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A family of uniformly accurate order Lobatto-Runge-Kutta collocation methods

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Abstract. We consider the construction of an interpolant for use with Lobatto-Runge-Kutta collocation methods. The main aim is to derive single symmetric continuous solution(interpolant) for uniform accuracy at the step points as well as at the off-step points whose uniform order six everywhere in the interval of consideration. We evaluate the continuous scheme at different off-step points to obtain multi-hybrid schemes which if desired can be solved simultaneously for dense approximations. The multi-hybrid schemes obtained were converted to Lobatto-Runge-Kutta collocation methods for accurate solution of initial value problems. The unique feature of the paper is the idea of using all the set of off-step collocation points as additional interpolation points while symmetry is retained naturally by integration identities as equal areas under the various segments of the solution graph over the interval of consideration. We show two possible ways of implementing the interpolant to achieve the aim and compare them on some numerical examples.

Mathematical subject classification: 65L05.

Key words: Block hybrid scheme, Continuous scheme, Lobatto-Runge-Kutta collocation method, Symmetric scheme.

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1 Introduction

In this paper we consider the construction of Lobatto-Runge-Kutta collocation methods due to their excellent stability and stiffly accurate characteristic properties for the direct integration of initial value problem, possibly stiff, of the form

$$y'(x) = f(x, y(x)), \quad y(x_0) = y_0, \quad (x_0 \leq x \leq b). \tag{1}$$

Here the unknown function y is a mapping $[x_0, b] \to R^N$, the right–hand side function f is $[x_0, b] \times R^N \to R^N$ and the initial vector $y(x_0)$ is given in R^N. For the solution of (1) we seek the following form of a continuous multi-step collocation approximation formula [5] which was a generalization of [4] defined for the interval $[x_0, b]$ by

$$y(x) = \sum_{j=0}^{t-1} \alpha_j(x) y_{n+j} + h \sum_{j=0}^{s-1} \beta_j(x) f_{n+j} \tag{2}$$

where t denotes the number of interpolation points x_j, $j = 0, 1, \ldots, t-1$ and s denotes the distinct collocation points $\overline{x}_j \in [x_0, b]$, $j = 0, 1, 2, \ldots$, $s-1$, belonging to the given interval. The step size h can be a variable, it is assumed in this paper as a constant for simplicity, with the given mesh $x_n : x_n = x_0 + nh, n = 0, 1, 2, \ldots, N$ where $h = x_{n+1} - x_n, N = (b - a)/h$ and a set of equally spaced points on the integration interval given by $x_0 < x_1 < \cdots < x_{n+1} = b$. Also we assumed that (1) has exactly one solution and $\alpha_j(x)$ and $h\beta_j(x)$ in (2) are to be represented by the polynomials:

$$\alpha_j(x) = \sum_{i=0}^{t+s-1} \alpha_{j,i+1} x^i, \quad h\beta_j(x) = \sum_{i=0}^{t+s-1} h\beta_{j,i+1} x^i, \tag{3}$$

with constant coefficients $\alpha_{j,i+1}$ and $\beta_{j,i+1}$ to be determined. Proceeding in the same way as is done for linear multi-step methods, we expand $y(x)$ in (2) using Taylor series method of expansion about x and collect powers in h to obtain the methods. This takes the following form:

Inserting (3) into (2) we have

$$
\begin{aligned}
y(x) &= \sum_{j=0}^{t-1}\sum_{i=0}^{t+s-1} \alpha_{j,i+1} x^i y_{n+j} + \sum_{j=0}^{s-1}\sum_{i=0}^{t+s-1} h\beta_{j,i+1} x^i f_{n+j} \\
&= \sum_{i=0}^{t+s-1} \left\{ \sum_{j=0}^{t-1} \alpha_{j,i+1} y_{n+j} + \sum_{j=0}^{s-1} h\beta_{j,i+1} f_{n+j} \right\} x^i .
\end{aligned} \tag{4}
$$

Writing

$$
a_i = \sum_{j=0}^{t-1} \alpha_{j,i+1} y_{n+j} + \sum_{j=0}^{s-1} h\beta_{j,i+1} f_{n+j}
$$

such that (4) reduces to

$$
y(x) = \sum_{i=0}^{t+s-1} a_i x^i \tag{5}
$$

which can now be express in the form

$$
y(x) = \left\{ \sum_{j=0}^{t-1} \alpha_{j,t+s-1} y_{n+j} + \sum_{j=0}^{s-1} h\beta_{j,t+s-1} f_{n+j} \right\} (1, x, x^2, \cdots, x^{t+s-1})^T .
$$

Thus, we can express equation (5) explicitly as follows:

$$
y(x) = (y_n, \cdots, y_{n+t-1}, f_n, \cdots, f_{n+s-1}) C^T (1, x, x^2, \cdots, x^{t+s-1})^T \tag{6}
$$

where

$$
C = \begin{pmatrix}
C_{1,1} & \cdots & C_{1,t} & C_{1,t+1} & \cdots & C_{1,t+s} \\
C_{2,1} & \cdots & C_{2,t} & C_{2,t+1} & \cdots & C_{2,t+s} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
C_{t,1} & \cdots & C_{t,t} & C_{t,t+1} & \cdots & C_{t,t+s} \\
C_{t+1,1} & \cdots & C_{t+1,t} & C_{t+1,t+1} & \cdots & C_{t+1,t+s} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
C_{t+s,1} & \cdots & C_{t+s,t} & C_{t+s,t+1} & \cdots & C_{t+s,t+s}
\end{pmatrix} = D^{-1} \tag{7}
$$

and

$$D = \begin{pmatrix} 1 & x_n & x_n^2 & \cdots & x_n^{t+s-1} \\ 1 & x_{n+1} & x_{n+1}^2 & \cdots & x_{n+1}^{t+s-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n+t-1} & x_{n+t-1}^2 & \cdots & x_{n+t-1}^{t+s-1} \\ 0 & 1 & 2x_n & \cdots & (t+s-1)x_n^{t+s-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 2x_{n+s} & \cdots & (t+s-1)x_{n+s}^{t+s-2} \end{pmatrix} \tag{8}$$

are matrices of dimensions $(t+s) \times (t+s)$. We call D the multistep collocation and interpolation matrix which has a very simple structure. It is similar to Vandermonde matrix, consisting of distinct elements, nonsingular, and of dimension $(s+t) \times (s+t)$. This matrix affects the efficiency, accuracy and stability properties of (2). The choice $C = D^{-1}$ leads to the determination of the constant coefficients $\alpha_{j,i+1}$ and $\beta_{j,i+1}$. It was shown in [5, 7] that the method (2) is convergent with order $p = t + s - 1$. We now examine in more detail how the constant coefficients $\alpha_{j,i+1}$ and $\beta_{j,i+1}$ of equation (2) are obtained for the new Lobatto-Runge-Kutta collocation methods.

Remark 1.1. $y(x)$ given in (6), is the proposed collocation and interpolation polynomial for (1). From the structure of the matrix D the inverse matrix exists because the rows are linearly independent as they have distinct values like the Vandermonde matrix. The class of linear multistep methods (2) becomes a special class of the multistep collocation method when $s = t + 1$ and $x \in [x_0, b]$ which can also be solved simultaneously to obtain Lobatto-Runge-Kutta collocation methods. This interesting connection between the multistep collocation and Runge-Kutta methods is well discussed in [9].

2 Derivation of Lobatto-Runge-Kutta collocation methods

In this section we consider some specific methods that involve square matrices D and C both of dimensions $(t+s) \times (t+s)$. From equation (7) $C = D^{-1}$ where $C = (c_{i,j}), ij = 1, 2, 3, \cdots, t+s$; $D = (d_{i,j}), ij = 1, 2, 3, \cdots, t+s$; and $I = (e_{i,j}), ij = 1, 2, 3, \cdots, t+s$, see [5] for an algorithm to obtain the elements of the matrices C, D and I. We shall derive multistep collocation method as continuous single finite difference formula of non-uniform order six based on Lobatto points see [8]. For $s = 4$, $t = 1$ and $\Theta = (x - x_n)$ convergence throughout the interval $[x_0, b]$, the matrix D of equation (8) takes the form:

$$D = \begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 \\ 0 & 1 & 2x_{n+u} & 3x_{n+u}^2 & 4x_{n+u}^3 \\ 0 & 1 & 2x_{n+v} & 3x_{n+v}^2 & 4x_{n+v}^3 \end{pmatrix} \tag{9}$$

where u and v are zeros of $L_m(x) = 0$, Lobatto polynomial [8] of degree m which after certain transformation, we obtain

$$\overline{x}_0 = x_{n+u},\ u = \left(\frac{1}{2} - \frac{\sqrt{5}}{10}\right), \overline{x}_1 = x_{n+v},\ v = \left(\frac{1}{2} + \frac{\sqrt{5}}{10}\right) \tag{10}$$

which are valid in the interval $[x_0, b]$. Inverting the matrix D in equation (9) once, using computer algebra, for example, Maple or Matlab software package we obtain the continuous scheme as:

$$y\big(\Theta + x_n\big) = \alpha_0(x) y_n + \big[\beta_0(x) f_n + \beta_1(x) f_{n+u} + \beta_2(x) f_{n+v} + \beta_3(x) f_{n+1}\big], \tag{11}$$

where

$$\alpha_0(x) = -1$$

$$\beta_0(x) = \left[\frac{-3\Theta^4 + 4(v + u + 1)h\Theta^3 - 6(vu + v + u)h^2\Theta^2 + 12vuh^3\Theta}{12vuh^3}\right]$$

$$\beta_1(x) = \left[\frac{-3\Theta^4 + 4(v + 1)h\Theta^3 - 6vh^2\Theta^2}{12u(v - u)(u - 1)h^3}\right]$$

$$\beta_2(x) = \left[\frac{3\Theta^4 - 4(u + 1)h\Theta^3 + 6uh^2\Theta^2}{12v(v - u)(v - 1)h^3}\right]$$

$$\beta_3(x) = \left[\frac{3\Theta^4 - 4(v + u)h\Theta^3 + 6vuh^2\Theta^2}{12(v - 1)(u - 1)h^3}\right].$$

We evaluate $y(x)$ in (11) at the following point $x = x_{n+1}$, we recovered the well known Lobatto IIIA with $s = 4$ and order $p = 6$, see [1] page 210, where $D_i's(i = 5, 7)$ are the error constants.

$$y_{n+1} = y_n + \frac{h}{12}\left[f_n + 5f_{n+u} + 5f_{n+v} + f_{n+1}\right]$$
$$\text{order } p = 6, \qquad D_7 = -6.613 \times 10^{-7}$$

$$\begin{aligned} y_{n+u} = y_n + \frac{h}{120}\big[&(11+\sqrt{5})f_n + (25-\sqrt{5})f_{n+u} \\ &+(25-13\sqrt{5})f_{n+v} + (-1+\sqrt{5})f_{n+1}\big] \end{aligned}$$
$$\text{order } p = 4, \qquad D_5 = -7.45 \times 10^{-5}$$

$$\begin{aligned} y_{n+v} = y_n + \frac{h}{120}\big[&(11-\sqrt{5})f_n + (25+13\sqrt{5})f_{n+u} \\ &+(25+\sqrt{5})f_{n+v} + (-1-\sqrt{5})f_{n+1}\big] \end{aligned}$$
$$\text{order } p = 4, \qquad D_5 = 7.45 \times 10^{-5}.$$

We converted the block hybrid scheme above to Lobatto-Runge-Kutta collocation method, written as:

$$y_n = y_{n-1} + h\left(\frac{1}{12}\right)F_1 + h\left(\frac{5}{12}\right)F_2 + h\left(\frac{5}{12}\right)F_3 + h\left(\frac{1}{12}\right)F_4 \tag{12}$$

The stage values at the nth step are computed as:

$$\begin{aligned} Y_1 &= y_{n-1} \\ Y_2 &= y_{n-1} + h\left(\frac{11}{120} + \frac{\sqrt{5}}{120}\right)F_1 + h\left(\frac{5}{24} - \frac{\sqrt{5}}{120}\right) \\ &\quad F_2 + h\left(\frac{5}{24} - \frac{13\sqrt{5}}{120}\right)F_3 + h\left(\frac{-1}{120} + \frac{\sqrt{5}}{120}\right)F_4 \\ Y_3 &= y_{n-1} + h\left(\frac{11}{120} - \frac{\sqrt{5}}{120}\right)F_1 + h\left(\frac{5}{24} + \frac{13\sqrt{5}}{120}\right) \\ &\quad F_2 + h\left(\frac{5}{24} + \frac{\sqrt{5}}{120}\right)F_3 + h\left(-\frac{1}{120} - \frac{\sqrt{5}}{120}\right)F_4 \\ Y_4 &= y_{n-1} + h\left(\frac{1}{12}\right)F_1 + h\left(\frac{5}{12}\right)F_2 + h\left(\frac{5}{12}\right)F_3 + h\left(\frac{1}{12}\right)F_4 \end{aligned}$$

with the stage derivatives as follows:

$$F_1 = f(x_{n-1} + h(0), Y_1)$$

$$F_2 = f\left(x_{n-1} + h\left(\frac{1}{2} - \frac{\sqrt{5}}{10}\right), Y_2\right)$$

$$F_3 = f\left(x_{n-1} + h\left(\frac{1}{2} + \frac{\sqrt{5}}{10}\right), Y_3\right)$$

$$F_4 = f(x_{n-1} + h(1), Y_4).$$

3 Uniformly accurate order six Lobatto-Runge-Kutta Collocation methods

By careful selection of interpolation and collocation points inside the interval $[x_0, b]$, leads to a single continuous finite difference method whose members are of uniform accuracies see [6] and [7]. For $s = 4, t = 3$ to yield uniformly accurate order six convergence (accuracy) throughout the interval $[x_0, b]$, the matrix D of equation (8) takes the form:

$$D = \begin{pmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 \\ 1 & x_{n+u} & x_{n+u}^2 & x_{n+u}^3 & x_{n+u}^4 & x_{n+u}^5 & x_{n+u}^6 \\ 1 & x_{n+v} & x_{n+v}^2 & x_{n+v}^3 & x_{n+v}^4 & x_{n+v}^5 & x_{n+v}^6 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 \\ 0 & 1 & 2x_{n+u} & 3x_{n+u}^2 & 4x_{n+u}^3 & 5x_{n+u}^4 & 6x_{n+u}^5 \\ 0 & 1 & 2x_{n+v} & 3x_{n+v}^2 & 4x_{n+v}^3 & 5x_{n+v}^4 & 6x_{n+v}^5 \\ 0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 & 5x_{n+1}^4 & 6x_{n+1}^5 \end{pmatrix} \tag{13}$$

where u and v are obtained in a similar manner as in equation (10) which are also valid in the interval $[x_0, b]$. Inverting the matrix D in equation (13) once, using MAPLE or MATLAB software package we obtain the continuous scheme as follows:

$$\begin{aligned} y\big(\Theta + x_n\big) &= \alpha_0(x)y_n + \alpha_1(x)y_{n+u} + \alpha_2(x)y_{n+v} \\ &+ \big[\beta_0(x)f_n + \beta_1(x)f_{n+u} + \beta_2(x)f_{n+v} + \beta_3(x)f_{n+1}\big] \end{aligned} \tag{14}$$

where

$$\alpha_0(x) = \left[\frac{\frac{-12}{5}\Theta^6 + \frac{36}{5}h\Theta^5 - \frac{198}{25}h^2\Theta^4 + \frac{96}{25}h^3\Theta^3 - \frac{18}{25}h^4\Theta^2 + \frac{6}{625}h^6}{u^3v^3h^6[3 - 2v - 2u + vu]}\right]$$

$$\alpha_1(x) = \left[\frac{\frac{\Theta^6(24-12\sqrt{5})}{10} - \frac{6h\Theta^5(50-26\sqrt{5})}{50} + \frac{3h^2\Theta^4(82-46\sqrt{5})}{50} - \frac{2h^3\Theta^3(36-24\sqrt{5})}{50} + \frac{3h^4\Theta^2(2-2\sqrt{5})}{50}}{u^3h^6(v-u)(v-u)(u-v)[3 - 2v - 2u + vu]}\right]$$

$$\alpha_2(x) = \left[\frac{\frac{-\Theta^6(24+12\sqrt{5})}{10} + \frac{6h\Theta^5(50+26\sqrt{5})}{50} - \frac{3h^2\Theta^4(82+46\sqrt{5})}{50} + \frac{2h^3\Theta^3(36+24\sqrt{5})}{50} - \frac{3h^4\Theta^2(2+2\sqrt{5})}{50}}{v^3h^6(v-u)(v-u)(u-v)[3 - 2v - 2u + vu]}\right]$$

$$\beta_0(x) = \left[\frac{\frac{-11}{5}\Theta^6 + \frac{34}{5}h\Theta^5 - \frac{197}{25}h^2\Theta^4 + \frac{106}{25}h^3\Theta^3 - \frac{131}{125}h^4\Theta^2 + \frac{12}{125}h^5\Theta}{u^2v^2h^5[6 - 4v - 4u + 2vu]}\right]$$

$$\beta_1(x) = \left[\frac{\frac{-(19+\sqrt{5})\Theta^6}{10} + \frac{2(125+11\sqrt{5})h\Theta^5}{50} - \frac{(229+31\sqrt{5})h^2\Theta^4}{50} + \frac{2(215+41\sqrt{5})h^3\Theta^3}{250} - \frac{(55+13\sqrt{5})h^4\Theta^2}{250}}{u^2h^5(v-u)(u-v)(u-1)[6 - 4v - 4u + 2vu]}\right]$$

$$\beta_2(x) = \left[\frac{\frac{(19-\sqrt{5})\Theta^6}{10} - \frac{2(125-11\sqrt{5})h\Theta^5}{50} + \frac{(229-31\sqrt{5})h^2\Theta^4}{50} - \frac{2(215-41\sqrt{5})h^3\Theta^3}{250} + \frac{(55-13\sqrt{5})h^4\Theta^2}{250}}{v^2h^5(v-u)(v-v)(v-1)[6 - 4v - 4u + 2vu]}\right]$$

$$\beta_3(x) = \left[\frac{\Theta^6 - 2h\Theta^5 + \frac{7}{5}h^2\Theta^4 - \frac{2}{5}h^3\Theta^3 + \frac{1}{25}h^4\Theta^2}{h^5(v-1)(u-1)[6 - 4v - 4u + 2vu]}\right].$$

We evaluate $y(x)$ in (14) and its first derivative at the point **w** midway between x_0 and b and at the point **r** midway between x_n and the point **w**, we obtain the following 4-block hybrid scheme with uniformly accurate order six:

$$y_{n+1} = y_n + \frac{h}{12}\left[f_n + 5f_{n+u} + 5f_{n+v} + f_{n+1}\right]$$
$$\text{order } p = 6, \qquad D_7 = -6.613 \times 10^{-7}$$

$$64y_{n+w} - 14y_n - 25y_{n+u} - 25y_{n+v}$$
$$= \frac{h}{12}\left[13f_n + (35 + 30\sqrt{5})f_{n+u} + (35 - 30\sqrt{5})f_{n+v} + f_{n+1}\right]$$
$$\text{order } p = 6, \qquad D_7 = -4.629 \times 10^{-6}$$

$$4096y_{n+r} - 46y_n - (2025 + 900\sqrt{5})y_{n+u} - (2025 - 900\sqrt{5})y_{n+v}$$
$$= \frac{h}{12}\left[37f_n - (625 + 270\sqrt{5})f_{n+u} - (625 - 270\sqrt{5})f_{n+v} + f_{n+1}\right]$$
$$\text{order } p = 6, \qquad D_7 = -6.613 \times 10^{-6}$$

$$\begin{aligned}
&150y_n - (75+50\sqrt{5})y_{n+u} - (75-50\sqrt{5})y_{n+v}\\
&= \frac{h}{12}\big[-119f_n + (935+390\sqrt{5})f_{n+u} - 2048f_{n+r}\\
&\qquad +(935-390\sqrt{5})f_{n+v} - 3f_{n+1}\big]\\
&\text{order } p=6, \qquad D_7 = 2.017\times 10^{-5}
\end{aligned}$$

$$\begin{aligned}
&25\sqrt{5}y_{n+u} - 25\sqrt{5}y_{n+v} = \frac{h}{12}\big[f_n - 55f_{n+u} - 192f_{n+w} - 55f_{n+v} + f_{n+1}\big]\\
&\text{order } p=6, \qquad D_7 = -2.645\times 10^{-6}.
\end{aligned}$$

Solving the block hybrid scheme simultaneously we obtain the following block accurate scheme:

$$\begin{aligned}
y_{n+1} &= y_n + \frac{h}{12}\big[f_n + 5f_{n+u} + 5f_{n+v} + f_{n+1}\big]\\
y_{n+r} &= y_n + \frac{h}{3072}\big[202f_n + 3456f_{n+r} - (1555+675\sqrt{5})f_{n+u}\\
&\quad + 216f_{n+w} - (1555-675\sqrt{5})f_{n+v} + 4f_{n+1}\big]\\
y_{n+u} &= y_n + \frac{h}{9000}\big[(585+3\sqrt{5})f_n + 10240f_{n+r}\\
&\quad - (4125+2115\sqrt{5})f_{n+u} + (1920-576\sqrt{5})f_{n+w}\\
&\quad - (4125-1785\sqrt{5})f_{n+v} + (5+3\sqrt{5})f_{n+1}\big]\\
y_{n+w} &= y_n + \frac{h}{576}\big[39f_n + 512f_{n+r} - (180+75\sqrt{5})f_{n+u}\\
&\quad + 96f_{n+w} - (180-75\sqrt{5})f_{n+v} + f_{n+1}\big]\\
y_{n+v} &= y_n + \frac{h}{9000}\big[(585-3\sqrt{5})f_n + 10240f_{n+r}\\
&\quad - (4125+1785\sqrt{5})f_{n+u} + (1920+576\sqrt{5})f_{n+w}\\
&\quad - (4125-2115\sqrt{5})f_{n+v} + (5-3\sqrt{5})f_{n+1}\big].
\end{aligned}$$

We converted the block scheme to Lobatto-Runge-Kutta collocation method:

$$y_n = y_{n-1} + h\left(\frac{1}{12}\right)F_1 + h\left(\frac{5}{12}\right)F_3 + h\left(\frac{5}{12}\right)F_5 + h\left(\frac{1}{12}\right)F_6 \tag{15}$$

with the stage values at the nth step calculated as:

$$
\begin{aligned}
Y_1 &= y_{n-1}\\
Y_2 &= y_{n-1} + h\left(\frac{101}{1536}\right)F_1 + h\left(\frac{9}{8}\right)F_2 - h\left(\frac{1555}{3072} + \frac{225\sqrt{5}}{1024}\right)F_3\\
&\quad + h\left(\frac{9}{128}\right)F_4 - h\left(\frac{1555}{3072} - \frac{225\sqrt{5}}{1024}\right)F_5 + h\left(\frac{1}{768}\right)F_6\\
Y_3 &= y_{n-1} + h\left(\frac{13}{200} + \frac{\sqrt{5}}{3000}\right)F_1 + h\left(\frac{256}{225}\right)F_2 - h\left(\frac{11}{24} + \frac{47\sqrt{5}}{200}\right)F_3\\
&\quad + h\left(\frac{16}{75} - \frac{8\sqrt{5}}{125}\right)F_4 - h\left(\frac{11}{24} - \frac{119\sqrt{5}}{600}\right)F_5 + h\left(\frac{1}{1800} + \frac{\sqrt{5}}{3000}\right)F_6\\
Y_4 &= y_{n-1} + h\left(\frac{13}{192}\right)F_1 + h\left(\frac{8}{9}\right)F_2 - h\left(\frac{5}{16} + \frac{25\sqrt{5}}{192}\right)F_3\\
&\quad + h\left(\frac{1}{6}\right)F_4 - h\left(\frac{5}{16} - \frac{25\sqrt{5}}{192}\right)F_5 + h\left(\frac{1}{576}\right)F_6\\
Y_5 &= y_{n-1} + h\left(\frac{13}{200} - \frac{\sqrt{5}}{3000}\right)F_1 + h\left(\frac{256}{225}\right)F_2 - h\left(\frac{11}{24} - \frac{119\sqrt{5}}{600}\right)F_3\\
&\quad + h\left(\frac{16}{75} + \frac{8\sqrt{5}}{125}\right)F_4 - h\left(\frac{11}{24} - \frac{47\sqrt{5}}{200}\right)F_5 + h\left(\frac{1}{1800} - \frac{\sqrt{5}}{3000}\right)F_6\\
Y_6 &= y_{n-1} + h\left(\frac{1}{12}\right)F_1 + h\left(\frac{5}{12}\right)F_3 + h\left(\frac{5}{12}\right)F_5 + h\left(\frac{1}{12}\right)F_6
\end{aligned}
$$

where the stage derivatives are:

$$
\begin{aligned}
F_1 &= f\big(x_{n-1} + h(0), Y_1\big)\\
F_2 &= f\left(x_{n-1} + h\left(\frac{1}{4}\right), Y_2\right)\\
F_3 &= f\left(x_{n-1} + h\left(\frac{1}{2} - \frac{\sqrt{5}}{10}\right), Y_3\right)\\
F_4 &= f\left(x_{n-1} + h\left(\frac{1}{2}\right), Y_4\right)
\end{aligned}
$$

$$F_5 = f\left(x_{n-1} + h\left(\frac{1}{2} + \frac{\sqrt{5}}{10}\right), Y_5\right)$$
$$F_6 = f\big(x_{n-1} + h(1), Y_6\big).$$

We again evaluate $y(x)$, that is equation (14) and its first derivative at the point **q** one third between the point x_0 and b or the interval $[x_0, b]$ and combine its members with the members of the point **w**, to obtain the following 4-block hybrid method with uniformly accurate order six:

$$y_{n+1} = y_n + \frac{h}{12}\big[f_n + 5f_{n+u} + 5f_{n+v} + f_{n+1}\big]$$

$$729y_{n+q} - 29y_n - (350 + 150\sqrt{5})y_{n+u} - (350 - 150\sqrt{5})y_{n+v}$$
$$= \frac{h}{12}\big[25f_n + (245 + 120\sqrt{5})f_{n+u} + (245 - 120\sqrt{5})f_{n+v} + f_{n+1}\big]$$
$$\text{order } p = 6, \qquad D_7 = -5.952 \times 10^{-6}$$

$$64y_{n+w} - 14y_n - 25y_{n+u} - 25y_{n+v}$$
$$= \frac{h}{12}\big[13f_n + (35 + 30\sqrt{5})f_{n+u} + (35 - 30\sqrt{5})f_{n+v} + f_{n+1}\big]$$
$$\text{order } p = 6, \qquad D_7 = -4.629 \times 10^{-6}$$

$$(50 + 50\sqrt{5})y_{n+u} + (50 - 50\sqrt{5})y_{n+v} - 100y_n$$
$$= \frac{h}{3}\big[22f_n + (110 + 60\sqrt{5})f_{n+u} - 243f_{n+q} + (110 - 60\sqrt{5})f_{n+v} + f_{n+1}\big]$$
$$\text{order } p = 6, \qquad D_7 = -2.292 \times 10^{-5}$$

$$25\sqrt{5}y_{n+u} - 25\sqrt{5}y_{n+v} = \frac{h}{12}\big[f_n - 55f_{n+u} - 192f_{n+w} - 55f_{n+v} + f_{n+1}\big]$$
$$\text{order } p = 6, \qquad D_7 = -2.645 \times 10^{-6}.$$

Solving the block hybrid scheme simultaneously we obtain the following block accurate scheme:

$$y_{n+1} = y_n + \frac{h}{12}\big[f_n + 5f_{n+u} + 5f_{n+v} + f_{n+1}\big]$$

$$y_{n+u} = y_n + \frac{h}{3000}\big[(215 + \sqrt{5})f_n + (1375 + 545\sqrt{5})f_{n+u} - 2430f_{n+q}$$

$$+ (960 - 192\sqrt{5}) f_{n+w} + (1375 - 655\sqrt{5}) f_{n+v} + (5 + \sqrt{5}) f_{n+1}\big]$$

$$y_{n+q} = y_n + \frac{h}{2916}\big[211 f_n + (1255 + 600\sqrt{5}) f_{n+u} - 2268 f_{n+q} + 512 f_{n+w} + (1255 - 600\sqrt{5}) f_{n+v} + 7 f_{n+1}\big]$$

$$y_{n+w} = y_n + \frac{h}{386}\big[28 f_n + (155 + 75\sqrt{5}) f_{n+u} - 243 f_{n+q} + 96 f_{n+w} + (155 - 75\sqrt{5}) f_{n+v} + f_{n+1}\big]$$

$$y_{n+v} = y_n + \frac{h}{3000}\big[(215 - \sqrt{5}) f_n + (1375 + 655\sqrt{5}) f_{n+u} - 2430 f_{n+q} + (960 + 192\sqrt{5}) f_{n+w} + (1375 - 545\sqrt{5}) f_{n+v} + (5 - \sqrt{5}) f_{n+1}\big]$$

We converted the block scheme to Lobatto-Runge-Kutta collocation method:

$$y_n = y_{n-1} + h\left(\frac{1}{12}\right) F_1 + h\left(\frac{5}{12}\right) F_2 + h\left(\frac{5}{12}\right) F_5 + h\left(\frac{1}{12}\right) F_6 \tag{16}$$

with the stage values at the nth step calculated as:

$$Y_1 = y_{n-1}$$

$$Y_2 = y_{n-1} + h\left(\frac{43}{600} + \frac{\sqrt{5}}{3000}\right) F_1 + h\left(\frac{11}{24} + \frac{109\sqrt{5}}{600}\right) F_2 - h\left(\frac{81}{100}\right) F_3 + h\left(\frac{8}{25} - \frac{8\sqrt{5}}{125}\right) F_4 + h\left(\frac{11}{24} - \frac{131\sqrt{5}}{600}\right) F_5 + h\left(\frac{1}{600} + \frac{\sqrt{5}}{3000}\right) F_6$$

$$Y_3 = y_{n-1} + h\left(\frac{211}{2916}\right) F_1 + h\left(\frac{1255}{2916} + \frac{50\sqrt{5}}{243}\right) F_2 - h\left(\frac{7}{9}\right) F_3 + h\left(\frac{128}{729}\right) F_4 + h\left(\frac{1255}{2916} - \frac{50\sqrt{5}}{243}\right) F_5 + h\left(\frac{7}{2916}\right) F_6$$

$$Y_4 = y_{n-1} + h\left(\frac{7}{96}\right) F_1 + h\left(\frac{155}{384} + \frac{25\sqrt{5}}{128}\right) F_2 - h\left(\frac{81}{128}\right) F_3 + h\left(\frac{1}{4}\right) F_4 + h\left(\frac{155}{384} - \frac{25\sqrt{5}}{128}\right) F_5 + h\left(\frac{1}{384}\right) F_6$$

$$Y_5 = y_{n-1} + h\left(\frac{43}{600} - \frac{\sqrt{5}}{3000}\right)F_1 + h\left(\frac{11}{24} + \frac{131\sqrt{5}}{600}\right)F_2 - h\left(\frac{81}{100}\right)F_3$$
$$+ h\left(\frac{8}{25} + \frac{8\sqrt{5}}{125}\right)F_4 + h\left(\frac{11}{24} - \frac{109\sqrt{5}}{600}\right)F_5 + h\left(\frac{1}{600} - \frac{\sqrt{5}}{3000}\right)F_6$$

$$Y_6 = y_{n-1} + h\left(\frac{1}{12}\right)F_1 + h\left(\frac{5}{12}\right)F_2 + h\left(\frac{5}{12}\right)F_5 + h\left(\frac{1}{12}\right)F_6$$

where the stage derivatives are:

$$\begin{aligned}
F_1 &= f\big(x_{n-1} + h(0), Y_1\big)\\
F_2 &= f\left(x_{n-1} + h\left(\frac{1}{2} - \frac{\sqrt{5}}{10}\right), Y_2\right)\\
F_3 &= f\left(x_{n-1} + h\left(\frac{1}{3}\right), Y_3\right)\\
F_4 &= f\left(x_{n-1} + h\left(\frac{1}{2}\right), Y_4\right)\\
F_5 &= f\left(x_{n-1} + h\left(\frac{1}{2} + \frac{\sqrt{5}}{10}\right), Y_5\right)\\
F_6 &= f\big(x_{n-1} + h(1), Y_6\big)
\end{aligned}$$

4 Numerical illustrations

In order to test the new derived methods we present some numerical results. The error of the results obtained from computed and exact values at some selected mesh points are shown in the following Tables.

Example 1: $y' = -100(y - x^3) + 3x^2,\ y(0) = 1,\ y(x) = x^3 + e^{-100x}$

x	Exact value	Lobatto IIIA [1]	New Method (15)	Lobatto IIIA [1]	New Method (15)
0.1	0.00104539	−0.056844812	−0.0573941594	5.579×10^{-2}	5.843×10^{-3}
0.2	0.00800000	0.0104361344	0.01140987773	2.436×10^{-3}	3.409×10^{-4}
0.3	0.02700000	0.0247384839	0.02680088338	2.261×10^{-3}	1.991×10^{-5}
0.4	0.06400000	0.0602693436	0.06401162582	3.730×10^{-3}	1.162×10^{-7}
0.5	0.12500000	0.1190965075	0.12499931840	5.903×10^{-3}	6.816×10^{-8}

Table 1 – Numerical solutions of example 1, with $h = 0.1$ and their absolute errors.

Example 2: $y' = -y,\ y(0) = 1,\ y(x) = e^{-x}$

x	Exact value	Lobatto IIIA [1]	New method (15)	Lobatto IIIA [1]	New Method (15)
0.1	0.904837418035959	0.90483742012	0.904837418035899	2.093×10^{-9}	6.057×10^{-14}
0.2	0.818730753077981	0.81873075686	0.818730753077872	3.789×10^{-9}	1.098×10^{-13}
0.3	0.740818220681717	0.74081822582	0.740818220681569	5.143×10^{-9}	1.488×10^{-13}
0.4	0.670320046035639	0.67032005224	0.670320046035460	6.204×10^{-9}	1.793×10^{-13}
0.5	0.606530659712633	0.60653066673	0.606530659712431	7.020×10^{-9}	2.024×10^{-13}

Table 2 – Numerical solutions of example 2, with $h = 0.1$ and their absolute errors.

Example 3: $y' = \frac{y}{4}\left(1 - \frac{y}{20}\right),\ y(0) = 1,\ y(x) = \frac{20}{1+19e^{-x/4}}$

x	Exact value	Lobatto IIIA [1]	New Method (15)	Lobatto IIIA [1]	New Method (15)
0.1	1.024018962351866	1.0240189495	1.02401896229202	1.285×10^{-8}	5.984×10^{-11}
0.2	1.048582996382734	1.0485829696	1.04858299626072	2.672×10^{-8}	1.219×10^{-10}
0.3	1.073702928838836	1.0737028876	1.07370292865234	4.117×10^{-8}	1.864×10^{-10}
0.4	1.099389726731483	1.0993896711	1.09938972647798	5.561×10^{-8}	2.534×10^{-10}
0.5	1.125654495329782	1.1256544242	1.12565449500686	7.104×10^{-8}	3.229×10^{-10}

Table 3 – Numerical solutions of example 3, with, $h = 0.1$ and their absolute errors.

5 Conclusions

Consequently the numerical results of Tables 1, 2 and 3 revealed the novelty of the uniformly accurate order six methods which in fact give results closer to the exact solutions at the expense of very low computational cost. Moreover, as the first row of the matrix A consists of zeros, the first stage of each method coincides with the initial value. And due to the requirement of stiff accuracy, the last stage also coincides with the expression for the final point, which implies that no further function evaluation is necessary to obtain y_{n+1} in each of the method, see [3].

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