\zeta^4 + 1691h^4\zeta^3 - 873h^5\zeta^2 + 108h^6}{108h^7} \right],$$

$$\phi_1(x) = \left[\frac{\zeta^7 - 10h\zeta^6 + 37h^2\zeta^5 - 60h^3\zeta^4 + 36h^4\zeta^3}{4h^7} \right],$$

$$\phi_2(x) = \left[\frac{-\zeta^7 + 11h\zeta^6 - 46h^2\zeta^5 + 90h^3\zeta^4 - 81h^4\zeta^3 + 27h^5\zeta^2}{4h^7} \right],$$

$$\phi_3(x) = \left[\frac{-11\zeta^7 + 102h\zeta^6 - 359h^2\zeta^5 + 600h^3\zeta^4 - 476h^4\zeta^3 + 144h^5\zeta^2}{108h^7} \right],$$

$$\psi_0(x) = \left[\frac{\zeta^7 - 12h\zeta^6 + 58h^2\zeta^5 - 144h^3\zeta^4 + 193h^4\zeta^3 - 132h^5\zeta^2 + 36h^6\zeta}{36h^6} \right],$$

$$\psi_1(x) = \left[\frac{\zeta^7 - 11h\zeta^6 + 47h^2\zeta^5 - 97h^3\zeta^4 + 96h^4\zeta^3 - 36h^5\zeta^2}{4h^6} \right],$$

$$\psi_2(x) = \left[\frac{\zeta^7 - 10h\zeta^6 + 38h^2\zeta^5 - 68h^3\zeta^4 + 57h^4\zeta^3 - 18h^5\zeta^2}{4h^6} \right],$$

$$\psi_3(x) = \left[\frac{\zeta^7 - 9h\zeta^6 + 31h^2\zeta^5 - 51h^3\zeta^4 + 40h^4\zeta^3 - 12h^5\zeta^2}{36h^6} \right].$$

Evaluating the continuous scheme (18) we obtain the block method first and then converted the block method into general linear method:

0	0	0	0	0	0	0	0	0	0	1
$\frac{25}{512}$	0	$\frac{225}{512}$	0	$\frac{-75}{512}$	0	$\frac{-5}{512}$	$\frac{61}{1536}$	$\frac{125}{512}$	$\frac{225}{512}$	$\frac{425}{1536}$
$\frac{155}{1305}$	$\frac{768}{1305}$	$\frac{-45}{1305}$	0	$\frac{-162}{1305}$	0	$\frac{-13}{1305}$	$\frac{31}{783}$	$\frac{5}{29}$	0	$\frac{617}{783}$
$\frac{3}{512}$	0	$\frac{81}{512}$	0	$\frac{-81}{512}$	0	$\frac{-3}{512}$	$\frac{13}{512}$	$\frac{243}{512}$	$\frac{243}{512}$	$\frac{13}{512}$
$\frac{1}{405}$	0	$\frac{81}{405}$	$\frac{256}{405}$	$\frac{81}{405}$	0	$\frac{1}{405}$	$\frac{-1}{81}$	0	1	$\frac{1}{81}$
$\frac{5}{512}$	0	$\frac{75}{512}$	0	$\frac{225}{512}$	0	$\frac{-25}{512}$	$\frac{425}{1536}$	$\frac{225}{512}$	$\frac{125}{512}$	$\frac{61}{1536}$
$\frac{-39}{3085}$	0	$\frac{-495}{3085}$	0	$\frac{-135}{3085}$	$\frac{2304}{3085}$	$\frac{465}{3085}$	0	$\frac{783}{617}$	$\frac{-135}{617}$	$\frac{31}{617}$
$\frac{-39}{3085}$	0	$\frac{-495}{3085}$	0	$\frac{-135}{3085}$	$\frac{2304}{3085}$	$\frac{465}{3085}$	0	$\frac{783}{617}$	$\frac{-135}{617}$	$\frac{31}{617}$
$\frac{5}{512}$	0	$\frac{75}{512}$	0	$\frac{225}{512}$	0	$\frac{-25}{512}$	$\frac{425}{1536}$	$\frac{225}{512}$	$\frac{125}{512}$	$\frac{61}{1536}$
$\frac{1}{405}$	0	$\frac{81}{405}$	$\frac{256}{405}$	$\frac{81}{405}$	0	$\frac{1}{405}$	$\frac{-1}{81}$	0	1	$\frac{1}{81}$
$\frac{155}{1305}$	$\frac{768}{1305}$	$\frac{-45}{1305}$	0	$\frac{-162}{1305}$	0	$\frac{-13}{1305}$	$\frac{31}{783}$	$\frac{5}{29}$	0	$\frac{617}{783}$

(19)

We plotted the region of absolute stability of the general linear method (19) using the method used in [9]:

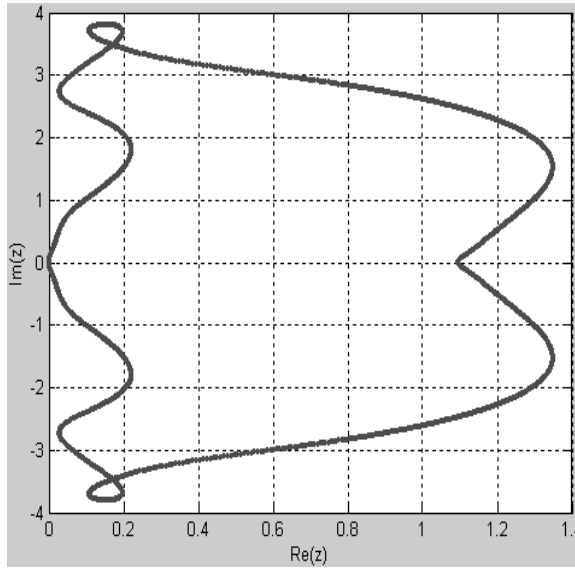


Figure 2 – Region of absolute stability of the general linear method (19).

For $k = 4$, the matrix D in (12) and the polynomial equation in (4) are respectively:

$$D = \begin{pmatrix}
 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 & x_n^9 \\
 1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & x_{n+1}^5 & x_{n+1}^6 & x_{n+1}^7 & x_{n+1}^8 & x_{n+1}^9 \\
 1 & x_{n+2} & x_{n+2}^2 & x_{n+2}^3 & x_{n+2}^4 & x_{n+2}^5 & x_{n+2}^6 & x_{n+2}^7 & x_{n+2}^8 & x_{n+2}^9 \\
 1 & x_{n+3} & x_{n+3}^2 & x_{n+3}^3 & x_{n+3}^4 & x_{n+3}^5 & x_{n+3}^6 & x_{n+3}^7 & x_{n+3}^8 & x_{n+3}^9 \\
 1 & x_{n+4} & x_{n+4}^2 & x_{n+4}^3 & x_{n+4}^4 & x_{n+4}^5 & x_{n+4}^6 & x_{n+4}^7 & x_{n+4}^8 & x_{n+4}^9 \\
 0 & 1 & 2\bar{x}_n & 3\bar{x}_n^2 & 4\bar{x}_n^3 & 5\bar{x}_n^4 & 6\bar{x}_n^5 & 7\bar{x}_n^6 & 8\bar{x}_n^7 & 9\bar{x}_n^8 \\
 0 & 1 & 2\bar{x}_{n+1} & 3\bar{x}_{n+1}^2 & 4\bar{x}_{n+1}^3 & 5\bar{x}_{n+1}^4 & 6\bar{x}_{n+1}^5 & 7\bar{x}_{n+1}^6 & 8\bar{x}_{n+1}^7 & 9\bar{x}_{n+1}^8 \\
 0 & 1 & 2\bar{x}_{n+2} & 3\bar{x}_{n+2}^2 & 4\bar{x}_{n+2}^3 & 5\bar{x}_{n+2}^4 & 6\bar{x}_{n+2}^5 & 7\bar{x}_{n+2}^6 & 8\bar{x}_{n+2}^7 & 9\bar{x}_{n+2}^8 \\
 0 & 1 & 2\bar{x}_{n+3} & 3\bar{x}_{n+3}^2 & 4\bar{x}_{n+3}^3 & 5\bar{x}_{n+3}^4 & 6\bar{x}_{n+3}^5 & 7\bar{x}_{n+3}^6 & 8\bar{x}_{n+3}^7 & 9\bar{x}_{n+3}^8 \\
 0 & 1 & 2\bar{x}_{n+4} & 3\bar{x}_{n+4}^2 & 4\bar{x}_{n+4}^3 & 5\bar{x}_{n+4}^4 & 6\bar{x}_{n+4}^5 & 7\bar{x}_{n+4}^6 & 8\bar{x}_{n+4}^7 & 9\bar{x}_{n+4}^8
 \end{pmatrix} \quad (20)$$

and

$$\bar{y}(x) = \phi_0(x)y_n + \phi_1(x)y_{n+1} + \phi_2(x)y_{n+2} + \phi_3(x)y_{n+3} + \phi_4(x)y_{n+4} \\ + [\psi_0(x)f_n + \psi_1(x)f_{n+1} + \psi_2(x)f_{n+2} + \psi_3(x)f_{n+3} + \psi_4(x)f_{n+4}] \quad (21)$$

where

$$\phi_0(x) = \left[\frac{25\zeta^9 - 494h\zeta^8 + 4130h^2\zeta^7 - 18980h^3\zeta^6 + 52025h^4\zeta^5}{3456h^9} \right],$$

$$\phi_1(x) = \left[\frac{5\zeta^9 - 92h\zeta^8 + 701h^2\zeta^7 - 2846h^3\zeta^6 + 6572h^4\zeta^5}{108h^9} \right],$$

$$\phi_2(x) = \left[\frac{\zeta^8 - 16h\zeta^7 + 102h^2\zeta^6 - 328h^3\zeta^5 + 553h^4\zeta^4 - 456h^5\zeta^3 + 144h^6\zeta^2}{16h^8} \right],$$

$$\phi_3(x) = \left[\frac{-5\zeta^9 + 88h\zeta^8 - 637h^2\zeta^7 + 2446h^3\zeta^6 - 5356h^4\zeta^5}{108h^9} \right],$$

$$\phi_4(x) = \left[\frac{-25\zeta^9 + 406h\zeta^8 - 2722h^2\zeta^7 + 9748h^3\zeta^6 - 20089h^4\zeta^5}{3456h^9} \right],$$

$$\psi_0(x) = \left[\frac{\zeta^9 - 20h\zeta^8 + 170h^2\zeta^7 - 800h^3\zeta^6 + 2273h^4\zeta^5 - 3980h^5\zeta^4}{576h^8} \right],$$

$$\psi_1(x) = \left[\frac{\zeta^9 - 19h\zeta^8 + 151h^2\zeta^7 - 649h^3\zeta^6 + 1624h^4\zeta^5}{36h^8} \right],$$

$$\psi_2(x) = \left[\frac{\zeta^9 - 18h\zeta^8 + 134h^2\zeta^7 - 532h^3\zeta^6 + 1209h^4\zeta^5 - 1562h^5\zeta^4 + 1056h^6\zeta^3 - 288h^7\zeta^2}{16h^8} \right],$$

$$\psi_3(x) = \left[\frac{\zeta^9 - 17h\zeta^8 + 119h^2\zeta^7 - 443h^3\zeta^6 + 944h^4\zeta^5 - 1148h^5\zeta^4 + 736h^6\zeta^3 - 192h^7\zeta^2}{36h^8} \right],$$

$$\psi_4(x) = \left[\frac{\zeta^9 - 16h\zeta^8 + 106h^2\zeta^7 - 376h^3\zeta^6 + 769h^4\zeta^5 - 904h^5\zeta^4 + 564h^6\zeta^3 - 144h^7\zeta^2}{576h^8} \right],$$

Evaluating the continuous scheme in (21) we obtain the block method which was converted to general linear method:

0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
$\frac{1225}{32768}$	0	$-\frac{1225}{2048}$	0	$-\frac{3675}{8192}$	0	$-\frac{245}{2048}$	0	$-\frac{175}{32768}$	$\frac{4675}{196608}$	$\frac{1519}{6144}$	$\frac{1225}{4096}$	$\frac{1225}{6144}$	$\frac{45325}{196608}$	
$\frac{741}{9488}$	$\frac{9216}{20755}$	$-\frac{111}{593}$	0	$-\frac{963}{2372}$	0	$-\frac{363}{2965}$	0	$-\frac{381}{66416}$	$\frac{481}{18976}$	$\frac{145}{593}$	$\frac{108}{593}$	0	$\frac{10399}{18976}$	
$\frac{75}{32768}$	0	$\frac{225}{2048}$	0	$-\frac{2025}{8192}$	0	$-\frac{75}{2048}$	0	$-\frac{45}{32768}$	$\frac{411}{65536}$	$\frac{175}{2048}$	$\frac{2025}{4096}$	$\frac{825}{2048}$	$\frac{725}{65536}$	
$\frac{11}{5184}$	0	$\frac{21}{108}$	$\frac{256}{405}$	$\frac{3}{16}$	0	$-\frac{1}{324}$	0	$-\frac{1}{2880}$	$\frac{5}{3456}$	$-\frac{1}{324}$	0	$\frac{1712}{1728}$	$\frac{113}{10368}$	
$\frac{45}{32768}$	0	$\frac{75}{2048}$	0	$\frac{2025}{8192}$	0	$-\frac{225}{2048}$	0	$-\frac{75}{32768}$	$\frac{725}{65536}$	$\frac{825}{2048}$	$\frac{2025}{4096}$	$\frac{175}{2048}$	$\frac{411}{65536}$	
$-\frac{3}{8560}$	0	$-\frac{1}{321}$	0	$\frac{81}{428}$	$\frac{1024}{1605}$	$\frac{21}{107}$	0	$\frac{11}{5136}$	$-\frac{113}{10272}$	0	$\frac{108}{107}$	$\frac{1}{321}$	$-\frac{5}{3424}$	
$\frac{175}{32768}$	0	$\frac{245}{2048}$	0	$\frac{3675}{8192}$	0	$\frac{1225}{2048}$	0	$-\frac{1225}{32768}$	$\frac{45325}{196608}$	$\frac{1225}{6144}$	$\frac{1225}{4096}$	$\frac{1519}{6144}$	$\frac{4675}{196608}$	
$-\frac{762}{72793}$	0	$\frac{616}{51995}$	0	$-\frac{7704}{10399}$	0	$-\frac{3552}{10399}$	$\frac{294912}{363965}$	$\frac{1482}{10399}$	0	$\frac{18976}{10399}$	$-\frac{3456}{10399}$	$-\frac{4640}{10399}$	$-\frac{481}{10399}$	
$-\frac{762}{72793}$	0	$\frac{616}{51995}$	0	$-\frac{7704}{10399}$	0	$-\frac{3552}{10399}$	$\frac{294912}{363965}$	$\frac{1482}{10399}$	0	$\frac{18976}{10399}$	$-\frac{3456}{10399}$	$-\frac{4640}{10399}$	$-\frac{481}{10399}$	
$\frac{175}{32768}$	0	$\frac{245}{2048}$	0	$\frac{3675}{8192}$	0	$\frac{1225}{2048}$	0	$-\frac{1225}{32768}$	$\frac{45325}{196608}$	$\frac{1225}{6144}$	$\frac{1225}{4096}$	$\frac{1519}{6144}$	$\frac{4675}{196608}$	
$-\frac{3}{8560}$	0	$-\frac{1}{321}$	0	$\frac{81}{428}$	$\frac{1024}{1605}$	$\frac{21}{107}$	0	$\frac{11}{5136}$	$-\frac{113}{10272}$	0	$\frac{108}{107}$	$\frac{1}{321}$	$-\frac{5}{3424}$	
$\frac{11}{5184}$	0	$\frac{21}{108}$	$\frac{256}{405}$	$\frac{3}{16}$	0	$-\frac{1}{324}$	0	$-\frac{1}{2880}$	$\frac{5}{3456}$	$-\frac{1}{324}$	0	$\frac{1712}{1728}$	$\frac{113}{10368}$	
$\frac{741}{9488}$	$\frac{9216}{20755}$	$-\frac{111}{593}$	0	$-\frac{963}{2372}$	0	$-\frac{363}{2965}$	0	$-\frac{381}{66416}$	$\frac{481}{18976}$	$\frac{145}{593}$	$\frac{108}{593}$	0	$\frac{10399}{18976}$	

5 Numerical illustrations

In order to test the methods of section 4 we present some numerical results. The absolute errors of the results obtained from computed and exact solutions at some selected mesh points are shown in Tables.

$$\text{Problem 5.1: } y' = 20x^2 - 20y + 2x, \quad y(0) = \frac{1}{3}, \quad y(x) = x^2 + \frac{1}{3}e^{20x}$$

Mesh values	Block Adams-Moulton [15]	New General Linear Method (19)
0.1	1.0629×10^{-2}	1.5230×10^{-5}
0.2	5.3890×10^{-3}	1.2075×10^{-5}
0.3	1.2320×10^{-2}	1.6286×10^{-4}
0.4	1.3008×10^{-3}	2.1996×10^{-5}
0.5	4.1148×10^{-4}	3.0206×10^{-6}
0.6	3.9430×10^{-4}	8.8700×10^{-7}
0.7	4.0724×10^{-5}	1.1990×10^{-7}
0.8	1.3629×10^{-5}	1.6357×10^{-8}
0.9	1.3672×10^{-5}	3.6327×10^{-9}
1.0	1.4145×10^{-6}	4.8996×10^{-10}

Table 1 – Absolute errors of numerical solutions of Problem 5.1 with, $h = 0.1$.

$$\text{Problem 5.2: } y' = -y, \quad y(0) = 1, \quad y(x) = e^{-x}$$

Mesh values	Block Adams-Moulton [15]	New General Linear Method (22)
0.1	2.1541×10^{-7}	2.5817×10^{-13}
0.2	6.9544×10^{-8}	2.3105×10^{-13}
0.3	2.8062×10^{-7}	7.4733×10^{-13}
0.4	4.1350×10^{-7}	1.2390×10^{-13}
0.5	2.8127×10^{-7}	1.1002×10^{-13}
0.6	4.1578×10^{-7}	1.1066×10^{-13}
0.7	4.9444×10^{-7}	4.1514×10^{-14}
0.8	3.7858×10^{-7}	3.6091×10^{-14}
0.9	4.6203×10^{-7}	1.2298×10^{-13}
1.0	5.0564×10^{-7}	6.3784×10^{-15}

Table 2 – Absolute errors of numerical solutions of Problem 5.2, with, $h = 0.1$.

Problem 5.3: $y' = -\lambda y$, $y(0) = 1$, $y(x) = e^{-\lambda x}$

Mesh values	Block Adams-Moulton [15]	New General Linear Method(16)
0.1	7.4622×10^{-2}	6.3759×10^{-3}
0.2	9.7739×10^{-2}	4.6451×10^{-3}
0.3	2.2659×10^{-4}	3.8855×10^{-4}
0.4	1.0870×10^{-2}	8.4175×10^{-5}
0.5	7.7732×10^{-5}	6.9921×10^{-6}
0.6	1.1401×10^{-3}	1.1690×10^{-6}
0.7	8.8455×10^{-6}	9.6903×10^{-8}
0.8	1.1914×10^{-4}	1.4728×10^{-8}
0.9	8.8900×10^{-7}	1.2197×10^{-9}
1.0	1.2448×10^{-5}	1.7723×10^{-10}

Table 3 – Absolute errors of numerical solutions of Problem 5.3, with stiffness ratio, $\lambda = 10000$.

The fourth problem is a system of standard test problem with the exact solutions for easy comparison purposes:

$$y_1' = -8y_1 + 7y_2, \quad y_1(0) = 1$$

$$y_2' = 42y_1 - 43y_2, \quad y_2(0) = 8$$

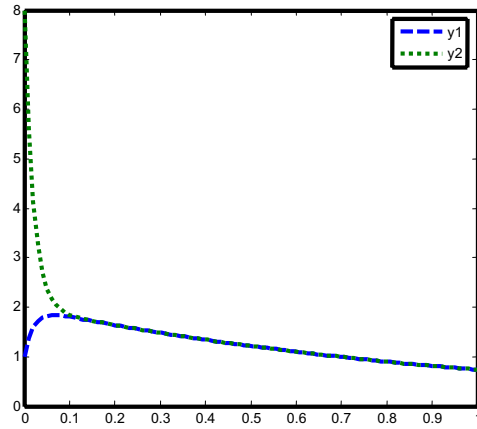
The coefficient matrix of this problem has two eigenvalues, $\lambda_1 = -1$ and $\lambda_2 = -50$. The stiffness ratio is $R = 50$. This is a mildly stiff linear problem with the exact solutions as:

$$y_1(x) = 2 \exp(-x) - \exp(-50x)$$

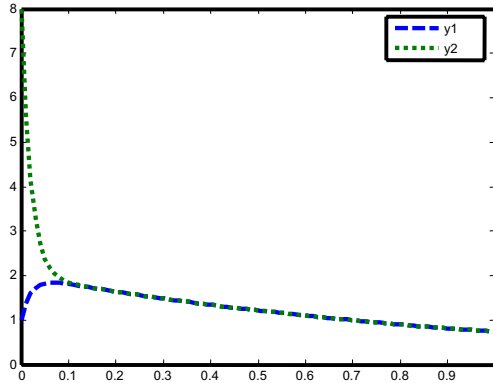
$$y_2(x) = 2 \exp(-x) + 6 \exp(-50x)$$

This problem shows that to solve stiff equations the stability of a good method should impose no limitation on the step size, and hence it requires a large stability region. From the plots of Figure 3, it is indicated that both the block Adams-Moulton methods(BAMMs)and the GLMs are good methods for stiff equations. Though, for the BMM as k increases the method becomes less stable (see Table 4). For the GLMs the numerical results in Table 4 show that the new GLMs are very promising and the implementation is reasonably efficient. Their performances are no doubt very excellent.

Solution of problem 4.4 using BAMM $k = 2$, with nfe = 100



Solution of problem 4.4 using BAMM $k = 3$, with nfe = 100



Solution of problem 4.4 using BAMM $k = 4$, with nfe = 100

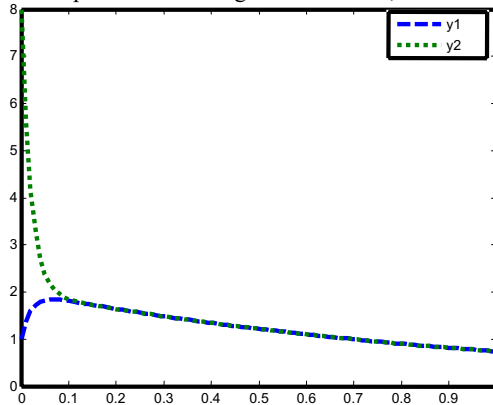
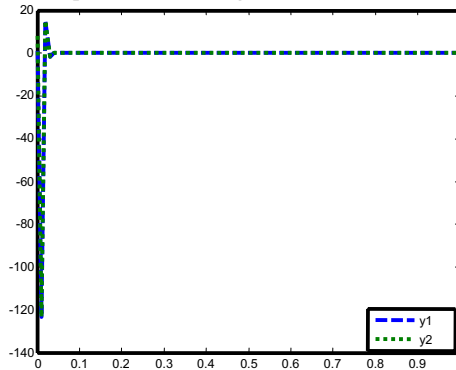
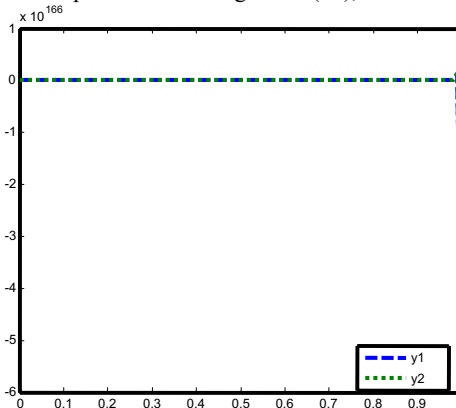


Figure 3 – Computed solutions of the system of equations in (4.4) using BAMMs [15] and the GMLs with the same number of functions evaluations (nfe).

Solution of problem 4.4 using GLM (16), with nfe = 100



Solution of problem 4.4 using GLM (19), with nfe = 100



Solution of problem 4.4 using GLM (22), with nfe = 100

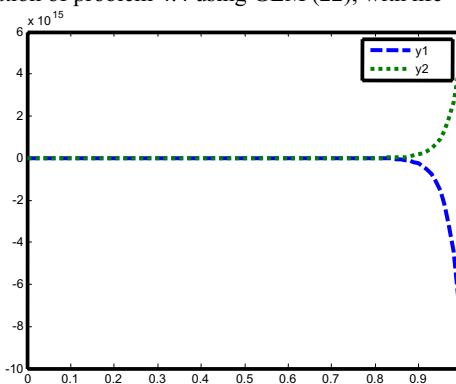
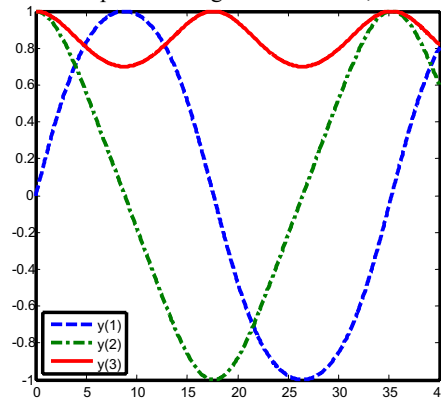


Figure 3 (continuation) – Computed solutions of the system of equations in (4.4) using BAMMs [15] and the GLMs with the same number of functions evaluations (nfe).

Mesh values	BAMM [15]	BAMM [15]	BAMM [15]	GLM (16)	GLM (19)	GLM (22)
10.0	2.05×10^{-4}	5.15×10^{-5}	9.82×10^{-5}	5.77×10^{-5}	1.14×10^{-17}	1.04×10^{-2}
20.0	2.73×10^{-6}	6.94×10^{-7}	8.05×10^{-7}	1.51×10^{-10}	4.92×10^{-33}	3.81×10^{-3}
30.0	7.94×10^{-9}	6.65×10^{-9}	7.04×10^{-7}	2.75×10^{-24}	2.12×10^{-50}	1.34×10^{-5}
40.0	2.24×10^{-8}	8.62×10^{-11}	8.71×10^{-7}	1.90×10^{-33}	9.18×10^{-66}	4.74×10^{-6}
50.0	2.57×10^{-8}	1.63×10^{-10}	9.89×10^{-7}	1.31×10^{-42}	3.96×10^{-83}	1.67×10^{-8}
60.0	2.80×10^{-8}	1.78×10^{-10}	1.07×10^{-6}	9.05×10^{-52}	1.71×10^{-100}	5.90×10^{-9}
70.0	2.96×10^{-8}	1.88×10^{-10}	1.13×10^{-6}	6.24×10^{-61}	7.40×10^{-116}	2.08×10^{-11}
80.0	3.06×10^{-8}	1.95×10^{-10}	1.17×10^{-6}	4.31×10^{-70}	3.19×10^{-133}	7.33×10^{-12}
90.0	3.12×10^{-8}	1.98×10^{-10}	1.20×10^{-6}	2.97×10^{-79}	1.38×10^{-150}	2.58×10^{-14}
100	3.14×10^{-8}	1.99×10^{-10}	1.20×10^{-6}	2.05×10^{-88}	5.96×10^{-166}	9.12×10^{-15}

Table 4 – Absolute errors of numerical solutions of the systems of equations.

Solution of example 4.5 using BAMM $k = 2$, with nfe = 100



Solution of example 4.5 using BAMM $k = 3$, with nfe = 100

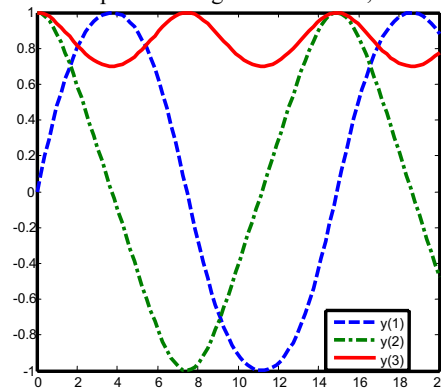
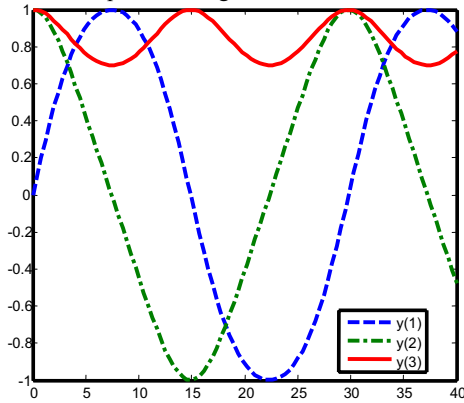
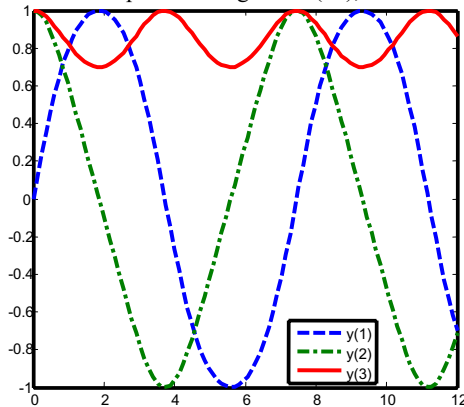


Figure 4 – Computed solutions of the system of equations in (4.5) using BAMMs [15] and the GLMs with indicated number of functions evaluations (nfe).

Solution of example 4.5 using BAMM $k = 4$, with nfe = 100

Solution of example 4.5 using GLM (16), with nfe = 100



Solution of example 4.5 using GLM (19), with nfe = 100

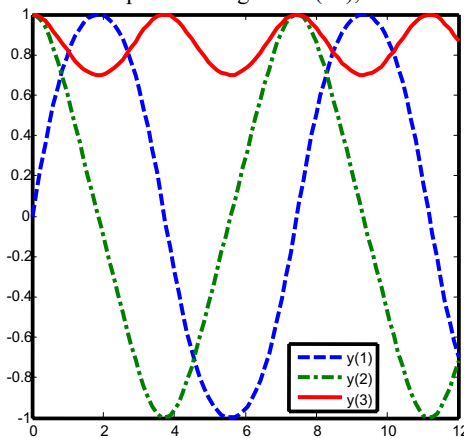
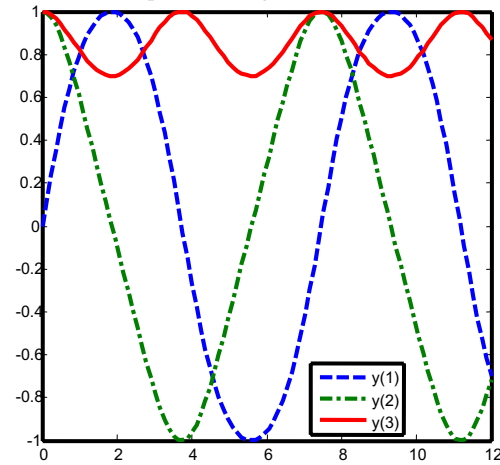


Figure 4 (continuation) – Computed solutions of the system of equations in (4.5) using BAMMs [15] and the GLMs with indicated number of functions evaluations (nfe).

Solution of example 4.5 using GLM (22), with nfe = 100



See ODE45, ODE23, ODE113 of Matlab Work

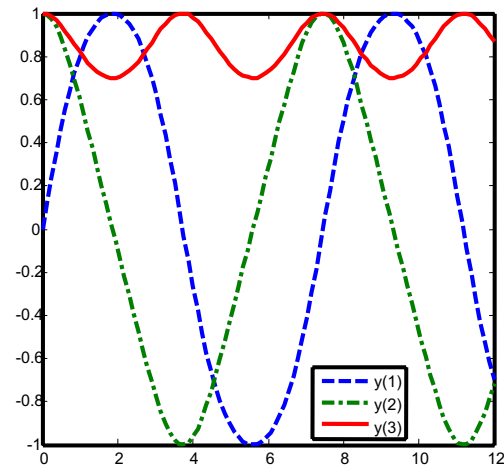


Figure 4 (continuation) – Computed solutions of the system of equations in (4.5) using BAMMs [15] and the GLMs with indicated number of functions evaluations (nfe).

In Figure 4 we report the graphical plots of the Euler equation of motion for a rigid body without external forces which is one of the standard test problems of the DETest set, see Hull et al. (1972):

$$\begin{aligned} y_1' &= y_2 y_3, & y_1(0) &= 0 \\ y_2' &= -y_1 y_3, & y_2(0) &= 1 \\ y_3' &= -5.1 y_1 y_2, & y_3(0) &= 1 \end{aligned}$$

Our graphical plots confirm that all the derived methods are promising in solving higher order equation written in form of first order system of initial value problems. Hence for fair comparison all the methods are comparable with the ode solvers as we can see from Figure 4.

6 Conclusion

The new feature considered in this paper is the use of matrix inversion procedure which extends some conventional multistep collocation at the step points. In this way acceptable stability for stiff problems as for the Runge-Kutta methods [3] is retained. All the derived methods obtained through this approach performed remarkably well in both stiff and non-stiff systems of initial value problem in ordinary differential equations (see Tables 1, 2, 3 and 4). The plots of the fourth test problem are system of differential equations written as first order initial value problems. We plotted these solutions and compare them with the exact values, we found out that it is difficult to distinguish between the computed solutions and the exact values on the interval of integration and there is remarkable agreement over very much longer intervals (see Fig. 3). Similarly the solutions of the fifth problem were compared with ODE solvers (see Fig. 4).

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