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Spectral properties of the preconditioned AHSS iteration method for generalized saddle point problems

ZHUO-HONG HUANG and TING-ZHU HUANG

School of Applied Mathematics, University of Electronic Science and Technology of China
Chengdu, Sichuan, 610054, P.R. China

E-mails: zhuohonghuang@yahoo.cn / tzhuang@uestc.edu.cn / tingzhuhuang@126.com

Abstract. In this paper, we study the distribution on the eigenvalues of the preconditioned matrices that arise in solving two-by-two block non-Hermitian positive semidefinite linear systems by use of the accelerated Hermitian and skew-Hermitian splitting iteration methods. According to theoretical analysis, we prove that all eigenvalues of the preconditioned matrices are very clustered with any positive iteration parameters α and β; especially, when the iteration parameters α and β approximate to 1, all eigenvalues approach 1. We also prove that the real parts of all eigenvalues of the preconditioned matrices are positive, i.e., the preconditioned matrix is positive stable. Numerical experiments show the correctness and feasibility of the theoretical analysis.

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Key words: PAHSS, generalized saddle point problem, splitting iteration method, positive stable.

1 Introduction

Let us first consider the nonsingular saddle point system $Ax = b$ as follows:

$$A = \begin{pmatrix} B & E \\ -E^* & D \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}, \tag{1}$$

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where, $B \in C^{n\times n}$ is Hermitian positive definite, $D \in C^{m\times m}$ is Hermitian positive semidefinite, $E \in C^{n\times m}$ $(n \geq m)$ has full column rank, $f \in C^n$, $g \in C^m$, and E^* denotes the conjugate transpose of E.

We review the Hermitian and skew-Hermitian splitting:

$$A = H + S,$$

where

$$H = \frac{1}{2}(A + A^*) = \begin{pmatrix} B & 0 \\ 0 & D \end{pmatrix} \quad \text{and} \quad S = \frac{1}{2}(A - A^*) = \begin{pmatrix} 0 & E \\ -E^* & 0 \end{pmatrix}. \tag{2}$$

Obviously, H is a Hermitian positive semidefinite matrix, and S is a skew-Hermitian matrix, see [1].

To solve the linear system (1), we have usually used efficient splittings of the coefficient matrix A. Many studies have shown that the Hermitian and skew-Hermitian splitting (HSS) iteration method is very efficient, see e.g., [1–15]. In particular, Benzi and Golub [2] considered the HSS iteration method and pointed out that it converges unconditionally to the unique solution of the saddle point linear system (1) for any iteration parameter. In the case of $D = 0$, Bai et al. [3] proposed the PHSS iteration method and showed the advantages of the PHSS iteration method over the HSS iteration method by solving the Stokes problem. Bai et al. [4] generalized the PHSS iteration method by introducing two iteration parameters and proved theoretically the convergence rate of the obtained AHSS iterative method is faster than that of the PHSS iteration method, when they are applied to solve the saddle point problems. Under the condition that B is symmetric and positive definite and $D = 0$, Simoncini and Benzi [5] estimated bounds on the spectral radius of the preconditioned matrix of the HSS iteration method and pointed out that any eigenvalue (denote by λ) of the preconditioned matrix approximates to 0 or 2, i.e., $\lambda \in (0, \varepsilon_1) \cup (\varepsilon_2, 2)$, with $\varepsilon_1, \varepsilon_2 > 0$ and $\varepsilon_1, \varepsilon_2 \to 0$, as $\alpha \to 0$; meanwhile, they pointed out that all eigenvalues are real while the iteration parameter $\alpha \leq \frac{1}{2}\lambda_n$, where λ_n is the smallest eigenvalue of B; Bychenkov [6] obtained more accurate result than that in [5] and believed that all eigenvalues are real, as the iteration parameter $\alpha \leq \lambda_n$. Chan, Ng and Tsing [7] studied the spectral analysis of the preconditioned matrix of the HSS iteration method for the generalized saddle point problem for the case of $D = \mu I_m$, researched the spectral properties of the preconditioned matrices, and gave sufficient conditions that all eigenvalues of the preconditioned matrix are real. Huang, Wu and Li [8] studied the spectral properties of the preconditioned matrix of the HSS iteration method for nonsymmetric generalized saddle problems and pointed out that the eigenvalues of the preconditioned matrix gather to $(0, 0)$ or $(2, 0)$ on the complex plane as the iteration parameter approaches 0. Benzi [9] presented a generalized HSS (GHSS) iteration method by splitting H into the sum of two Hermitian positive semidefinite matrices.

In [1, 2], the following Hermitian and skew-Hermitian splitting iteration method was used to solve the large sparse non-Hermitian positive semidefinite linear system (1) with $D = 0$:

$$\begin{cases} \big(\alpha I_{n+m} + H\big)x^{k+\frac{1}{2}} = \big(\alpha I_{n+m} - S\big)x^k + b, \\ \big(\alpha I_{n+m} + S\big)x^{k+1} = \big(\alpha I_{n+m} - H\big)x^{k+\frac{1}{2}} + b, \end{cases} \tag{3}$$

where α is a given positive constant and I_{n+m} is the identity matrix of order $n + m$. The equation (3) can be rewritten as

$$x^{k+1} = M(\alpha)x^k + N(\alpha)b,$$

where

$$M(\alpha) = \big(\alpha I_{n+m} + S\big)^{-1}\big(\alpha I_{n+m} - H\big)\big(\alpha I_{n+m} + H\big)^{-1}\big(\alpha I_{n+m} - S\big),$$
$$N(\alpha) = 2\alpha\big(\alpha I_{n+m} + S\big)^{-1}\big(\alpha I_{n+m} + H\big)^{-1}.$$

By simple manipulation, the authors of [1, 2] obtained the preconditioner of the following form

$$P(\alpha) = [N(\alpha)]^{-1} = (2\alpha)^{-1}\big(\alpha I_{n+m} + S\big)\big(\alpha I_{n+m} + H\big).$$

According to theoretical analysis, they proved the spectral radius $\rho(M(\alpha)) < 1$ and the optimal iteration parameter

$$\alpha^* = \arg\min_{\alpha}\left\{\max_{\gamma_{\min}\le\lambda\le\gamma_{\max}}\left|\frac{\alpha-\lambda}{\alpha+\lambda}\right|\right\} = \sqrt{\gamma_{\min}\gamma_{\max}},$$

where $\gamma_{\min}$, $\gamma_{\max}$ and λ denote the minimum, the maximum and the arbitrary eigenvalue of H, respectively.

We review the accelerated Hermitian and skew-Hermitian splitting (AHSS) iteration method established in [4] and first consider the simpler case that $D = 0$. Let $U \in C^{n\times n}$ be nonsingular such that $U^*BU = I_n$, and $V \in C^{m\times m}$ be also nonsingular. We denote by

$$\tilde{E} = U^*EV, \qquad F = (V^*)^{-1}V^{-1},$$

$$P = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \quad \text{and} \quad \tilde{A} = P^*AP = \begin{pmatrix} I_n & \tilde{E} \\ -\tilde{E}^* & 0 \end{pmatrix}.$$

Then, the linear system (1) is equivalent to

$$\tilde{A}\tilde{x} = \tilde{b},$$

where

$$\tilde{A} = \tilde{H} + \tilde{S},$$

with

$$\tilde{H} = \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \tilde{S} = \begin{pmatrix} 0 & \tilde{E} \\ -\tilde{E}^* & 0 \end{pmatrix}.$$

Therefore, the AHSS iteration method proposed in [4] can be written as follows:

$$\begin{cases} (\Lambda + \tilde{H})\tilde{x}^{k+\frac{1}{2}} = (\Lambda - \tilde{S})x^k + \tilde{b}, \\ (\Lambda + \tilde{S})\tilde{x}^{k+1} = (\Lambda - \tilde{H})x^{k+\frac{1}{2}} + \tilde{b}, \end{cases} \tag{4}$$

where

$$\Lambda = \begin{pmatrix} \alpha I_n & 0 \\ 0 & \beta I_m \end{pmatrix},$$

with α and β are any positive constants.

Further, by straightforward computation, it is easy to see that

$$A = M(\alpha,\beta) - N(\alpha,\beta) \quad \text{and} \quad \tilde{A} = \tilde{M}(\alpha,\beta) - \tilde{N}(\alpha,\beta),$$

where

$$M(\alpha,\beta) = \begin{pmatrix} \frac{\alpha+1}{2}B & \frac{\alpha+1}{2\alpha}E \\ -\frac{1}{2}E^* & \frac{\beta}{2}F \end{pmatrix}, \qquad N(\alpha,\beta) = \begin{pmatrix} \frac{\alpha-1}{2}B & -\frac{\alpha-1}{2\alpha}E \\ \frac{1}{2}E^* & \frac{\beta}{2}F \end{pmatrix}$$

and

$$\tilde{M}(\alpha,\beta)=\begin{pmatrix}\frac{\alpha+1}{2}I_n & \frac{\alpha+1}{2\alpha}\tilde{E}\\ -\frac{1}{2}\tilde{E}^* & \frac{\beta}{2}I_m\end{pmatrix},\quad \tilde{N}(\alpha,\beta)=\begin{pmatrix}\frac{\alpha-1}{2}I_n & -\frac{\alpha-1}{2\alpha}\tilde{E}\\ \frac{1}{2}\tilde{E}^* & \frac{\beta}{2}I_m\end{pmatrix}.$$

Now we consider the general case that $D \neq 0$, Bai and Golub in [4] further extended the AHSS iteration method to solve the generalized saddle point problems and proposed the AHSS preconditioner of the following form

$$M(\alpha,\beta)=\frac{1}{2}\begin{pmatrix}\alpha I_n + B & \frac{1}{\alpha}(\alpha I_n + B)E\\ -\frac{1}{\beta}(\beta I_m + D)E^* & \beta I_m + D\end{pmatrix}. \tag{5}$$

In this paper, we use the AHSS iteration method to solve the generalized saddle point system (1) with D being positive semidefinite. According to the analysis, we easily know that the preconditioner proposed in this paper is different from the AHSS preconditioner in (5). We prove that all eigenvalues of the preconditioned matrices are very clustered with any positive iteration parameters α and β; especially when the iteration parameters α and β approximate to 1, all eigenvalues approach 1. We also prove that the real parts of all eigenvalues of the preconditioned matrix are positive, i.e., the preconditioned matrix is positive stable. Numerical experiments show the correctness and feasibility of the theoretical analysis.

2 Spectral analysis for PAHSS iteration method

Bai [16] studied algebraic properties of the AHSS iteration method for solving the general saddle point problem (1) when $D = 0$ and obtained the optimal parameters. By theoretical analysis and numerical experiments, we easily see that the AHSS iteration method is considerably robust and efficient. For the large sparse generalized saddle-point problems, Bai in [17] only introduced the AHSS iteration methods and the AHSS preconditioner in (5). In this paper, we propose a preconditioner based on the AHSS iteration method, called the preconditioned AHSS (PAHSS) preconditioner, which is different from the AHSS preconditioner in (5), and study in detail the related spectral properties of the PAHSS iteration method. So, the study in this paper is a complement and an extention of that in [17]. In this section, our main contribution is to use the PAHSS iteration method to solve the generalized saddle point problem (1) when D is positive semidefinite and to analyze the spectral properties of the preconditioned matrix. First, we consider the special case that D is a Hermitian positive definite matrix, and we begin our analysis by giving some notations.

Assume that $UB_1^{-1}E(D_1^{-1})^*V^* = \Sigma$ is the singular value decomposition [17, 18], both $U \in C^{n\times n}$ and $V \in C^{m\times m}$ are unitary matrices, where $B = B_1^* B_1$, $D = D_1^* D_1$ and

$$\Sigma_1 = \begin{pmatrix} \Sigma \\ 0 \end{pmatrix}, \qquad \Sigma = \mathrm{diag}\{\sigma_1, \sigma_2, \dots, \sigma_m\} \in C^{m\times m},$$

where σ_i $(i = 1, 2, \dots, m)$ denote the singular values of $B_1^{-1}E(D_1^{-1})^*$.

We apply the following preconditioned AHSS iteration method to solve the generalized saddle point system (1),

$$\begin{cases} (\Lambda + H)x^{k+\frac{1}{2}} = (\Lambda - S)x^k + b, \\ (\Lambda + S)x^{k+1} = (\Lambda - H)x^{k+\frac{1}{2}} + b, \end{cases} \tag{6}$$

where H and S are defined as in (2), and

$$\Lambda = \begin{pmatrix} \alpha B & \\ & \beta D \end{pmatrix},$$

with α and β being any positive constants. When the iterative parameter $\alpha = \beta$, we can easily know that the iteration method (6) reduces to the PHSS iteration method [3]. Bai, Golub and Li proposed in [19] the preconditioned HSS iteration method. When we properly select the matrix P or the parameter matrix in [19], we easily know the AHSS iteration method is a special case of that in [19].

By simple calculation, the iteration scheme (6) can be equivalently written as

$$x^{k+1} = \Phi(\alpha, \beta)x^k + \Psi(\alpha, \beta)b, \tag{7}$$

where

$$\Phi(\alpha, \beta) = (\Lambda + S)^{-1}(\Lambda - H)(\Lambda + H)^{-1}(\Lambda - S),$$
$$\Psi(\alpha, \beta) = 2(\Lambda + S)^{-1}(\Lambda + H)^{-1}\Lambda.$$

After straightforward operations, we obtain the following preconditioner

$$\begin{aligned} M(\alpha,\beta) &= \Psi(\alpha,\beta)^{-1} \\ &= (2\Lambda)^{-1}(\Lambda+H)(\Lambda+S) \\ &= \frac{1}{2}\begin{pmatrix} (\alpha+1)B & \frac{\alpha+1}{\alpha}E \\ -\frac{\beta+1}{\beta}E^* & (\beta+1)D \end{pmatrix}. \end{aligned}$$

It is straightforward to show that

$$A = M(\alpha,\beta) - N(\alpha,\beta),$$

where

$$\begin{aligned} N(\alpha,\beta) &= (2\Lambda)^{-1}(\Lambda-H)(\Lambda-S) \\ &= \frac{1}{2}\begin{pmatrix} (\alpha-1)B & -\frac{\alpha-1}{\alpha}E \\ \frac{\beta-1}{\beta}E^* & (\beta-1)D \end{pmatrix}. \end{aligned}$$

In the following, we denote by

$$T = \begin{pmatrix} UB_1^{-1} & \\ & VD_1^{-1} \end{pmatrix}. \tag{8}$$

Then, we obtain the following two equalities:

$$\tilde{M}(\alpha,\beta) = TM(\alpha,\beta)T^* = \frac{1}{2}\begin{pmatrix} (\alpha+1)I_n & \frac{\alpha+1}{\alpha}\Sigma_1 \\ -\frac{\beta+1}{\beta}\Sigma_1^* & (\beta+1)I_m \end{pmatrix}, \tag{9}$$

$$\tilde{N}(\alpha,\beta) = TN(\alpha,\beta)T^* = \frac{1}{2}\begin{pmatrix} (\alpha-1)I_n & -\frac{\alpha-1}{\alpha}\Sigma_1 \\ \frac{\beta-1}{\beta}\Sigma_1^* & (\beta-1)I_m \end{pmatrix}. \tag{10}$$

According to (9), we further get

$$\tilde{M}(\alpha,\beta)^{-1} = \begin{pmatrix} \frac{2}{\alpha+1}\left(I - \frac{1}{\alpha\beta}\Sigma_1\tilde{S}(\alpha,\beta)^{-1}\Sigma_1^*\right) & -\frac{2}{\alpha(\beta+1)}\Sigma_1\tilde{S}(\alpha,\beta)^{-1} \\ \frac{2}{\beta(\alpha+1)}\tilde{S}(\alpha,\beta)^{-1}\Sigma_1^* & \frac{2}{\beta+1}\tilde{S}(\alpha,\beta)^{-1} \end{pmatrix}, \tag{11}$$

where

$$\tilde{S}(\alpha,\beta) = I_m + \frac{1}{\alpha\beta}\Sigma^2.$$

Subsequently, by analysing the eigenproblem

$$[M(\alpha,\beta)]^{-1}N(\alpha,\beta)x = \lambda x, \tag{12}$$

we obtain the following related properties for the eigenvalues λ of the iteration matrix $[M(\alpha,\beta)]^{-1}N(\alpha,\beta)$.

Theorem 2.1. *Consider the linear system* (1)*, let B and D be Hermitian positive definite matrices, $E \in C^{n\times m}$ have full column rank, and α, β be positive constants. If σ_k $(k = 1, 2, \dots, m)$ are the singular values of the matrix $B_1^{-1}E(D_1^{-1})^*$, where $B = B_1^* B_1$, and $D = D_1^* D_1$, then the eigenvalues of the iteration matrix $[M(\alpha, \beta)]^{-1}N(\alpha, \beta)$ of the $PAHSS$ iteration method are $\frac{\alpha-1}{\alpha+1}$ with multiplicity $n - m$, and, for $k = 1, 2, \dots, m$,* the remainder eigenvalues are

$$\lambda_{k\pm} = \frac{(\alpha\beta - 1)(\alpha\beta - \sigma_k^2) \pm \sqrt{(\alpha - \beta)^2(\alpha\beta + \sigma_k^2)^2 - 4\alpha\beta\sigma_k^2(\alpha\beta - 1)^2}}{(\alpha + 1)(\beta + 1)(\alpha\beta + \sigma_k^2)}.$$

Proof. Equivalently, the eigenvalue problem (12) can be written as the following generalized eigenvalue problem:

$$N(\alpha, \beta)x = \lambda M(\alpha, \beta)x. \tag{13}$$

Then, according to (8) we obtain

$$T^* N(\alpha, \beta)TT^{-1}x = \lambda T^* M(\alpha, \beta)TT^{-1}x.$$

Therefore, according to the formulas (9) and (10), the generalized eigenvalue problem (13) is equivalent to

$$\tilde{N}(\alpha, \beta)\tilde{x} = \lambda\tilde{M}(\alpha, \beta)\tilde{x},$$

where

$$\tilde{x} = T^{-1}x = \begin{pmatrix} BU^{-1}u \\ CV^{-1}v \end{pmatrix},$$

i.e.,

$$[\tilde{M}(\alpha, \beta)]^{-1}\tilde{N}(\alpha, \beta)\tilde{x} = \lambda\tilde{x}.$$

By straightforward computation, we see that

$$\tilde{M}(\alpha, \beta)^{-1}\tilde{N}(\alpha, \beta) = \begin{pmatrix} T(11) & 0 & T(13) \\ 0 & \frac{\alpha-1}{\alpha+1}I_{n-m} & 0 \\ T(31) & 0 & T(33) \end{pmatrix}, \tag{14}$$

where

$$T(11) = \frac{\alpha-1}{\alpha+1} I_n - \frac{2(\alpha\beta-1)}{\alpha\beta(\alpha+1)(\beta+1)} \tilde{S}(\alpha,\beta)^{-1}\Sigma^2,$$
$$T(13) = -\frac{\alpha-1}{\alpha(\alpha+1)}\Sigma + \frac{\alpha-1}{\alpha^2\beta(\alpha+1)} \tilde{S}(\alpha,\beta)^{-1}\Sigma^3 - \frac{\beta-1}{\alpha(\beta+1)}\Sigma\tilde{S}(\alpha,\beta)^{-1},$$
$$T(31) = \frac{2(\alpha\beta-1)}{\beta(\alpha+1)(\beta+1)} \tilde{S}(\alpha,\beta)^{-1}\Sigma,$$
$$T(33) = -\frac{\alpha-1}{\alpha\beta(\alpha+1)} \tilde{S}(\alpha,\beta)^{-1}\Sigma^2 + \frac{\beta-1}{\beta+1}\tilde{S}(\alpha,\beta)^{-1}.$$

For the convenience of our statements, we denote by

$$\tilde{\Gamma}(\alpha,\beta) = \begin{pmatrix} T(11) & T(13) \\ T(31) & T(33) \end{pmatrix}.$$

By [17, Lemma 2.6], we obtain the kth ($k = 1, 2, \ldots, m$) block submatrix of $\tilde{\Gamma}(\alpha,\beta)$:

$$\tilde{\Gamma}(\alpha,\beta)_k = \begin{pmatrix} T(11)_k & T(13)_k \\ T(31)_k & T(33)_k \end{pmatrix} \triangleq \frac{1}{(\alpha+1)(\beta+1)(\alpha\beta+\sigma_k^2)}\Theta(\alpha,\beta)_k,$$

where

$$T(11)_k = \frac{\alpha-1}{\alpha+1} - \frac{2(\alpha\beta-1)}{\alpha\beta(\alpha+1)(\beta+1)}\sigma_k^2\left(1+\frac{1}{\alpha\beta}\sigma_k^2\right)^{-1},$$
$$T(13)_k = -\frac{\alpha-1}{\alpha(\alpha+1)}\sigma_k + \frac{\alpha-1}{\alpha^2\beta(\alpha+1)}\left(1+\frac{1}{\alpha\beta}\sigma_k^2\right)^{-1}\sigma_k^3 - \frac{\beta-1}{\alpha(\beta+1)}\sigma_k\left(1+\frac{1}{\alpha\beta}\sigma_k^2\right)^{-1},$$
$$T(31)_k = \frac{2(\alpha\beta-1)}{\beta(\alpha+1)(\beta+1)}\left(1+\frac{1}{\alpha\beta}\sigma_k^2\right)^{-1}\sigma_k,$$
$$T(33)_k = -\frac{\alpha-1}{\alpha\beta(\alpha+1)}\sigma_k^2\left(1+\frac{1}{\alpha\beta}\sigma_k^2\right)^{-1} + \frac{\beta-1}{\beta+1}\left(1+\frac{1}{\alpha\beta}\sigma_k^2\right)^{-1},$$
$$\Theta(\alpha,\beta)_{k_{11}} = (\alpha-1)(\beta+1)\alpha\beta - (\alpha+1)(\beta-1)\sigma_k^2,$$
$$\Theta(\alpha,\beta)_{k_{22}} = -(\alpha-1)(\beta+1)\sigma_k^2 + (\alpha+1)(\beta-1)\alpha\beta,$$
$$\Theta(\alpha,\beta)_k = \begin{pmatrix} \Theta(\alpha,\beta)_{k_{11}} & -2\beta(\alpha\beta-1)\sigma_k \\ 2\alpha(\alpha\beta-1)\sigma_k & \Theta(\alpha,\beta)_{k_{22}} \end{pmatrix}.$$

We denote by $\tilde{\lambda}$ an arbitrary eigenvalue of $\Theta(\alpha, \beta)_k$. Then it holds that

$$\tilde{\lambda}^2 - 2(\alpha\beta - 1)(\alpha\beta - \sigma_k^2)\tilde{\lambda} + (\alpha^2 - 1)(\beta^2 - 1)(\alpha\beta + \sigma_k^2)^2 = 0.$$

Further, for $k = 1, 2, \ldots, m$, we have

$$\tilde{\lambda}_{k\pm} = (\alpha\beta - 1)(\alpha\beta - \sigma_k^2) \pm \sqrt{(\alpha - \beta)^2(\alpha\beta + \sigma_k^2)^2 - 4\alpha\beta\sigma_k^2(\alpha\beta - 1)^2}. \qquad (15)$$

Therefore, we complete the proof of Theorem 2.1. □

Theorem 2.2. *Let the conditions of Theorem* 2.1 *be satisfied. Then all eigenvalues of* $[M(\alpha, \beta)]^{-1}N(\alpha, \beta)$ *are real, provided* α *and* β *meet one of the following cases:*

i) $\alpha < 1,\ \beta > 1$ *or* $\alpha > 1,\ \beta < 1$*;*

ii) $\alpha = 1, \beta \neq 1$ *or* $\beta = 1, \alpha \neq 1$*;*

iii) $0 < \alpha, \beta < 1$ *or* $\alpha, \beta > 1$, $\alpha \neq \beta$, *and* $\sigma_k < \sigma_-$, *or* $\sigma_k > \sigma_+$, $k = 1, 2, \ldots, m$,

where σ_- *and* σ_+ *are the roots of the quadratic equation:*

$$|\alpha - \beta|\sigma^2 - 2\sqrt{\alpha\beta}|\alpha\beta - 1|\sigma + \alpha\beta|\alpha - \beta| = 0,$$

with

$$\sigma_- = \sqrt{\alpha\beta}\left[|\alpha\beta - 1| - \sqrt{(\alpha^2 - 1)(\beta^2 - 1)}\right] / |\alpha - \beta|,$$

$$\sigma_+ = \sqrt{\alpha\beta}\left[|\alpha\beta - 1| + \sqrt{(\alpha^2 - 1)(\beta^2 - 1)}\right] / |\alpha - \beta|.$$

Proof. According to (15), we obtain that $\tilde{\lambda}_{k\pm}$ $(k = 1, 2, \ldots, m)$ are all real if and only if

$$(\alpha - \beta)^2\left(\alpha\beta + \sigma_k^2\right)^2 - 4\alpha\beta\sigma_k^2(\alpha\beta - 1)^2 \geq 0,$$

i.e.,

$$|\alpha - \beta|\left(\alpha\beta + \sigma_k^2\right) \geq 2\sqrt{\alpha\beta}\sigma_k|\alpha\beta - 1|.$$

Further, we have

$$|\alpha-\beta|\sigma_k^2-2\sqrt{\alpha\beta}|\alpha\beta-1|\sigma_k+\alpha\beta|\alpha-\beta|\geq 0.$$

Consider the following inequality

$$|\alpha-\beta|\sigma^2-2\sqrt{\alpha\beta}|\alpha\beta-1|\sigma+\alpha\beta|\alpha-\beta|\geq 0. \tag{16}$$

On one hand, if α and β satisfy the conditions i) and ii), then

$$4\alpha\beta(\alpha\beta-1)^2-4\alpha\beta(\alpha-\beta)^2=4\alpha\beta(\alpha^2-1)(\beta^2-1)\leq 0.$$

It is easy to obtain the inequality (16). On the other hand, if $0<\alpha,\beta<1$ or $\alpha,\beta>1$ and $\alpha\neq\beta$, then we have

$$4\alpha\beta(\alpha\beta-1)^2-4\alpha\beta(\alpha-\beta)^2=4\alpha\beta(\alpha^2-1)(\beta^2-1)>0.$$

So, for all σ_k $(k=1,2,\ldots,m)$, we can find out some positive constants α and β such that $\sigma_k<\sigma_-$, or $\sigma_k>\sigma_+$. Then, the inequality (16) is obtained.

Hence, as α and β meet one of the cases i), ii) and iii), we obtain $\tilde{\lambda}_{k\pm}$ $(k=1,2,\ldots,m)$ are all real, i.e., $\lambda_{k\pm}=\frac{\tilde{\lambda}_{k\pm}}{(\alpha+1)(\beta+1)(\alpha\beta+\sigma_k^2)}$ $(k=1,2,\ldots,m)$ are all real.

So, we complete the proof of Theorem 2.2. □

Theorem 2.3. *Let the conditions of Theorem* 2.1 *be satisfied. Denote by* ρ_{PAHSS} *the spectral radius of the iteration matrices* $[M(\alpha,\beta)]^{-1}N(\alpha,\beta)$. *Then, we have*

1) *if the iteration parameters α and β satisfy one of the following conditions*

   i) $\alpha>\beta\geq 1$,
   ii) $\alpha<\beta\leq 1$,
   iii) $\beta>1,\alpha<1,$ *and* $\alpha\beta\leq 1$,
   vi) $\beta<1,\alpha>1,$ *and* $\alpha\beta\geq 1$,
   v) $\alpha=\beta\neq 1$,

   then, $\rho_{PAHSS}=\dfrac{|\alpha-1|}{\alpha+1}$,

2) *if the iteration parameters* α *and* β *satisfy one of the following conditions*

i) $\beta > \alpha \geq 1$,

ii) $\beta < \alpha \leq 1$,

iii) $\beta > 1, \alpha < 1,$ *and* $\alpha\beta > 1$,

iv) $\beta < 1, \alpha > 1,$ *and* $\alpha\beta < 1$,

then, $\rho_{PAHSS} \leq \dfrac{|\beta - 1|}{\beta + 1}$.

Proof. In order to complete the above proves, we first estimate the bounds of $\tilde{\lambda}$ and λ ($\tilde{\lambda}$ and λ defined as in Theorem 2.1). As one of the conditions i), ii) or iii) in Theorem 2.2 are satisfied, then, we easily obtain the following result

$$\begin{aligned} |\tilde{\lambda}| &= |(\alpha\beta - 1)(\alpha\beta - \sigma_k^2) \pm \sqrt{(\alpha - \beta)^2(\alpha\beta + \sigma_k^2)^2 - 4\alpha\beta\sigma_k^2(\alpha\beta - 1)^2}| \\ &\leq |(\alpha\beta - 1)(\alpha\beta - \sigma_k^2)| + \sqrt{(\alpha - \beta)^2(\alpha\beta + \sigma_k^2)^2 - 4\alpha\beta\sigma_k^2(\alpha\beta - 1)^2} \\ &\leq |(\alpha\beta - 1)(\alpha\beta - \sigma_k^2)| + \sqrt{(\alpha - \beta)^2(\alpha\beta + \sigma_k^2)^2} \\ &\leq |(\alpha\beta - 1)(\alpha\beta + \sigma_k^2)| + |(\alpha - \beta)(\alpha\beta + \sigma_k^2)|. \end{aligned}$$

Further, we obtain

$$\begin{aligned} |\lambda| &= \frac{|\tilde{\lambda}|}{(\alpha + 1)(\beta + 1)(\alpha\beta + \sigma_k^2)} \\ &\leq \frac{|(\alpha\beta - 1)(\alpha\beta + \sigma_k^2)| + |(\alpha - \beta)(\alpha\beta + \sigma_k^2)|}{(\alpha + 1)(\beta + 1)(\alpha\beta + \sigma_k^2)} \\ &= \frac{|\alpha\beta - 1| + |\alpha - \beta|}{(\alpha + 1)(\beta + 1)}. \end{aligned} \tag{17}$$

If $0 < \alpha, \beta < 1$, or $\alpha, \beta > 1$, and $\sigma_k \in [\sigma_-, \sigma_+], (k = 1, 2, \ldots, m)$, ($\sigma_-, \sigma_+$ defined as in Theorem 2.2), it is obvious that

$$\begin{aligned} |\tilde{\lambda}| &= \sqrt{(\alpha\beta - 1)^2(\alpha\beta - \sigma_k^2)^2 + \left[\sqrt{4\alpha\beta\sigma_k^2(\alpha\beta - 1)^2 - (\alpha - \beta)^2(\alpha\beta + \sigma_k^2)^2}\right]^2} \\ &= \sqrt{(\alpha\beta - 1)^2(\alpha\beta + \sigma_k^2)^2 - (\alpha - \beta)^2(\alpha\beta + \sigma_k^2)^2} \\ &= \sqrt{(\alpha^2 - 1)(\beta^2 - 1)(\alpha\beta + \sigma_k^2)^2}. \end{aligned}$$

Further, we have

$$|\lambda| = \sqrt{\frac{(\alpha - 1)(\beta - 1)}{(\alpha + 1)(\beta + 1)}}. \tag{18}$$

Secondly, since $f_1(x) = \frac{x-1}{1+x}$ $(x > 1)$ and $f_2(x) = \frac{1-x}{1+x}$ $(0 < x < 1)$ are monotone increasing function and monotone decreasing function, respectively, then

$$\frac{\beta - 1}{1 + \beta} < \frac{\alpha - 1}{1 + \alpha}, \quad \text{with} \quad \beta < \alpha, \tag{19}$$

and

$$\frac{1 - \alpha}{1 + \alpha} < \frac{1 - \beta}{1 + \beta}, \quad \text{with} \quad \alpha > \beta. \tag{20}$$

According to Theorem 2.1, the iteration matrices $[M(\alpha, \beta)]^{-1} N(\alpha, \beta)$ have $n - m$ eigenvalues $\frac{\alpha-1}{\alpha+1}$. Then, for any other eigenvalues of the iteration matrices, we complete the proves of the conclusions in 1) by the following four cases:

(i) If $\alpha > \beta \geq 1$, then, by (17), we obtain

$$|\lambda| \leq \frac{|\alpha\beta - 1| + |\alpha - \beta|}{(\alpha + 1)(\beta + 1)} = \frac{\alpha\beta - 1 + \beta - \alpha}{(\alpha + 1)(\beta + 1)} = \frac{\alpha - 1}{1 + \alpha},$$

and by (18), we have

$$|\lambda| = \sqrt{\frac{(\alpha - 1)(\beta - 1)}{(\alpha + 1)(\beta + 1)}} \leq \frac{\alpha - 1}{1 + \alpha} \quad \text{(by (19))}.$$

(ii) If $\alpha < \beta \leq 1$, then, by (17), we have

$$|\lambda| \leq \frac{|\alpha\beta - 1| + |\alpha - \beta|}{(\alpha + 1)(\beta + 1)} = \frac{1 - \alpha\beta + \beta - \alpha}{(\alpha + 1)(\beta + 1)} = \frac{1 - \alpha}{1 + \alpha}, \tag{21}$$

and by (18), we get

$$|\lambda| = \sqrt{\frac{(1 - \alpha)(1 - \beta)}{(\alpha + 1)(\beta + 1)}} \leq \frac{1 - \alpha}{1 + \alpha} \quad \text{(by (20))}.$$

(iii) If $\beta > 1, \alpha < 1$, and $\alpha\beta \leq 1$, or $\beta < 1, \alpha > 1$, and $\alpha\beta > 1$, by (17), the inequality (21) can be straightforwardly obtained.

(iv) If $\alpha = \beta \neq 1$, according to (18), we have

$$|\lambda_{k_\pm}| = \frac{\sqrt{(\alpha^2-1)^2(\alpha^2-\sigma_k^2)^2 + 4\alpha^2\sigma_k^2(\alpha^2-1)^2}}{(\alpha+1)^2(\alpha^2+\sigma_k^2)} = \frac{|\alpha-1|}{1+\alpha}.$$

Therefore, combining the above proves, we obtain

$$\rho_{PAHSS} = \frac{|\alpha-1|}{\alpha+1}.$$

It is similar to prove the conclusion in 2). Therefore, we complete the proves. □

Corollary 2.1. *Let the conditions of Theorem* 2.1 *be satisfied.* ρ_{PAHSS} *defined as in Theorem* 2.3. *To solve the generalized saddle point problem* (1)*, the PAHSS method unconditionally converges to the unique solution for any positive iteration parameters* α *and* β*, i.e.,*

$$\rho_{PAHSS} < 1.$$

Proof. According to Theorem 2.3, we straightforwardly obtain the above result. □

Remark 2.1. According to Theorem 2.3, on the one hand, as the iteration parameters α and β approach 1, the spectral radius ρ_{PAHSS} approximates to 0, on the other hand, when the iteration parameter α is fixed, for different value β, we have

$$\rho_{PAHSS\min} = \frac{|\alpha-1|}{\alpha+1},$$

where we denote by $\rho_{PAHSS\min}$ the smallest spectral radius of the iteration matrix $[M(\alpha,\beta)]^{-1}N(\alpha,\beta)$ with different value β. □

In the following, we study the spectral properties of the preconditioned matrix. Since

$$M(\alpha,\beta)^{-1}A(\alpha,\beta) = I - M(\alpha,\beta)^{-1}N(\alpha,\beta),$$

then

$$\lambda[M(\alpha,\beta)^{-1}A(\alpha,\beta)] = 1 - \lambda[M(\alpha,\beta)^{-1}N(\alpha,\beta)], \quad \text{(see e.g., [3])}.$$

Thus

$$\lambda[M(\alpha,\beta)^{-1}A(\alpha,\beta)] = 1 - \frac{\alpha-1}{\alpha+1} = \frac{2}{\alpha+1}, \tag{22}$$

with multiplicity $n-m$, the remainder eigenvalues of the preconditioned matrices are

$$\begin{aligned}\lambda[M(\alpha,\beta)^{-1}A(\alpha,\beta)] &= 1 - \frac{(\alpha\beta-1)(\alpha\beta-\sigma_k^2) \pm \Gamma_k}{(\alpha+1)(\beta+1)(\alpha\beta+\sigma_k^2)} \\ &= \frac{\alpha\beta(\alpha\beta+\alpha+\beta+2) + (2\alpha\beta+\alpha+\beta)\sigma_k^2 \pm \Gamma_k}{(\alpha+1)(\beta+1)(\alpha\beta+\sigma_k^2)},\end{aligned}$$

where

$$\Gamma_k = \sqrt{(\alpha-\beta)^2(\alpha\beta+\sigma_k^2)^2 - 4\alpha\beta\sigma_k^2(\alpha\beta-1)^2}, \quad k = 1, 2, \ldots, m.$$

For the convenience of our statements, we denote

$$\hat{\lambda}_{k_+} = \frac{\alpha\beta(\alpha\beta+\alpha+\beta+2) + (2\alpha\beta+\alpha+\beta)\sigma_k^2 + \Gamma_k}{(\alpha+1)(\beta+1)(\alpha\beta+\sigma_k^2)}, \tag{23}$$

and

$$\hat{\lambda}_{k_-} = \frac{\alpha\beta(\alpha\beta+\alpha+\beta+2) + (2\alpha\beta+\alpha+\beta)\sigma_k^2 - \Gamma_k}{(\alpha+1)(\beta+1)(\alpha\beta+\sigma_k^2)}. \tag{24}$$

According to the above analysis, we obtain the following results:

To generalized saddle point problem (1), Bai [16] proved the preconditioned matrix $[M(\alpha,\beta)]^{-1}A$ is positive stable (*cf*. [20] for the definition of positive stable matrix). In the following, we also obtain the same property.

Theorem 2.4. *Let the conditions of Theorem* 2.1 *be satisfied. Then, for any positive constants α and β, the real parts of $\hat{\lambda}_{k_-}$ and $\hat{\lambda}_{k_+}$ $(k = 1, 2, \ldots, m)$ are all positive, i.e., the preconditioned matrices $[M(\alpha,\beta)]^{-1}A(\alpha,\beta)$ are positive stable.*

Proof. Obviously, $\frac{2}{\alpha+1}$ is positive real. Denote by $Re(\hat{\lambda})$ the real part of $\hat{\lambda}$, if $0 < \alpha, \beta < 1$ or $\alpha, \beta > 1$ and $\alpha \neq \beta$, for any $k = 1, 2, \ldots, m$, $\sigma_k \in [\sigma_-, \sigma_+]$, then according to (23) or (24), we get

$$Re(\hat{\lambda}) \geq 0.$$

If α and β meet one of the cases i), ii) or iii) in Theorem 2.2, then, we have

$$(\alpha - \beta)^2(\alpha\beta + \sigma_k^2)^2 - 4\alpha\beta\sigma_k^2(\alpha\beta - 1)^2 \geq 0.$$

Thus

$$Re(\hat{\lambda}_{k_\pm}) = \hat{\lambda}_{k_\pm}.$$

Obviously

$$Re(\hat{\lambda}_{k_+}) \geq 0.$$

According to (24), we obtain

$$\begin{aligned}
\hat{\lambda}_{k_-} &= \frac{\alpha\beta(\alpha\beta + \alpha + \beta + 2) + (2\alpha\beta + \alpha + \beta)\sigma_k^2 - \Gamma_k}{(\alpha + 1)(\beta + 1)(\alpha\beta + \sigma_k^2)} \\
&\geq \frac{\alpha\beta(\alpha\beta + \alpha + \beta + 2) + (2\alpha\beta + \alpha + \beta)\sigma_k^2 - \sqrt{(\alpha + \beta)^2(\alpha\beta + \sigma_k^2)^2}}{(\alpha + 1)(\beta + 1)(\alpha\beta + \sigma_k^2)} \\
&= \frac{2\alpha\beta(\sigma_k^2 + 1)}{(\alpha + 1)(\beta + 1)(\alpha\beta + \sigma_k^2)} \\
&> 0,
\end{aligned}$$

i.e.,

$$Re(\hat{\lambda}_{k_-}) \geq 0.$$

Therefore, we complete the proof of Theorem 2.5. □

Theorem 2.5. *Let* $\hat{\lambda}_{k_+}$, *and* $\hat{\lambda}_{k_-}$ *defined as in* (23) *and* (24), *respectively. Then, we obtain the following properties of the eigenvalues of the preconditioned matrices* $M(\alpha, \beta)^{-1}A(\alpha, \beta)$

1) *As the iteration parameters* α *and* β *approximate to 1, all eigenvalues of the preconditioned matrices* $M(\alpha,\beta)^{-1}A(\alpha,\beta)$ *approach 1.*

2) *For any positive iteration parameters* α *and* β*, the moduluses of all eigenvalues of the preconditioned matrices* $M(\alpha,\beta)^{-1}A(\alpha,\beta)$ *cluster in the interval* $(0,2)$.

Proof. According to Theorem 2.3 and Theorem 2.4, we can easily obtain the above results. □

Further, we consider the general case with D being Hermitian positive semidefinite, then, we generalize our conclusions by taking steps similar to those taken in [3, Theorem 5.1]. Denote the Moore-Penrose generalized inverse of B_1 and D_1 by B_1^+ and D_1^+, respectively, and the positive singular values of the matrix $B_1^+E(D_1^+)^*$ by σ_i $(i=1,2,\ldots,m)$. By the similar analysis, we can obtained the similar results with the above spectral properties.

3 Numerical examples

In this section, we use two examples to illustrate the feasibility and effectiveness of the PAHSS iteration method for the generalized saddle point problems. We perform the numerical examples by using MATLAB with machine precision 10^{-16} and using

$$\|r_k\|_2/\|r_0\|_2 = \|b - Ax^{(k)}\|_2/\|b\|_2 < 10^{-6}$$

as a stopping criterion, where r_k is the residual at the kth iterate. Bai, Golub and Pan [3] considered the Stokes problem:

$$\begin{cases} -\mu\triangle u + \nabla\omega &= \tilde{f}, \quad \text{in } \Omega, \\ \nabla\cdot u &= \tilde{g}, \quad \text{in } \Omega, \\ u &= 0, \quad \text{on } \partial\Omega, \\ \int_\Omega \omega(x)dx &= 0, \end{cases}$$

where $\Omega=(0,1)\times(0,1)\subset R^2$, $\partial\Omega$ is the boundary of Ω, Δ is the componentwise Laplace operator, u is a vector-valued function representing the velocity, and ω is a scalar function represeting the pressure. By discretizing the above equation, linear system (1) be obtained with $A\in R^{(3m^2)\times(3m^2)}$ and $D=0$.

Example 1 [3]**.** Consider the following linear system:

$$A = \begin{pmatrix} B & E \\ -E^* & D \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix},$$

where

$$B = \begin{pmatrix} I \bigotimes T + T \bigotimes I & 0 \\ 0 & I \bigotimes T + T \bigotimes I \end{pmatrix} \in R^{2m^2 \times 2m^2},$$

$$E = \begin{pmatrix} I \bigotimes F \\ I \bigotimes F \end{pmatrix} \in R^{2m^2 \times m^2},$$

and

$$T = \frac{\mu}{h^2} \, tridiag\,(-1, 2, -1) \in R^{m^2 \times m^2},$$
$$F = \frac{1}{h} \, tridiag\,(-1, -1, 0) \in R^{m^2 \times m^2},$$

in this example, we assume

$$D = \begin{pmatrix} I \bigotimes T + T \bigotimes I \end{pmatrix} \in R^{m^2 \times m^2},$$

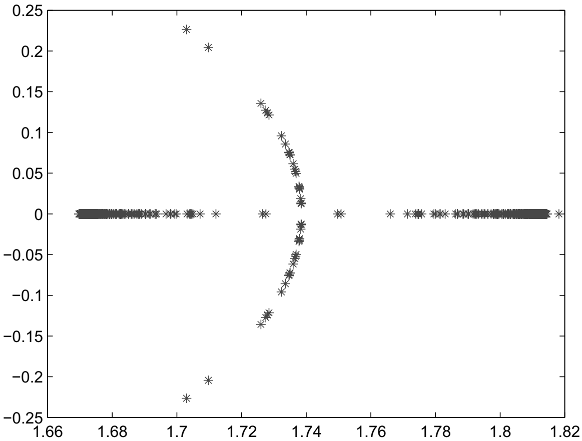
where $h = \frac{1}{m+1}$ is the discretization meshsize, $\bigotimes$ is the Kronecker product symbol. Then, we confirm the correctness and accuracy of our theoretical analysis by solving the generalized saddle problem.

Example 2 [12]**.** Consider the following linear system:

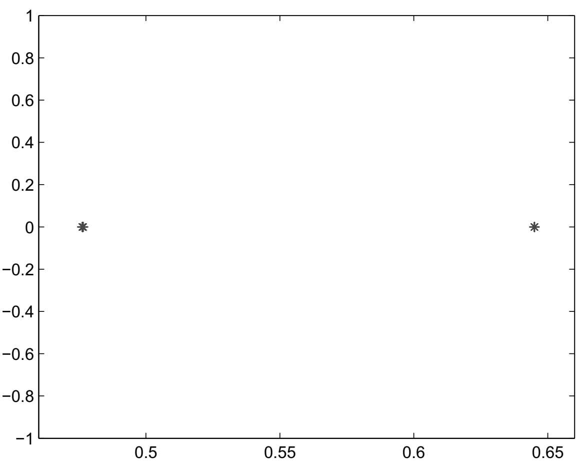
$$A = \begin{pmatrix} W & F \\ -F^T & N \end{pmatrix},$$

where $W = (w_{k,j}) \in R^{q \times q}$, $N = (n_{k,j}) \in R^{n-q \times n-q}$, $F = (f_{k,j}) \in R^{(n-q) \times q}$ and $2q > n$, where

$$w_{k,j} = \begin{cases} k+1, & \text{for} \quad j = k, \\ 1, & \text{for} \quad |k-j| = 1, \quad k, j = 1, 2, \ldots, q, \\ 0, & \text{otherwise,} \end{cases}$$



(a) $\alpha = 0.1, \ \ \beta = 0.2$



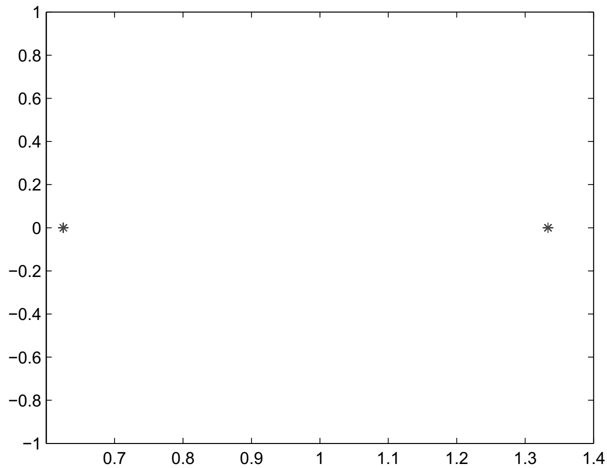
(b) $\alpha = 2.1, \ \ \beta = 3.2$

Figure 1 – The distribution of the eigenvalues of $[\tilde{M}(\alpha, \beta)]^{-1}\tilde{A}$ for Example 1.

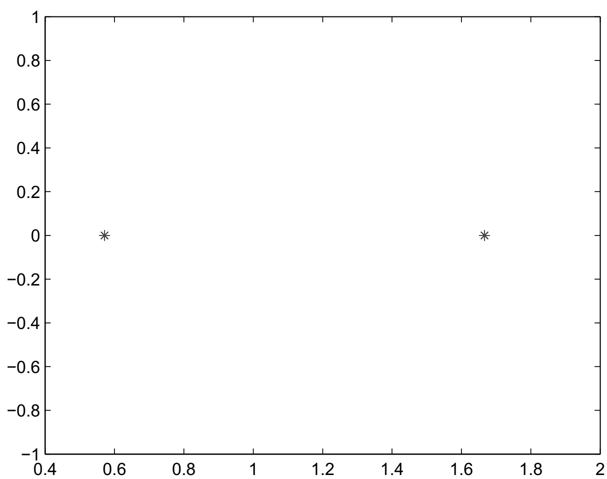
$$n_{k,j} = \begin{cases} k+1, & \text{for} \quad j = k, \\ 1 & \text{for} \quad |k-j| = 1, \quad k, j = 1, 2, \ldots, n-q, \\ 0, & \text{otherwise}, \end{cases}$$

$$f_{k,j} = \begin{cases} j, & \text{for} \quad k = j + 2q - n, \\ 0, & \text{otherwise}, \end{cases} \qquad k = 1, 2, \ldots, q, \quad j = 1, 2, \ldots, n-q.$$

In Figures 1, 2, 4 and 5, for different iteration parameters α and β, we depict the distribution on the eigenvalues of the preconditioned matrices $[\tilde{M}(\alpha, \beta)]^{-1}\tilde{A}$.



(a) $\alpha = 0.5, \quad \beta = 2.2$

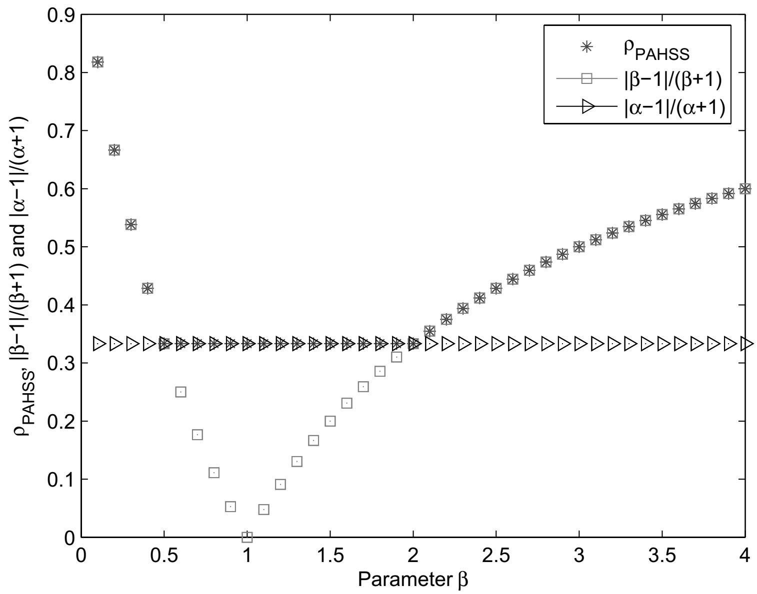


(b) $\alpha = 2.5, \quad \beta = 0.2$

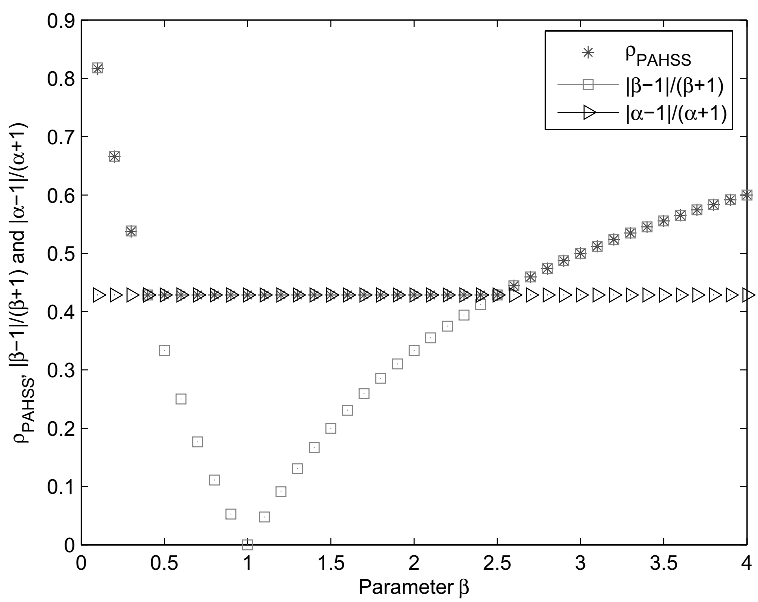
Figure 2 – The distribution of the eigenvalues of $[\tilde{M}(\alpha, \beta)]^{-1}\tilde{A}$ for Example 1.

From these images, we see that the eigenvalues λ of the preconditioned matrix are quite clustered.

In Tables 1 and 2, we can know that the smallest real part $Re(\lambda)_{\min}$ of the eigenvalues of the preconditioned matrix $[\tilde{M}(\alpha, \beta)]^{-1}\tilde{A}$ are all positive, as the iteration parameters α and β take different values. Further, we know that the real parts of all eigenvalues of the preconditioned matrices are all positive, therefore, we numerically verify the accuracy of the Theorem 2.4.



(a) $\alpha = 2$



(b) $\alpha = 0.4$

Figure 3 – The relation of ρ_{PAHSS}, $\frac{|\alpha-1|}{\alpha+1}$ and $\frac{|\beta-1|}{\beta+1}$ for Example 1.

μ	1			10		
n	19			24		
α	0.9	0.2	3	0.9	0.2	3
β	0.2	2	5	0.2	2	5
$Re(\lambda)_{\min}$	1.0526	0.6664	0.33	1.0526	0.6664	0.34

Table 1 – The smallest real parts of the eigenvalues of the preconditioned matrix.

n	2500			3000		
q	1500			2000		
α	0.9	0.2	3	0.9	0.2	3
β	0.2	2	5	0.2	2	5
$Re(\lambda)_{\min}$	0.7055	0.485	0.3339	0.8307	0.5097	0.3336

Table 2 – The smallest real parts of the eigenvalues of the preconditioned matrix.

In Figures 3 and 6, we plot the curves of the spectral radius denote by ρ_{PAHSS}, $\frac{|\alpha-1|}{\alpha+1}$ and $\frac{|\beta-1|}{\beta+1}$ with the change of β. From subfigure a, we know that $\rho_{PAHSS} \leq \frac{|\beta-1|}{\beta+1}$, as $\beta \in [0.1, 0.5]$, $\rho_{PAHSS} = \frac{|\alpha-1|}{\alpha+1}$, as $\beta \in [0.5, 2]$, and $\rho_{PAHSS} \leq \frac{|\beta-1|}{\beta+1}$, as $\beta \in [2, 4]$, From subfigure b, we know that $\rho_{PAHSS} \leq \frac{|\beta-1|}{\beta+1}$, as $\beta \in [0.1, 0.4]$, $\rho_{PAHSS} = \frac{|\alpha-1|}{\alpha+1}$, as $\beta \in [0.4, 2.5]$, and $\rho_{PAHSS} \leq \frac{|\beta-1|}{\beta+1}$, as $\beta \in [2.5, 4]$. Therefore, through the two images, we verify the efficiency and accuracy of Theorem 2.3.

In Tables 3 and 4, by using GMRES(l) ($l = 5, 10, 20, 100$) iterative methods with PAHSS preconditioning, we compare between the preconditioner proposed in this paper and the preconditioner in (5) by the iteration numbers (denote by "IT") and the solution of times in seconds (denote by "CPU"). From the two tables, we can easily see that the superiority of the PAHSS iteration method is very evident.

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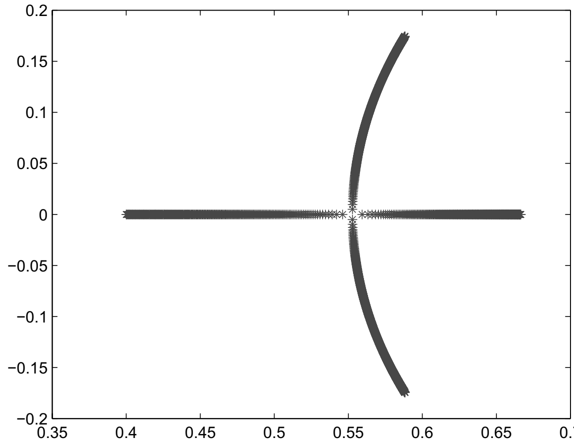
	m	8	12	16	20	24	28
PAHSS-GMRES(5)	CPU	**0.094**	**0.625**	**3.656**	**14.578**	**47.563**	**122.75**
	IT	**7**	**5**	**7**	**8**	**6**	**7**
AHSS-GMRES(5)	CPU	0.250	4.578	31.954	89.234	408.265	886.438
	IT	12	22	27	21	36	31
PAHSS-GMRES(10)	CPU	**0.078**	**0.485**	**2.969**	**12.047**	**38.313**	**102.125**
	IT	**10**	**11**	**3**	**5**	**7**	**8**
AHSS-GMRES(10)	CPU	0.094	1.375	11.61	44.094	163	445.39
	IT	10	11	11	7	15	12
PAHSS-GMRES(20)	CPU	**0.078**	**0.485**	**2.735**	**10.765**	**30.593**	**77.687**
	IT	**17**	**11**	**12**	**13**	**13**	**14**
AHSS-GMRES(20)	CPU	0.078	0.921	6.453	27.547	89.641	213.891
	IT	17	20	10	25	20	20
PAHSS-GMRES(100)	CPU	**0.078**	**0.484**	**2.625**	**10.50**	**30.141**	**76.781**
	IT	**10**	**11**	**12**	**13**	**13**	**14**
AHSS-GMRES(100)	CPU	0.078	0.906	5.016	19.719	67.047	160.328
	IT	17	20	22	25	30	30

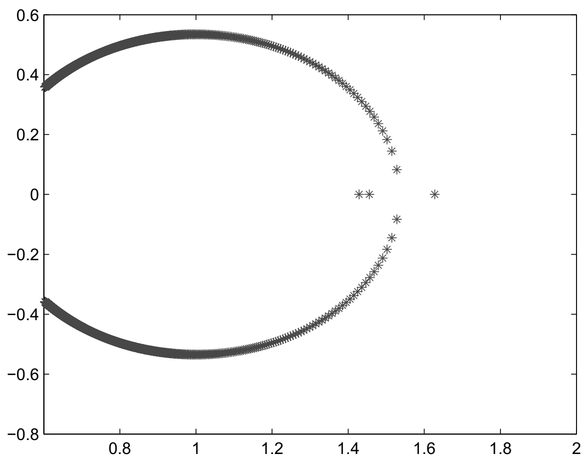
Table 3 – Example 1: $\mu = 10$, $\alpha = 0.5$ and $\beta = 2.2$.

	n	800	1000	1400	1600	1800	2000
	q	500	600	800	1000	1200	1500
PAHSS-GMRES(5)	CPU	**4.594**	**8.171**	**22.609**	**28.047**	**53.860**	**135.578**
	IT	**6**	**7**	**8**	**5**	**7**	**6**
AHSS-GMRES(5)	CPU	61.766	129.125	343.625	504.656	702.907	869.437
	IT	29	31	33	37	43	43
PAHSS-GMRES(10)	CPU	**3.687**	**7.734**	**20.219**	**26.062**	**45.454**	**114.25**
	IT	**4**	**5**	**6**	**4**	**10**	**9**
AHSS-GMRES(10)	CPU	56.969	117.422	299.219	441.328	644.203	805.25
	IT	20	18	25	22	27	23
PAHSS-GMRES(20)	CPU	**3.703**	**6.984**	**17.688**	**22.891**	**47.422**	**118.062**
	IT	**13**	**14**	**14**	**12**	**10**	**8**
AHSS-GMRES(20)	CPU	53.281	114.547	294.766	450.875	614.984	775.422
	IT	23	13	21	25	21	13
PAHSS-GMRES(100)	CPU	**3.5**	**6.922**	**18.093**	**22.469**	**46.984**	**115.922**
	IT	**13**	**14**	**14**	**13**	**10**	**9**
AHSS-GMRES(100)	CPU	35.453	76.031	204.547	310.625	409.828	507.25
	IT	97	3	13	27	34	38

Table 4 – Example 2: $\mu = 10$, $\alpha = 0.5$ and $\beta = 2.2$.



(a) $\alpha = 4, \quad \beta = 2$



(b) $\alpha = 0.4, \quad \beta = 0.2$

Figure 4 – The distribution of the eigenvalues of $[\tilde{M}(\alpha, \beta)]^{-1}\tilde{A}$ for Example 2.

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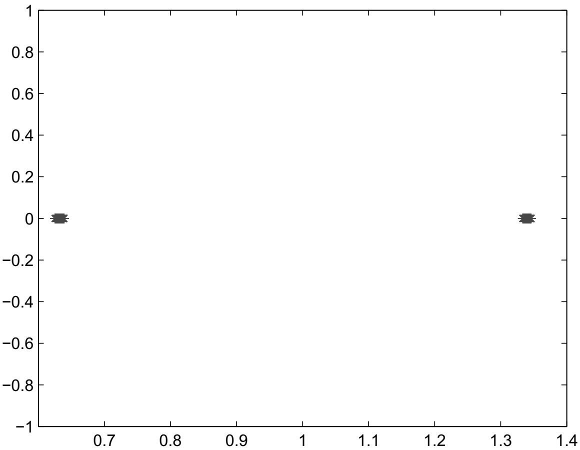
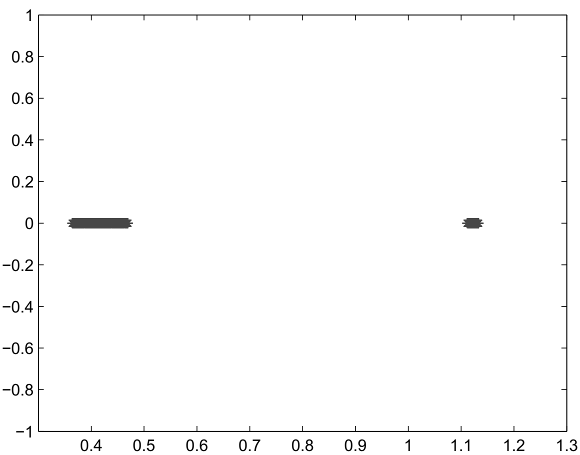


(a) $\alpha = 0.5, \quad \beta = 2.2$



(b) $\alpha = 4.5, \quad \beta = 0.8$

Figure 5 – The distribution of the eigenvalues of $[\tilde{M}(\alpha, \beta)]^{-1}\tilde{A}$ for Example 2.

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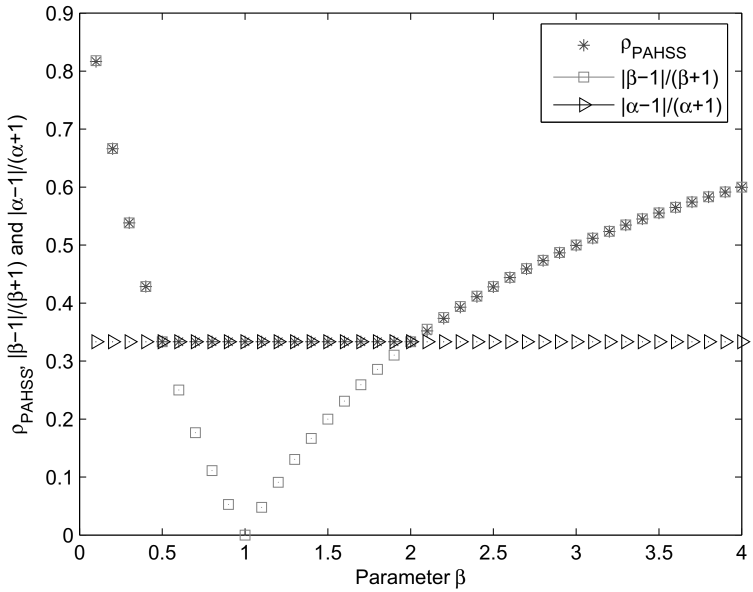
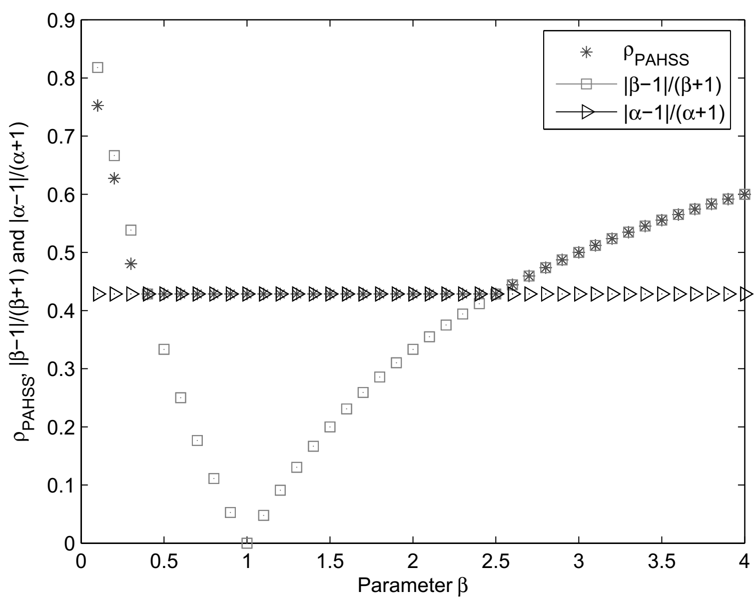


(a) $\alpha = 2$



(b) $\alpha = 0.4$

Figure 6 – The relation of ρ_{PAHSS}, $\frac{|\alpha-1|}{\alpha+1}$ and $\frac{|\beta-1|}{\beta+1}$ for Example 2.

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