

Finding the closest Toeplitz matrix*

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Abstract. The constrained least-squares $n \times n$ -matrix problem where the feasibility set is the subspace of the Toeplitz matrices is analyzed. The general, the upper and lower triangular cases are solved by making use of the singular value decomposition. For the symmetric case, an algorithm based on the alternate projection method is proposed. The implementation does not require the calculation of the eigenvalue of a matrix and still guarantees convergence. Encouraging preliminary results are discussed.

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1 Introduction

In this work the constrained least-squares Toeplitz matrix is analyzed. Let us begin considering the general constrained least squares matrix problem

$$\left\{ \begin{array}{ll} \min & \|AX - B\|_F^2 \\ s.t & \\ & X \in \mathcal{P}, \end{array} \right. \quad (1)$$

where $A, B \in \mathbb{R}^{m \times n}$, with $m > n$ and $\mathcal{P} \subset \mathbb{R}^{n \times n}$ has a particular pattern. Throughout this paper $\|A\|_F^2$ denotes the Frobenius norm of $A \in \mathbb{R}^{m \times n}$ defined as

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$$\|A\|_F^2 = \text{trace}(A^T A) = \sum_{i=1}^m \sum_{j=1}^n a_{ij}^2.$$

In the last two decades an extensive research activity has been done on this kind of problems which arise in many areas. For instance, in statistics, the problem of finding the nearest symmetric positive definite patterned matrix to a sample covariance matrix is a classical example, [11]. The symmetric positive semidefinite case has analyzed by Higham [9].

In the investigation of elastic structures, the determination of the strain matrix which relates forces f_i and displacement d_i according to $Xf_i = d_i$ is stated as (1). The orthogonal case has been studied by Schonemann [13] and the Higham [8] and the symmetric case by Higham [10]. These problems are known in the literature as the orthogonal and symmetric Procrustes problems respectively. When \mathcal{P} is the subspace of persymmetric matrices (that is, it is symmetric about the NE-SW diagonal) the resulting problem has been studied by Eberle [4, 5].

Now in this work we are interested in solving the constrained least squares Toeplitz matrix problem

$$\left\{ \begin{array}{ll} \min & \|AX - B\|_F^2 \\ \text{s.t} & \\ & X \in \mathcal{T}, \end{array} \right. \quad (2)$$

where $\mathcal{T} \subset \mathbb{R}^{n \times n}$ is the subspace of Toeplitz matrices.

Let us recall that a matrix $T = (t_{ij}) \in \mathbb{R}^{n \times n}$ is said to be a Toeplitz matrix if it is constant along its diagonals, i.e there exist scalars $r_{-n+1}, \dots, r_0, \dots, r_{n-1}$, such that $t_{ij} = r_{j-i}$, for all i, j :

$$T = \begin{bmatrix} r_0 & r_1 & r_2 & r_3 & \cdots & r_{n-2} & r_{n-1} \\ r_{-1} & r_0 & r_1 & r_2 & r_3 & \cdots & r_{n-2} \\ r_{-2} & r_{-1} & r_0 & r_1 & r_2 & \cdots & r_{n-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ r_{-(n-3)} & r_{-(n-4)} & \cdots & r_{-1} & r_0 & r_1 & r_2 \\ r_{-(n-2)} & r_{-(n-3)} & \cdots & \cdots & \cdots & r_0 & r_1 \\ r_{-(n-1)} & r_{-(n-2)} & \cdots & \cdots & \cdots & r_{-1} & r_0 \end{bmatrix}.$$

The rest of the paper is organized as follows. In section 2, the general Toeplitz problem is studied. The special cases of triangular and symmetric Toeplitz problems are analyzed in section 3 and 4 respectively. The numerical results are presented in section 5 and our conclusions and final remarks in section 6.

2 The general problem

In this section the feasibility set, \mathcal{T} , of problem (2) is characterized.

Given the matrices $A, B \in \mathbb{R}^{m \times n}$, the subspace \mathcal{T} can be represented by

$$\mathcal{T} = \left\{ X \in \mathbb{R}^{n \times n} : X = \sum_{p=-(n-1)}^{(n-1)} \alpha_p G_p, \right\},$$

where $\alpha_p \in \mathbb{R}$ and $G_p \in \mathbb{R}^{n \times n}$ are matrices defined as follows

$$(G_p)_{ij} = \begin{cases} 1 & \text{if } j = i + p \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

We must observe that the matrices G_p are characterized by the following fact: for each entry ij , there exists an unique index p , $-(n-1) \leq p \leq (n-1)$, such that $(G_p)_{ij} = 1$. Clearly they form a basis for the subspace \mathcal{T} .

2.1 The transformed problem

We will transform the problem (2) in a simpler one. To do this transformation let A be the matrix have the following singular value decomposition

$$A = P \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} Q^T,$$

where $P \in \mathbb{R}^{m \times m}$ and $Q \in \mathbb{R}^{n \times n}$, are orthogonal matrices and $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_n)$, $\sigma_1 \geq \dots \geq \sigma_n \geq 0$.

Since the Frobenius norm remains invariant under orthogonal transformations

it is obtained

$$\begin{aligned}
 \|AX - B\|_F^2 &= \left\| P \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} Q^T X - B \right\|_F^2 = \left\| P^T \left(P \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} Q^T X - B \right) \right\|_F^2 \\
 &= \left\| \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} Q^T X - P^T B \right\|_F^2 = \left\| \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} Q^T X - C \right\|_F^2 \\
 &= \left\| (\Sigma Q^T)X - C_1 \right\|_F^2 + \|C_2\|_F^2 = \|T - C_1\|_F^2 + \|C_2\|_F^2
 \end{aligned}$$

with

$$T = \Sigma Q^T X \in \mathbb{R}^{n \times n},$$

$$C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = P^T B,$$

$$C_1 \in \mathbb{R}^{n \times n}.$$

then the problem 2 is equivalent to

$$\begin{cases} \min & \|T - C_1\|_F^2 \\ s.t & \\ & T \in \mathcal{T}', \end{cases} \quad (4)$$

where \mathcal{T}' is the following subspace:

$$\mathcal{T}' = \left\{ T \in \mathbb{R}^{n \times n} : T = \Sigma Q^T \sum_{l=-(n-1)}^{(n-1)} \alpha_l G_l \right\}.$$

2.2 Characterization of the solution on \mathcal{T}'

We introduce the notation $P_{\mathcal{U}}(A)$ meaning the projection of the matrix A on the set \mathcal{U} . The following theorem characterizes the projection on the subspace of matrices \mathcal{T}' .

Theorem 2.1. *If $C_1 \in \mathbb{R}^{n \times n}$, then the unique solution of the problem (4) is*

$$P_{T'}(C_1) = \Sigma Q^T \sum_{l=-(n-1)}^{(n-1)} \alpha^*_l G_l, \quad \alpha^*_l = \frac{\sum_{i=1}^n \sum_{j>l}^n (C_1)_{ij} Q_{(j-l)i}}{\sum_{i=1}^n \sum_{j>l}^n (Q_{(j-l)i})^2 \sigma_i^2},$$

for all $-(n - 1) \leq l \leq (n - 1)$.

Proof. The objective function $f : \mathbb{R}^{2n-1} \rightarrow \mathbb{R}$ is given by

$$\begin{aligned} f(\alpha) &= f(\alpha_{-(n-1)}, \dots, \alpha_0, \dots, \alpha_{(n-1)}) \\ &= \frac{1}{2} \|T - C_1\|_F^2 = \frac{1}{2} \|\Sigma Q^T X - C_1\|_F^2 \\ &= \frac{1}{2} \|\Sigma Q^T \sum_{l=-(n-1)}^{(n-1)} \alpha_l G_l - C_1\|_F^2 \\ &= \frac{1}{2} \sum_{i,j=1}^n \left(\left(\Sigma Q^T \sum_{l=-(n-1)}^{(n-1)} \alpha_l G_l \right)_{ij} - (C_1)_{ij} \right)^2. \end{aligned}$$

Since f is twice continuously differentiable, we compute $\frac{\partial f}{\partial \alpha_p}(\alpha)$ for all p such that $-(n - 1) \leq p \leq (n - 1)$

$$\begin{aligned} \frac{\partial f}{\partial \alpha_p} &= \frac{\partial}{\partial \alpha_p} \left(\frac{1}{2} \sum_{i,j=1}^n \left(\left(\Sigma Q^T \sum_{l=-(n-1)}^{(n-1)} \alpha_l G_l \right)_{ij} - (C_1)_{ij} \right)^2 \right) \\ &= \left(\sum_{i,j=1}^n \left(\Sigma Q^T \sum_{l=-(n-1)}^{(n-1)} \alpha_l G_l \right)_{ij} - (C_1)_{ij} \right) \\ &\quad \frac{\partial}{\partial \alpha_p} \left(\left(\Sigma Q^T \sum_{l=-(n-1)}^{(n-1)} \alpha_l G_l \right)_{ij} \right). \end{aligned} \tag{5}$$

The factor in the right hand side of (5) can be written as

$$\begin{aligned} \frac{\partial}{\partial \alpha_p} \left(\left(\Sigma Q^T \sum_{l=-(n-1)}^{(n-1)} \alpha_l G_l \right)_{ij} \right) &= \frac{\partial}{\partial \alpha_p} \left(\sum_{k=1}^n (\Sigma Q^T)_{ik} \left(\sum_{l=-(n-1)}^{(n-1)} \alpha_l G_l \right)_{kj} \right) \\ &= \sum_{k=1}^n (\Sigma Q^T)_{ik} \frac{\partial}{\partial \alpha_p} \left(\left(\sum_{l=-(n-1)}^{(n-1)} \alpha_l G_l \right)_{kj} \right) \\ &= \sum_{k=1}^n (\Sigma Q^T)_{ik} (G_p)_{kj} = (\Sigma Q^T G_p)_{ij}, \end{aligned}$$

and (5) becomes in

$$\frac{\partial f}{\partial \alpha_p} = \left(\sum_{i,j=1}^n \left(\Sigma Q^T \sum_{l=-(n-1)}^{(n-1)} \alpha_l G_l \right)_{ij} - (C_1)_{ij} \right) (\Sigma Q^T G_p)_{ij}. \quad (6)$$

Having in mind that

$$\left(\Sigma Q^T \sum_{l=-(n-1)}^{(n-1)} \alpha_l G_l \right)_{ij}$$

is the inner product between the i^{th} row of (ΣQ^T) and the j^{th} column of $\sum_{l=-(n-1)}^{(n-1)} \alpha_l G_l$, we obtain

$$\left(\Sigma Q^T \sum_{l=-(n-1)}^{(n-1)} \alpha_l G_l \right)_{ij} = \sigma_i \sum_{k=(j-n)}^{(j-1)} Q_{(j-k)i} \alpha_k.$$

Similarly, the element $(\Sigma Q^T G_p)_{ij}$, is the inner product between the i^{th} row of ΣQ^T and the column j of G_p . Then:

$$(\Sigma Q^T G_p)_{ij} = \sigma_i Q_{(j-p)i}.$$

Because of the sparse structure of G_p the entries $(G_p)_{ij} = 1$ if $j > p$.

If $p > 0$, then $(G_p)_{ij} = 1$ for all j such that $p + 1 \leq j \leq n$.

If $p \leq 0$, then $(G_p)_{ij} = 1$ for all j such that $1 \leq j \leq (n + p)$.

Therefore (6) can be written as

$$\begin{aligned} & \left(\sum_{i=1}^n \sum_{j>p}^n \left(\Sigma Q^T \sum_{l=-(n-1)}^{(n-1)} \alpha_l G_l \right)_{ij} \right) (\Sigma Q^T G_p)_{ij} \\ &= \sum_{i=1}^n \sum_{j>p}^n \left(\sigma_i \sum_{k=(j-n)}^{(j-1)} Q_{(j-k)i} \alpha_k \right) (\sigma_i Q_{(j-p)i}), \\ &= \sum_{j>p}^n \sum_{k=(j-n)}^{(j-1)} \alpha_k \sum_{i=1}^n (Q_{(j-k)i} Q_{(j-p)i}) \sigma_i^2. \end{aligned}$$

Defining

$$\sum_{i=1}^n (Q_{(j-k)i} Q_{(j-p)i}) \sigma_i^2 = v^T \theta,$$

with

$$v^T = (Q_{(j-k)1} Q_{(j-p)1}, Q_{(j-k)2} Q_{(j-p)2}, \dots, Q_{(j-k)n} Q_{(j-p)n}),$$

$$\theta = (\sigma_1^2, \sigma_2^2, \dots, \sigma_n^2)^T,$$

we have that the sum of all the components of v is the inner product between two columns of the orthogonal matrix Q and then it is zero when $k \neq p$. Therefore for $k \neq p$ we have:

$$0 = \left(\sum_{i=1}^n v_i \right) \sigma_n^2 \leq v^T \theta = \sum_{i=1}^n v_i \sigma_i^2 \leq \left(\sum_{i=1}^n v_i \right) \sigma_1^2 = 0,$$

then $v^T \theta = 0$ for all $k \neq p$ and

$$\begin{aligned} \sum_{j>p}^n \sum_{k=(j-n)}^{(j-1)} \alpha_k \sum_{i=1}^n (Q_{(j-k)i} Q_{(j-p)i}) \sigma_i^2 &= \sum_{j=1}^n \sum_{k=(j-n)}^{(j-1)} \alpha_k v^T \theta \\ &= \sum_{j>p}^n \alpha_p \sum_{i=1}^n (Q_{(j-p)i})^2 \sigma_i^2. \end{aligned}$$

Then (6) becomes

$$\begin{aligned} \frac{\partial f}{\partial \alpha_p} &= \sum_{i,j=1}^n \left(\Sigma Q^T \sum_{l=-(n-1)}^{(n-1)} \alpha_l G_l \right)_{ij} (\Sigma Q^T G_p)_{ij} - (C_1)_{ij} (\Sigma Q^T G_p)_{ij} \\ &= \alpha_p \sum_{j>p}^n \sum_{i=1}^n (Q_{(j-p)i})^2 \sigma_i^2 - \sum_{j>p}^n \sum_{i=1}^n (C_1)_{ij} Q_{(j-p)i}, \end{aligned}$$

for all $p = 1, \dots, n$. Therefore, from the first order necessary condition

$$\frac{\partial f}{\partial \alpha_p} = 0, \quad p = 1, \dots, n,$$

we obtain

$$\alpha_p^* = \frac{\sum_{i=1}^n \sum_{j>p}^n (C_1)_{ij} Q_{(j-p)i}}{\sum_{i=1}^n \sum_{j>p}^n (Q_{(j-p)i})^2 \sigma_i^2}. \quad (7)$$

We observe that the denominator of (7) is non zero, because if it were zero for any i, j , it would be

$$Q_{(j-p)i} \sigma_i = 0, \quad \Sigma Q^T = 0 \quad \text{and} \quad P \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} Q^T = A = 0.$$

Now, we need to verify that α_p^* is a minimizer of f . We compute

$$\frac{\partial^2 f}{\partial \alpha_p^2}(\alpha^*) = \sum_{j=1}^n \sum_{i=1}^n (Q_{(j-p)i} \sigma_i)^2 > 0, \quad \forall i, j,$$

and

$$\frac{\partial^2 f}{\partial \alpha_p \partial \alpha_q} = 0, \quad \text{if } p \neq q.$$

It implies that the Hessian matrix $\nabla^2 f(\alpha_p^*)$ is positive definite and therefore α_p^* is the minimizer of f . \square

3 The triangular Toeplitz problem

In this section we consider the case when the feasibility is the subspace of triangular Toeplitz matrices. We begin considering the upper triangular case.

3.1 The upper triangular Toeplitz problem

Let the minimization problem be

$$\begin{cases} \min & \|AX - B\|_{\mathbb{F}}^2 \\ \text{s.t} & X \in \mathcal{T}_u, \end{cases} \tag{8}$$

where $A, B \in \mathbb{R}^{m \times n}$ and \mathcal{T}_u is the subspace of upper triangular Toeplitz matrices. If $X \in \mathcal{T}_u$, then

$$X = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \cdots & \cdots & \cdots & \alpha_n \\ 0 & \alpha_1 & \alpha_2 & \cdots & \cdots & \cdots & \alpha_{n-1} \\ 0 & 0 & \alpha_1 & \alpha_2 & \cdots & \cdots & \alpha_{n-2} \\ \vdots & \cdots & \cdots & \cdots & & & \vdots \\ \vdots & & & \cdots & \cdots & \cdots & \vdots \\ 0 & & & & \cdots & \alpha_1 & \alpha_2 \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \alpha_1 \end{bmatrix}.$$

The subspace \mathcal{T}_u , can be written as:

$$\mathcal{T}_u = \left\{ X \in \mathbb{R}^{n \times n} : X = \sum_{p=1}^n \alpha_p G_p, \right\}$$

where $\alpha_p \in \mathbb{R}$ and the matrices $G_p \in \mathbb{R}^{n \times n}$ are defined for all $1 \leq i, j, p \leq n$ as follow

$$(G_p)_{ij} = \begin{cases} 1 & \text{if } j = i + p - 1 \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

We remark that the matrices G_p are characterized by the property that for each entry ij , there exists an unique $p, 1 \leq p \leq n$ such that $(G_p)_{ij} = 1$.

The set $\{G_1, G_2, \dots, G_n\}$, is a basis for the subspace of the upper triangular Toeplitz matrices of $\mathbb{R}^{n \times n}$. Similarly to the general Toeplitz case, the singular value decomposition is used in order to obtain a solution of an equivalent problem

$$\begin{cases} \min & \|T - C_1\|_{\mathbb{F}}^2 \\ \text{s.t} & \\ & T \in \mathcal{T}'_u, \end{cases} \quad (10)$$

with

$$A = P \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} Q^T,$$

$P \in \mathbb{R}^{m \times m}$ y $Q \in \mathbb{R}^{n \times n}$, are orthogonal matrices, $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_n)$ with $\sigma_1 \geq \dots \geq \sigma_n \geq 0$, and the subspace

$$\mathcal{T}'_u = \left\{ T \in \mathbb{R}^{n \times n} : T = \Sigma Q^T \sum_{l=1}^n \alpha_l G_l \right\}.$$

3.1.1 Characterization of the solution on \mathcal{T}'_u

The following theorem, similar to Theorem 2.1 for the dense case characterizes the projection on \mathcal{T}'_u .

Theorem 3.1. *Let be $C_1 \in \mathbb{R}^{n \times n}$, then the unique solution of the problem*

$$\begin{cases} \min & \|T - C_1\|_{\mathbb{F}}^2 \\ \text{s.t} & \\ & T \in \mathcal{T}'_u \end{cases} \quad (11)$$

is given by

$$P_{\mathcal{T}'_u}(C_1) = \Sigma Q^T \sum_{l=1}^n \alpha^*_l G_l, \quad \alpha^*_l = \frac{\sum_{i=1}^n \sum_{j \geq l}^n (C_1)_{ij} Q_{(j-l+1)i}}{\sum_{i=1}^n \sum_{j \geq l}^n (Q_{(j-l+1)i})^2 \sigma_i^2},$$

for all $1 \leq l \leq n$.

Proof. Similar to the proof of theorem 2.1 having in mind that the first derivative is

$$\frac{\partial f}{\partial \alpha_p}(\alpha) = \left(\sum_{i,j=1}^n \left(\Sigma Q^T \sum_{l=1}^n (\alpha_l G_l) \right)_{ij} - (C_1)_{ij} \right) (\Sigma Q^T G_p)_{ij}.$$

The element $(\Sigma Q^T (\sum_{l=1}^n (\alpha_l G_l)))_{ij}$ is the inner product between the i^{th} row of (ΣQ^T) and the j^{th} column of $\sum_{l=1}^n (\alpha_l G_l)$, then it results

$$\left(\Sigma Q^T \sum_{l=1}^n (\alpha_l G_l) \right)_{ij} = \sigma_i \sum_{k=1}^j Q_{(j-k+1)i} \alpha_k.$$

Analogously, the element $(\Sigma Q^T G_p)_{ij}$, is the inner between the row i of ΣQ^T and the column j of G_p , then

$$(\Sigma Q^T G_p)_{ij} = \begin{cases} \sigma_i Q_{(j-p+1)i} & \text{si } j \geq p \\ 0 & \text{si } j < p. \end{cases} \quad \square$$

3.2 The lower triangular Toeplitz problem

Let us consider the problem

$$\begin{cases} \min & \|AX - B\|_F^2 \\ s.t & \\ & X \in \mathcal{T}_1, \end{cases} \quad (12)$$

where $A, B \in \mathbb{R}^{m \times n}$ and \mathcal{T}_1 is the subspace of the lower triangular Toeplitz matrices. If $X \in \mathcal{T}_1$, then

$$X = \begin{bmatrix} \beta_1 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ \beta_2 & \beta_1 & 0 & \cdots & \cdots & \cdots & 0 \\ \beta_3 & \beta_2 & \beta_1 & 0 & \cdots & \cdots & 0 \\ \vdots & \cdots & \cdots & \cdots & & & \vdots \\ \vdots & & & \cdots & \cdots & \cdots & \vdots \\ \beta_{n-1} & & & & \cdots & \beta_1 & 0 \\ \beta_n & \beta_{n-1} & \cdots & \cdots & \beta_3 & \beta_2 & \beta_1 \end{bmatrix},$$

The subspace \mathcal{T}_1 , can be expressed as

$$\mathcal{T}_1 = \left\{ X \in \mathbb{R}^{n \times n} : X = \sum_{p=1}^n \beta_p G_p, \right\}$$

where $\beta_p \in \mathbb{R}$ and the matrices $G_p \in \mathbb{R}^{n \times n}$ are defined as

$$(G_p)_{ij} = \begin{cases} 1 & \text{if } j = i - p + 1 \\ 0 & \text{otherwise.} \end{cases} \quad (13)$$

for all $1 \leq i, j, p \leq n$. The set $\{G_1, G_2, \dots, G_n\}$, is a basis for the subspace \mathcal{T}_1 . Again, applying the singular value decomposition of A the problem can be transformed in

$$\begin{cases} \min & \|T - C_1\|_{\mathbb{F}}^2 \\ \text{s.t} & \\ & T \in \mathcal{T}'_1, \end{cases} \quad (14)$$

with

$$A = P \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} Q^T,$$

where $P \in \mathbb{R}^{m \times m}$ y $Q \in \mathbb{R}^{n \times n}$, are the orthogonal matrices, $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_n)$, $\sigma_1 \geq \dots \geq \sigma_n \geq 0$, and the subspace

$$\mathcal{T}'_1 = \left\{ T \in \mathbb{R}^{n \times n} : T = \Sigma Q^T \sum_{l=1}^n \alpha_l G_l \right\}.$$

As in the general and upper triangular Toeplitz cases the projection is characterized on \mathcal{T}'_1 .

Theorem 3.2. *If $C_1 \in \mathbb{R}^{n \times n}$, then the unique solution of the problem*

$$\begin{cases} \min & \|T - C_1\|_{\mathbb{F}}^2 \\ \text{s.t} & \\ & T \in \mathcal{T}'_1 \end{cases} \quad (15)$$

is given by

$$P_{\mathcal{T}'}(C_1) = \Sigma Q^T \sum_{l=1}^n \beta^*_l G_l, \quad \beta^*_l = \frac{\sum_{i=1}^n \sum_{j<l} (C_1)_{ij} Q_{(j+l-1)i}}{\sum_{i=1}^n \sum_{j<l} (Q_{(j+l-1)i})^2 \sigma_i^2},$$

for all $1 \leq l \leq n$.

Proof. Similar to the proof of Teorema 2.1. □

4 The symmetric Toeplitz problem

This section is concerned with the problem

$$\begin{cases} \min & \|AX - B\|_{\mathbb{F}}^2 \\ \text{s.t} & \\ & X \in \mathcal{T} \cap S, \end{cases} \tag{16}$$

where $A, B \in \mathbb{R}^{m \times n}$, \mathcal{T} is the subspace of the Toeplitz matrices and S is the subspace of the symmetric matrices of $\mathbb{R}^{n \times n}$. Clearly, the solution is a matrix which is symmetric not only with respect to the main diagonal but also to the SW-NE diagonal. Since the feasibility region is the intersection of two subspace and because we know how to solve the persymmetric Procrustes problem [5] and the general Toeplitz problem, see section 2, we develop an algorithm based on the alternating projection method [3, 1, 6, 7]. Therefore the solution of (16) is obtained projecting alternating on the subspaces \mathcal{T} y S .

Given the problem

$$\begin{cases} \min & \|AX - B\|_{\mathbb{F}}^2 \\ \text{s.t} & \\ & X \in S, \end{cases} \tag{17}$$

according with Escalante and Raydán [7] it can be transformed, via the singular

value decomposition of A in the following equivalent problem

$$\begin{cases} \min & \|Z - C_1\|_F^2 \\ \text{s.t} & \\ & Z \in S', \end{cases} \quad (18)$$

where

$$Z = \Sigma Y, Y = Q^T X Q,$$

$$C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = P^T B Q,$$

$$C_1 \in \mathbb{R}^{n \times n}.$$

and the feasibility region

$$S' = \{Z \in \mathbb{R}^{n \times n} : Z = \Sigma Y, Y = Y^T\}.$$

Hence if the solution of the last problem is

$$P_{S'}(C_1) = \Sigma Y_* = \Sigma(Q^T X_* Q),$$

where Y_* is obtained by the methods proposed by Higham for the symmetric case [10], the solution of the first problem is

$$X_* = Q \Sigma^{-1} P_{S'}(C_1) Q^T.$$

On the other hand, we know how to solve the general Toeplitz problem, see section 2. Combining both results, to solve the problem 16 is equivalent to solve

$$\begin{cases} \min & \|Z - C_1\|_F^2 \\ \text{s.t} & \\ & Z \in \mathcal{T}'' \cap S'. \end{cases} \quad (19)$$

where

$$\mathcal{T}'' = \left\{ Z \in \mathbb{R}^{n \times n} : Z = \Sigma Q^T \sum_{l=-(n-1)}^{(n-1)} \alpha_l G_l Q \right\}.$$

Remark. The set \mathcal{T}'' , comes from the fact that the symmetric and Toeplitz problems have the same objective function.

The alternating projection algorithm for the symmetric Toeplitz case (19) is as follows:

Algorithm 4.1. Given $A, B \in \mathbb{R}^{m \times n}$. $X_0 = C_1 \in \mathbb{R}^{n \times n}$, where C_1 is obtained via the singular value decomposition of A , for $i = 0, 1, 2, \dots$, until convergence repeat

$$X_i = P_{S'}(X_i)$$

$$X_{i+1} = P_{\mathcal{T}''}(X_i).$$

The algorithm terminates when two consecutive projection on the same subspace are close enough. The convergence is guaranteed by the general theory established by von Neumann [12] for the alternating projection algorithm.

5 Numerical experience

In this section we show some examples. The algorithms have been implemented in Fortran, by using FORTRAN POWER STATION (1993) in an environment PC with a processor Pentium(R2) Intelmmx(TM) Technology, 31.0 MB de RAM. and 32 bits of virtual memory. The LINPACK library dsydc subroutine have been used to obtain the singular value decomposition.

Example 5.1. The general problem (2) has been solved with the following matrices

$$A = \begin{bmatrix} 1 & 1 & 2 & -2 & 2 & 1 \\ 0 & 2 & -1 & -2 & -3 & 2 \\ 0 & 2 & 1 & -1 & 2 & 2 \\ 1 & -1 & -1 & 1 & -1 & -1 \\ 2 & 2 & -1 & 2 & 0 & 1 \\ 3 & -1 & 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 & 1 \\ 1 & -1 & -2 & 0 & -1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & -1 & 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 2 & -1 & 0 & 1 \\ -1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & -1 & 1 \\ -1 & 1 & 2 & -1 & 2 & 2 \\ 1 & 1 & 2 & 0 & -1 & 1 \\ 1 & 2 & -1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ -1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & -1 & 0 \end{bmatrix}.$$

The solution X_* is

$$\begin{bmatrix} -3.49 \times 10^{-2} & 9.34 \times 10^{-3} & 1.37 \times 10^{-2} & 1.34 \times 10^{-2} & 1.68 \times 10^{-2} & 1.66 \times 10^{-2} \\ -5.43 \times 10^{-2} & -3.49 \times 10^{-2} & 9.34 \times 10^{-3} & 1.37 \times 10^{-2} & 1.34 \times 10^{-2} & 1.68 \times 10^{-2} \\ -7.2 \times 10^{-3} & -5.43 \times 10^{-2} & -3.49 \times 10^{-2} & 9.34 \times 10^{-3} & 1.37 \times 10^{-2} & 1.34 \times 10^{-2} \\ 1.46 \times 10^{-2} & -7.2 \times 10^{-3} & -5.43 \times 10^{-2} & -3.49 \times 10^{-2} & 9.34 \times 10^{-3} & 1.37 \times 10^{-2} \\ 8.75 \times 10^{-4} & 1.46 \times 10^{-2} & -7.2 \times 10^{-3} & -5.43 \times 10^{-2} & -3.49 \times 10^{-2} & 9.34 \times 10^{-3} \\ -2.268 \times 10^{-2} & 8.75 \times 10^{-4} & 1.46 \times 10^{-2} & -7.2 \times 10^{-3} & -5.43 \times 10^{-2} & -3.49 \times 10^{-2} \end{bmatrix}.$$

We project on the sets \mathcal{T}_u y \mathcal{T}_1 .

a) The projection on \mathcal{T}_u is

$$\begin{bmatrix} -3.53 \times 10^{-2} & 8.49 \times 10^{-3} & 2.03 \times 10^{-2} & 1.15 \times 10^{-2} & 4.34 \times 10^{-2} & 0 \\ 0 & -3.53 \times 10^{-2} & 8.49 \times 10^{-3} & 2.03 \times 10^{-2} & 1.15 \times 10^{-2} & 4.34 \times 10^{-2} \\ 0 & 0 & -3.53 \times 10^{-2} & 8.49 \times 10^{-3} & 2.03 \times 10^{-2} & 1.15 \times 10^{-2} \\ 0 & 0 & 0 & -3.53 \times 10^{-2} & 8.49 \times 10^{-3} & 2.03 \times 10^{-2} \\ 0 & 0 & 0 & 0 & -3.53 \times 10^{-2} & 8.49 \times 10^{-3} \\ 0 & 0 & 0 & 0 & 0 & -3.53 \times 10^{-2} \end{bmatrix}.$$

b) The projection on \mathcal{T}_1 is

$$\begin{bmatrix} -4.06 \times 10^{-2} & 0 & 0 & 0 & 0 & 0 \\ -2.97 \times 10^{-2} & -4.06 \times 10^{-2} & 0 & 0 & 0 & 0 \\ -1.79 \times 10^{-2} & -2.97 \times 10^{-2} & -4.06 \times 10^{-2} & 0 & 0 & 0 \\ 6.02 \times 10^{-2} & -1.79 \times 10^{-2} & -2.97 \times 10^{-2} & -4.06 \times 10^{-2} & 0 & 0 \\ -1.78 \times 10^{-3} & 6.02 \times 10^{-2} & -1.79 \times 10^{-2} & -2.97 \times 10^{-2} & -4.06 \times 10^{-2} & 0 \\ -2.27 \times 10^{-2} & -1.78 \times 10^{-3} & 6.02 \times 10^{-2} & -1.79 \times 10^{-2} & -2.97 \times 10^{-2} & -4.06 \times 10^{-2} \end{bmatrix}.$$

Example 5.2. We illustrate the behavior of the algorithm 4.1 for the symmetric case carrying out the following example

$$A = \begin{bmatrix} 0 & 1 & 3 & -1 \\ -1 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -1 & -1 & 3 \\ 0 & 1 & 0 & 0 \\ 2 & 2 & 2 & 2 \\ 0 & 1 & 1 & 0 \end{bmatrix}.$$

The solution is

$$X_* = \begin{bmatrix} 1.729 \times 10^{-8} & 2.325 \times 10^{-9} & 3.810 \times 10^{-9} & 2.598 \times 10^{-7} \\ 2.325 \times 10^{-9} & 1.729 \times 10^{-8} & 2.325 \times 10^{-9} & 4.041 \times 10^{-9} \\ 3.810 \times 10^{-9} & 2.325 \times 10^{-9} & 1.729 \times 10^{-8} & 2.657 \times 10^{-9} \\ 2.598 \times 10^{-7} & 4.041 \times 10^{-9} & 2.657 \times 10^{-9} & 3.458 \times 10^{-8} \end{bmatrix}$$

The algorithm terminates when two consecutive projections on the subspace of the Toeplitz matrices is less or equal to a fixed tolerance. We set $\epsilon = 10^{-6}$ for this experiment.

6 Concluding remarks

We have studied the Procrustes Toeplitz problem, analyzing the general, the triangular and the symmetric cases. In the last case an algorithm based on the alternating projection method is proposed. We illustrate our results with some examples.

The Toeplitz systems $Tx = b$ where $T \in \mathbb{R}^{n \times n}$ is a Toeplitz matrix, are present in many disciplines (signal and image processing, times series analyzes, queuing networks, partial differential equations problems and control problems). When these systems are solved via preconditioned conjugate gradient methods, the search of adequate preconditioners is the main issue. Among the possible preconditioners for a Toeplitz matrix, Chan proposes the preconditioner $c(T)$ defined to be the minimizer of $\|C - T\|_F^2$. Therefore a possible direction of future work is the application of the results obtained to answer questions related to Toeplitz systems. The reader interested in this subject can review the survey of Chan and Ng [2] and the references therein.

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