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New conservation laws for inviscid Burgers equation

IGOR LEITE FREIRE

Centro de Matemática, Computação e Cognição, Universidade Federal do ABC – UFABC
Rua Santa Adélia, 166, Bairro Bangu, 09210-170 Santo André, SP, Brasil

E-mails: igor.freire@ufabc.edu.br / igor.leite.freire@gmail.com

Abstract. In this paper it is shown that the inviscid Burgers equation is nonlinearly self-adjoint. Then, from Ibragimov's theorem on conservation laws, local conserved quantities are obtained.

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Key words: Lie point symmetry, Ibragimov's Theorem, conservation laws, inviscid Burgers equation, nonlinear self-adjoint equations, weak self-adjoint equations.

1 Introduction

In a previous work [4] the class of the self-adjoint equations (in the sense of the definition introduced by Ibragimov in [12]) of the type

$$u_t + f(t, x, u, u_x) = 0$$

was determined. In particular, it was proved that the inviscid Burgers equation

$$u_t + a(u)u_x = 0 \tag{1}$$

is (quasi) self-adjoint. In addition, from Ibragimov's theorem on conservation laws [13], conservation laws for projectable Lie point symmetries (see [21]) of (1) were established.

Recently Maria Luz Gandarias [8] and Nail Ibragimov [17, 18, 19] have generalized the previous concepts of self-adjoint equations [12, 13, 14].

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The recent developments allow one to find new conservation laws for equation (1). Thus in this paper the results obtained in [4] are complemented by using the new concepts [8, 17, 18, 19] combined with the powerful result [13].

The paper is organized as the follows: in the next section it is revisited Ibragimov's theorem on conservation laws and the concepts of self-adjoint equations. In the Section 3 the results regarding (1) obtained in [4] are discussed with the point of view of the new developments. In the following, some new conservation laws are established illustrating the results.

2 Revisiting previous results

2.1 Ibragimov's theorem on conservation laws

In what follows, it is assumed the summation over repeated indices, $x = (x^1, \ldots, x^n)$ and $u = (u^1, \ldots, u^m)$ denotes the indepedent variables and the depedents variables, respectively. The set of kth order derivatives and all differential functions of finite order shall be denoted by $\partial^k u$ and $\mathcal{A}$, respectively.

Ibragimov's theorem on conservation laws is

Theorem 2.1. *Any symmetry (Lie point, Lie-Bäcklund, nonlocal symmetry)*

$$X = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \eta_i^\alpha \frac{\partial}{\partial u_i^\alpha} + \eta_{ij}^\alpha \frac{\partial}{\partial u_{ij}^\alpha} + \cdots, \tag{2}$$

where $\xi^i,\ \eta^\alpha \in \mathcal{A}$, $\eta_i^\alpha = D_i(\eta^\alpha - \xi^j u_j^\alpha) + \xi^j u_{ij}^\alpha$, $\eta_{ij}^\alpha = D_i D_j(\eta^\alpha - \xi^k u_k^\alpha) + \xi^k u_{kij}^\alpha$, *etc, of the system of equations*

$$F_\alpha(x, u, \partial u, \ldots, \partial^s u) = 0 \tag{3}$$

with n *independent variables* $x = (x^1, \ldots, x^n)$ *and* m *dependent variables* $u = (u^1, \ldots, u^m)$ *is inherited by the adjoint equation. Specifically the operator*

$$Y = \xi^i \frac{\partial}{\partial x^i} + \eta^\alpha \frac{\partial}{\partial u^\alpha} + \eta_*^\beta \frac{\partial}{\partial v^\beta} \tag{4}$$

with an appropriately chosen coefficient η_* *is admitted by the system of equations* (3) *and its adjoint system*

$$F_\alpha^*(x, u, v, \partial u, \partial v, \ldots, \partial^s u, \partial^s v) := \frac{\delta(v^\beta F_\beta)}{\delta u^\alpha} = 0. \tag{5}$$

Furthermore, the combined system (3) *and* (5) *has the conservation law* $D_i C^i = 0$*, where*

$$\begin{aligned} C^i \;=\; & \xi^i \mathcal{L} + W_\alpha \left[\frac{\partial \mathcal{L}}{\partial u_i^\alpha} - D_j \left(\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} \right) + D_j D_k \frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} - \cdots \right] \\ & + D_j(W^\alpha) \left[\frac{\partial \mathcal{L}}{\partial u_{ij}^\alpha} - D_k \left(\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} \right) + \cdots \right] \\ & + D_j D_k (W^\alpha) \left[\frac{\partial \mathcal{L}}{\partial u_{ijk}^\alpha} - \cdots \right] + \cdots \end{aligned} \tag{6}$$

and $W^\alpha = \eta^\alpha - \xi^i u_i^\alpha$.

2.2 Quasi-self-adjoint and self-adjoint equations

The following definitions were introduced in [12, 13, 14].

Definition 2.2. *An equation* $F = 0$ *is said to be self-adjoint if there exists a function* $\phi = \phi(x, u, \ldots)$ *such that* $F^*|_{v=u} = \phi F$*. Thus* $F^*|_{v=u} = 0$ *if and only if* $F = 0$.

Definition 2.3. *An equation* $F = 0$ *is said to be quasi-self-adjoint if there exists a function* $\phi = \phi(x, u, \ldots)$ *such that* $F^*|_{v=\varphi(u)} = \phi F$*, with* $\varphi'(u) \neq 0$*. Thus* $F^*|_{v=\varphi(u)} = 0$ *if and only if* $F = 0$.

Whenever an equation is quasi-self-adjoint or self-adjoint, formulae (6) allow one to construct local conservation laws for the considered equation insteading of v by $\varphi(u)$ or u, respectively.

Recently many authors have been employing these concepts in order to establish conservation laws for equations and systems. For instance, Ibragimov, Torrisi and Tracinà determined the class of quasi-self-adjoint system derived from a $(2 + 1)$ generalized Burgers equation in [16]. Bruzón, Gandarias and Ibragimov determined a class of self-adjoint differential equations in [3]. Conservation laws for the Camassa-Holm equation was obtained by Ibragimov, Khamitova and Valenti in [20]. Further examples can be found in [4, 5, 15].

2.3 Conservation laws for inviscid Burgers equation

The adjoint equation to (1) is (see [4])

$$v_t + a(u)v_x = 0. \tag{7}$$

Let $F = v_t + a(u)u_x$. Then it is easy to see that $F^*|_{v=\varphi(u)} = -\varphi'(u)(u_t + a(u)u_x)$. Thus (1) is quasi-self-adjoint, for all smooth function $\varphi = \varphi(u)$. In particular, this holds for $\varphi = u$ and equation (1) is also self-adjoint.

From Ibragimov's theorem on conservation laws, a conserved vector for equation (1) is

$$\begin{aligned} C^0 &= [\eta + (\tau a(u) - \xi)\, u_x]\, \varphi(u), \\ C^1 &= [\eta a(u) - (\tau a(u) - \xi)\, u_t]\, \varphi(u), \end{aligned} \tag{8}$$

where

$$X = \tau(x,t,u)\frac{\partial}{\partial t} + \xi(x,t,u)\frac{\partial}{\partial x} + \eta(x,t,u)\frac{\partial}{\partial u}$$

is a Lie point symmetry generator of (1).

3 New conservation laws for equation (1)

Definitions 1 and 2 have been extended to

Definition 3.1. *An equation $F = 0$ is said to be nonlinearly self-adjoint if the equation obtained from the adjoint equation* (5) *by the substitution $v = \varphi(x,u)$ with a certain function $\varphi(x,u) \neq 0$ is identical with the original equation* (3)*, that is,*

$$F^*\big|_{v=\varphi(x,u)} = \phi(x,u,\ldots)F, \tag{9}$$

for some $\phi \in \mathcal{A}$.

Whenever (9) *holds for a certain function φ such that $\varphi_u \neq 0$ and $\varphi_{x^i} \neq 0$, the equation $F = 0$ is called weak self-adjoint.*

Remark 3.1. The concept of nonlinearly self-adjoint equations was introduced by Ibragimov, see [17, 18, 19]. On the other hand, the notion of weak self-adjoint equation was introduced by Gandarias in [8].

With respect to these new concepts, weak self-adjointness for evolution equations were obtained in [9, 10, 11]. Namely, in [10] Gandarias, Redondo and Bruzón applied the new concept to a class of equations arising in financial mathematics. In [9] Gandarias established local conservation laws for a porous medium equation using the self-adjointness of the equation under consideration. More recently Gandarias and Bruzón [11] found a class of weak self-adjoint forced KdV equations.

Nonlinearly self-adjointness have been focused by Freire and Sampaio in [6], where the authors determined a class of nonlinear self-adjoint equations of fifth-order. In [2] Bozhkov, Freire and Ibragimov showed that the nonlinear self-adjointness of the Novikov equation (further details, see [2] and references therein) implies in the strictly self-adjointness of that equation. In [7] new classes of self-adjoint equations up to fifth-order were found.

Taking the nonlinearly self-adjointness for differential equations into account, substituting $v = \psi(x, t, u)$ into (7) and using (1), it is obtained

$$\psi_t + a(u)\psi_x = 0. \tag{10}$$

A solution to (10) is $\psi(x, t, u) = \phi(z) + \varphi(u)$, with $z = x - ta(u)$.

From Ibragimov's theorem on conservation laws, a local conservation law for equation (1) is given by

$$\begin{aligned} C^0 &= [\eta + (\tau a(u) - \xi)\, u_x]\, (\phi(z) + \varphi(u)),\\ C^1 &= [\eta - (\tau a(u) - \xi)\, u_t]\, (\phi(z) + \varphi(u)), \end{aligned} \tag{11}$$

where $z = x - ta(u)$.

In fact, the new conserved vector obtained from the self-adjointness' new concept is $C = (C^0, C^1)$, where

$$\begin{aligned} C^0 &= [\eta + (\tau a(u) - \xi)\, u_x]\, \phi(x - ta(u)),\\ C^1 &= [\eta a(u) - (\tau a(u) - \xi)\, u_t]\, \phi(x - ta(u)). \end{aligned} \tag{12}$$

Observe that (12) possibilites to find an infinite number of new conservation laws for a fixed Lie point symmetry generator of equation (1).

4 Examples

Here it will be established some conservation laws illustrating the results obtained previously. Consider the Lie point symmetry generator

$$X_4 = t\frac{\partial}{\partial t} - \frac{a(u)}{a'(u)}\frac{\partial}{\partial u} \tag{13}$$

of equation (1). Here it is used the same notation employed in [4]. Substituting the components of (13) into (12) it is obtained

$$\begin{aligned} C^0 &= \left[-\frac{a(u)}{a'(u)} + ta(u)\,u_x\right]\phi(x - ta(u)),\\ C^1 &= \left[-\frac{a(u)^2}{a'(u)} - ta(u)\,u_t\right]\phi(x - ta(u)). \end{aligned} \tag{14}$$

Setting $\phi(z) = z$ into (14), it is found that the conserved vector is $C = (C^0, C^1)$, where

$$\begin{aligned} C^0 &= -\frac{a(u)}{a'(u)}(x - ta(u)) - tA(u) + D_x\left(txA(u) - t^2\alpha(u)\right),\\ C^1 &= -\frac{a(u)^2}{a'(u)}(x - ta(u)) + xA(u) - 2t\alpha(u) + D_t\left(t^2\alpha(u) - txA(u)\right) \end{aligned}$$

α is given by $A'(u) = a(u)$ and β is a function such that $\alpha'(u) = a(u)^2$. Transfering the terms $D_x(\ldots)$ from C^0 to C^1 and simplifying, it is obtained the conserved vector $C = (C^0, C^1)$, with components given by

$$\begin{aligned} C^0 &= -\frac{a(u)}{a'(u)}(x - ta(u)) - tA(u),\\ C^1 &= -\frac{a(u)^2}{a'(u)}(x - ta(u)) + xA(u) - 2t\alpha(u). \end{aligned}$$

Recently new Lie point symmetries of the inviscid Burgers equations were found, see [1]. In the next example we use the new generator (it is employed the same notation of the original paper [1])

$$Z_{11} = (x - ta(u))\frac{\partial}{\partial x}$$

for establishing another conservation laws for (1). Let $\phi = c_1(x - ta(u)) + c_2$, $\varphi = c_3u^p + c_4/u,\ p \neq -1$, where $c_1,\ c_2, c_3, c_4 \in \mathbb{R}$ are arbitrary constants,

$A'(u) = a(u)$, $B'(u) = a(u)u^p$, $U' = a(u)/u$ and $\alpha' = a(u)^2$. From (10) it is obtained

$$\begin{aligned} C^0 &= -c_1(x - ta(u))^2 u_x - c_2(x - ta(u))u_x \\ &\quad - c_3(x - ta(u))u_x - c_4 \frac{x - ta(u)}{u} u_x, \\ C^1 &= c_1(x - ta(u))^2 u_t + c_2(x - ta(u))u_t \\ &\quad + c_3(x - ta(u))u_t + c_4 \frac{x - ta(u)}{u} u_t. \end{aligned}$$

Since

$$\begin{aligned} C^0 &= c_1\big[D_x(2xtA - x^2u - t^2\alpha) + 2xu - 2tA\big] + c_2\left[D_x(tA - xu) + u\right] \\ &\quad + c_3\Big[D_x\Big(tB - x\tfrac{u^{p+1}}{p+1}\Big) + \tfrac{u^{p+1}}{p+1}\Big] + c_4\Big[D_x\Big(x\ \ln|u| + \tfrac{ta(u)}{u}\Big) + \ln|u|\Big], \\ C^1 &= c_1\big[D_t(x^2u + t^2\alpha - 2xtA) + 2xA - 2t\alpha\big] + c_2\left[D_t(xu - tA) + A\right] \\ &\quad + c_3\Big[D_t\Big(x\tfrac{u^{p+1}}{p+1} - tB\Big) + B\Big] + c_4\left[D_t\left(x\ \ln|u| - tU\right) + U\right], \end{aligned}$$

by transfering the terms $D_t(\ldots)$ from C^1 to C^0, it is obtained

$$\begin{aligned} C^0 &= c_1(2xu - 2tA) + c_2u + c_3\frac{u^{p+1}}{p+1} + c_4 \ln|u|, \\ C^1 &= c_1(2xA - 2t\alpha) + c_2A + c_3B + c_4U. \end{aligned}$$

It is easy to see that the conserved vector obtained is a linear combination of the conserved vectors

$$\begin{aligned} &D_1 = (2xu - 2tA, 2xA - 2t\alpha), \quad D_2 = (u, A), \\ &D_3 = \left(\frac{u^{p+1}}{p+1}, B\right), \quad D_4 = (\ln|u|, U). \end{aligned}$$

5 Conclusion

In this paper the previous results on conservation laws obtained by the author in [4] are generalized using the recent new developments due to Maria L. Gandarias [8] and Nail H. Ibragimov [17, 19]. The main result is the new conserved vector (12). In particular, the results obtained here possibilite one to construct an infinite number of new conservation laws for equation (1).

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