Fast convergences towards Euler-Mascheroni constant

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Abstract. The aim of this paper is to introduce a new family of sequences which faster converge to the Euler-Mascheroni constant. Finally, numerical computations are given.

1 Introduction

One of the most important constants in mathematics is defined as the limit of the sequence

$$\gamma_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln n,$$

denoted $\gamma = 0.57721566490153286\ldots$. It is now known as the Euler-Mascheroni constant, in honour of the Swiss mathematician Leonhard Euler (1707-1783) and of the Italian mathematician Lorenzo Mascheroni (1750-1800).

The sequence $(\gamma_n)_{n \geq 1}$ and the constant γ have numerous applications in many areas of mathematics, such as analysis, theory of probability, special functions, and number theory. As a consequence, many authors are preoccupied to improve the speed of convergence of the sequence $(\gamma_n)_{n \geq 1}$, which is very slowly, if we take into account that it converges toward its limit like n^{-1}.

More precisely, we mention the following results related to the speed of convergence of the sequence $(\gamma_n)_{n \geq 1}$:

$$\frac{1}{2(n+1)} < \gamma_n - \gamma < \frac{1}{2n} \quad \text{(Young)}$$

(see, [14, 15, 28]). We also refer here to the papers [1, 2, 5-12, 20-27], where important improvements of the speed of convergence of γ_n were established.

The complete asymptotic expansion of the sequence $(\gamma_n)_{n \geq 1}$ is

$$\gamma_n \sim \gamma + \frac{1}{2n} - \sum_{k=1}^{\infty} \frac{B_{2k}}{2k} \frac{1}{n^{2k}},$$

where the Bernoulli numbers B_{2k} are defined by

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} t^k.$$

As $B_2 = 1/6$, $B_4 = -1/30$, $B_6 = 1/42$, $B_8 = -1/30$, we imply

$$\gamma_n \sim \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - \frac{1}{252n^6} + \frac{1}{240n^8} - \cdots .$$

DeTemple [3-4] introduced the sequence

$$R_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \ln\left(n + \frac{1}{2}\right)$$

which converges to γ like n^{-2}, since

$$\frac{1}{24(n+1)^2} < R_n - \gamma < \frac{1}{24n^2}.$$

Recently, Mortici [17] introduced the sequences

$$u_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{\left(6 - 2\sqrt{6}\right)n} - \ln\left(n + \frac{1}{\sqrt{6}}\right) \quad (1)$$

and

$$v_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{\left(6 + 2\sqrt{6}\right)n} - \ln\left(n - \frac{1}{\sqrt{6}}\right), \quad (2)$$

which converges as n^{-3}, since

$$\lim_{n\to\infty} n^3 (u_n - \gamma) = -\frac{\sqrt{6}}{108} \quad \text{and} \quad \lim_{n\to\infty} n^3 (v_n - \gamma) = \frac{\sqrt{6}}{108}.$$

See [17, Theorem 2.1]. Furthermore, the arithmetic mean of the sequences $(u_n)_{n\geq1}$ and $(v_n)_{n\geq1}$,

$$z_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1} + \frac{1}{2n} - \frac{1}{2}\ln\left(n^2 - \frac{1}{6}\right)$$

converges to γ as n^{-4}.

We open here a new direction to accelerate the sequence $(\gamma_n)_{n\geq1}$, that is to consider an additional term of the form

$$M_n = \gamma_n - \gamma + \ln\frac{P(n)}{Q(n)},$$

where P, Q are polynomials of the same degree, having the leading coefficient equal to one. Precisely, we introduce the sequences

$$v_n = \gamma_n + \ln\frac{n - \frac{1}{12}}{n + \frac{5}{12}} \quad \text{and} \quad \mu_n = \gamma_n + \ln\frac{n^2 + \frac{33}{140}n + \frac{37}{1680}}{n^2 + \frac{103}{140}n + \frac{61}{336}}$$

whose speeds of convergence increase to n^{-3}, respective n^{-5}, since

$$\lim_{n\to\infty} n^3 (v_n - \gamma) = \frac{-7}{288} \quad \text{and} \quad \lim_{n\to\infty} n^5 (\mu_n - \gamma) = \frac{3959}{806\,400}.$$

Our study is based on the following result, which represents a powerful tool for constructing some asymptotic expansions, or to accelerate some convergences.

Lemma 1. *If $(\omega_n)_{n\geq1}$ is convergent to zero and there exists the limit*

$$\lim_{n\to\infty} n^k(\omega_n - \omega_{n+1}) = l \in [-\infty, \infty], \tag{3}$$

with $k > 1$, then there exists the limit:

$$\lim_{n\to\infty} n^{k-1}\omega_n = \frac{l}{k-1}.$$

For proofs and further applications, see [13-19]. The sequences (1)-(2) were introduced in [17] using Lemma 1. Clearly the sequence $(\omega_n)_{n\geq1}$ converges more quickly when the value of k satisfying (3) is larger.

2   First degree term

In this section we define the sequence

$$\omega_n = \gamma_n - \gamma + \ln \frac{n+a}{n+b}$$

to find the values a, b which provide the fastest sequence $(\omega_n)_{n\geq 1}$. First

$$\omega_n - \omega_{n+1} = -\frac{1}{n+1} - \ln \frac{n}{n+1} + \ln \frac{n+a}{n+b} - \ln \frac{n+1+a}{n+1+b},$$

and we are concentrated to compute a limit of the form (3). In this sense, we used a computer software to obtain the following representation in power series:

$$\omega_n - \omega_{n+1} = \left(a - b + \frac{1}{2}\right)\frac{1}{n^2} + \left(-a^2 - a + b^2 + b - \frac{2}{3}\right)\frac{1}{n^3}$$
$$+ \left(a^3 + \frac{3}{2}a^2 + a - b^3 - \frac{3}{2}b^2 - b + \frac{3}{4}\right)\frac{1}{n^4} + O\left(\frac{1}{n^5}\right). \tag{4}$$

We can state the following

Theorem 2.

i) *If $a - b + \frac{1}{2} \neq 0$, then the speed of convergence of the sequence $(\omega_n)_{n\geq 1}$ is n^{-1}, since*

$$\lim_{n\to\infty} n^2 (\omega_n - \omega_{n+1}) = a - b + \frac{1}{2} \quad \text{and} \quad \lim_{n\to\infty} n\omega_n = a - b + \frac{1}{2} \neq 0.$$

ii) *If $a - b + \frac{1}{2} = 0$ and $-a^2 - a + b^2 + b - \frac{2}{3} \neq 0$, then the speed of convergence of the sequence $(\omega_n)_{n\geq 1}$ is n^{-2}, since*

$$\lim_{n\to\infty} n^3 (\omega_n - \omega_{n+1}) = -a^2 - a + b^2 + b - \frac{2}{3}$$

and

$$\lim_{n\to\infty} n^2\omega_n = \frac{1}{2}\left(-a^2 - a + b^2 + b - \frac{2}{3}\right) \neq 0.$$

iii) *If $a - b + \frac{1}{2} = 0$ and $-a^2 - a + b^2 + b - \frac{2}{3} = 0$ (equivalent with $a = -1/12$, $b = 5/12$), then the speed of convergence of the sequence $(\omega_n)_{n\geq 1}$ is n^{-3}, since*

$$\lim_{n\to\infty} n^4 (\omega_n - \omega_{n+1}) = \frac{-7}{96} \quad \text{and} \quad \lim_{n\to\infty} n^3\omega_n = \frac{-7}{288}.$$

The proof of Theorem 2 easily follows from Lemma 1 and (4).

For $a = -1/12$, $b = 5/12$, the relation (4) becomes

$$\omega_n - \omega_{n+1} = -\frac{7}{96n^4} + O\left(\frac{1}{n^5}\right)$$

and so, iii) is completely proved.

3 Second degree term

Now we define the sequence

$$\lambda_n = \gamma_n - \gamma + \ln\frac{n^2 + an + b}{n^2 + cn + d}$$

to find the values a, b, c, d which provide the fastest sequence $(\lambda_n)_{n\geq 1}$. First

$$\lambda_n - \lambda_{n+1} = -\frac{1}{n+1} - \ln\frac{n}{n+1} + \ln\frac{n^2 + an + b}{n^2 + cn + d}$$

$$- \ln\frac{(n+1)^2 + a(n+1) + b}{(n+1)^2 + c(n+1) + d},$$

and we are concentrated to compute a limit of the form (3).

In this sense, we used again the computer software to obtain the following representation in power series:

$$\lambda_n - \lambda_{n+1} = \left(a - c + \frac{1}{2}\right)\frac{1}{n^2} - \left(a^2 + a - c^2 - c - 2b + 2d + \frac{2}{3}\right)\frac{1}{n^3}$$

$$+\left(a - 3b - c + 3d - 3ab + 3cd + \frac{3}{2}a^2 + a^3 - \frac{3}{2}c^2 - c^3 + \frac{3}{4}\right)\frac{1}{n^4}$$

$$-\left(a^4 + 2a^3 - 4a^2b + 2a^2 - 6ab + a + 2b^2 - 4b - c^4\right.$$

$$\left.-2c^3 + 4c^2d - 2c^2 + 6cd - c - 2d^2 + 4d + \frac{4}{5}\right)\frac{1}{n^5} \tag{1}$$

$$+\left(a - 5b - 5a^3b + 5d - c - \frac{5}{2}c^2 - \frac{10}{3}c^3 - \frac{5}{2}c^4 - c^5 + \frac{5}{2}a^4\right.$$

$$+10cd - 5d^2 + 10c^2d + 5c^3d - 5cd^2 + a^5 + 5ab^2$$

$$\left.-10ab + 5b^2 + \frac{10}{3}a^3 - 10a^2b + \frac{5}{2}a^2 + \frac{5}{6}\right)\frac{1}{n^6} + O\left(\frac{1}{n^7}\right).$$

We can state the following

Theorem 3.

   i) *Let us denote the coefficients of* (5) *by*

$$\alpha = a - c + \frac{1}{2}$$

$$\beta = -\left(a^2 + a - c^2 - c - 2b + 2d + \frac{2}{3}\right)$$

$$\delta = a - 3b - c + 3d - 3ab + 3cd + \frac{3}{2}a^2 + a^3 - \frac{3}{2}c^2 - c^3 + \frac{3}{4}$$

$$\eta = -\left(a^4 + 2a^3 - 4a^2b + 2a^2 - 6ab + a + 2b^2 - 4b - c^4\right.$$

$$\left. - 2c^3 + 4c^2d - 2c^2 + 6cd - c - 2d^2 + 4d + \frac{4}{5}\right)$$

   ii) *If* $\alpha \neq 0$, *then the speed of convergence of the sequence* $(\lambda_n)_{n\geq 1}$ *is* n^{-1}, *since*

$$\lim_{n\to\infty} n^2 (\lambda_n - \lambda_{n+1}) = \alpha \quad \text{and} \quad \lim_{n\to\infty} n\lambda_n = \alpha \neq 0.$$

   iii) *If* $\alpha = 0$ *and* $\beta \neq 0$, *then the speed of convergence of the sequence* $(\lambda_n)_{n\geq 1}$ *is* n^{-2}, *since*

$$\lim_{n\to\infty} n^3 (\lambda_n - \lambda_{n+1}) = \beta \quad \text{and} \quad \lim_{n\to\infty} n^2\lambda_n = \frac{\beta}{2} \neq 0.$$

   iv) *If* $\alpha = \beta = 0$ *and* $\delta \neq 0$, *then the speed of convergence of the sequence* $(\lambda_n)_{n\geq 1}$ *is* n^{-3}, *since*

$$\lim_{n\to\infty} n^4 (\lambda_n - \lambda_{n+1}) = \delta \quad \text{and} \quad \lim_{n\to\infty} n^3\lambda_n = \frac{\delta}{3} \neq 0.$$

   v) *If* $\alpha = \beta = \delta = 0$ *and* $\eta \neq 0$, *then the speed of convergence of the sequence* $(\lambda_n)_{n\geq 1}$ *is* n^{-4}, *since*

$$\lim_{n\to\infty} n^5 (\lambda_n - \lambda_{n+1}) = \eta \quad \text{and} \quad \lim_{n\to\infty} n^4\lambda_n = \frac{\eta}{4} \neq 0.$$

   vi) *If* $a = \beta = \delta = \eta = 0$ *(equivalent with* $a = 33/140$, $b = 37/1680$, $c = 103/140$, $d = 61/336$), *then the speed of convergence of the sequence* $(\lambda_n)_{n\geq 1}$ *is* n^{-5}, *since*

$$\lim_{n\to\infty} n^6 (\lambda_n - \lambda_{n+1}) = \frac{3959}{161\,280} \quad \text{and} \quad \lim_{n\to\infty} n^5\lambda_n = \frac{3959}{806\,400}.$$

The proof of Theorem 3 easily follows from Lemma 1 and (5).

For $a = 33/140$, $b = 37/1680$, $c = 103/140$, $d = 61/336$, the relation (5) becomes

$$\lambda_n - \lambda_{n+1} = \frac{3959}{161\,280n^6} + O\left(\frac{1}{n^7}\right).$$

and so, vi) is completely proved.

4 Concluding remarks

As least theoretically, further sequences of the form

$$M_n = \gamma_n - \gamma + \ln \frac{P(n)}{Q(n)}$$

can be defined, where $\deg P = \deg Q = k \geq 3$. As above,

$$M_n - M_{n+1} = -\frac{1}{n+1} - \ln \frac{n}{n+1} + \ln \frac{P(n)\,Q(n+1)}{Q(n)\,P(n+1)} \qquad (6)$$

and if we expand (6) into a power series of n^{-1}, then the $2k$ coefficients of the polynomials P and Q are the unique solution of the system obtained by imposing that the first $2k$ coefficients of the power series (6) vanish. In this case,

$$M_n - M_{n+1} = \frac{\theta}{n^{2k+2}} + O\left(\frac{1}{n^{2k+3}}\right),$$

with $\theta \neq 0$. By Lemma 1, $(M_n)_{n \geq 1}$ tends to zero as $n^{-(2k+1)}$, since

$$\lim_{n \to \infty} n^{2k+1} M_n = \frac{\theta}{2k+1}.$$

Finally, we offer some numerical computations to prove the superiority of our sequences $(v_n)_{n \geq 1}$ and $(\mu_n)_{n \geq 1}$ over the classical sequence $(\gamma_n)_{n \geq 1}$ and the DeTemple sequence $(R_n)_{n \geq 1}$. Remark that already μ_1 approximates γ with seven exact decimals.

n	$\gamma_n - \gamma$	$R_n - \gamma$	$\gamma - v_n$	$\mu_n - \gamma$
10	4.9167×10^{-2}	3.7733×10^{-4}	2.2748×10^{-5}	$\mathbf{4.3237 \times 10^{-8}}$
50	9.9667×10^{-3}	1.6337×10^{-5}	1.9192×10^{-7}	$\mathbf{1.5330 \times 10^{-11}}$
100	4.9917×10^{-3}	4.1252×10^{-6}	2.4147×10^{-8}	$\mathbf{4.8499 \times 10^{-13}}$
300	1.6657×10^{-3}	4.6142×10^{-7}	8.9825×10^{-10}	$\mathbf{2.0122 \times 10^{-15}}$

Our approach works by small degrees k of $P(t)$ and $Q(t)$, respectively, compared with the obtained order $2k + 1$ of the speed of convergence, but for larger values of k, it becomes quite difficult (even for computer softwares) to compute the coefficients of the polynomials and the limit $\theta / (2k + 1)$ of the sequences $n^{2k+1} M_n$.

We propose now a refined numerical method to compute the coefficients based on an explicit formula for the series expansion of $M_n - M_{n+1}$ around $1/n$.

Starting with formula (6), we have

$$M_n - M_{n+1} = -\frac{1}{n+1} - \ln\frac{n}{n+1} + \ln\frac{P(n)}{Q(n)} - \ln\frac{P(n+1)}{Q(n+1)}. \qquad (7)$$

As in the cases studied above, we assume that the polynomials P and Q have rational coefficients. Let us consider the factorizations

$$P(t) = t^k + a_{k-1}t^{k-1} + \cdots + a_0 = (t - \alpha_1)(t - \alpha_2)\cdots(t - \alpha_k) \qquad (8)$$

$$Q(t) = t^k + b_{k-1}t^{k-1} + \cdots + b_0 = (t - \beta_1)(t - \beta_2)\cdots(t - \beta_k), \qquad (9)$$

where the roots α_ν and β_ν are complex numbers. Then (7) takes the form

$$M_n - M_{n+1} = -\frac{1}{n+1} - \ln\frac{n}{n+1} + \sum_{j=1}^{k}\left(\ln\frac{n - \alpha_j}{n - \beta_j} - \ln\frac{n - (\alpha_j - 1)}{n - (\beta_j - 1)}\right). \qquad (10)$$

Next, we have for real numbers α, β that

$$\ln\frac{n - \alpha}{n - \beta} = \ln\left(1 + \frac{\beta - \alpha}{n - \beta}\right) = \sum_{\nu=1}^{\infty}\frac{(-1)^{\nu-1}}{\nu}\left(\frac{\beta - \alpha}{n - \beta}\right)^{\nu}$$

$$= \sum_{\nu=1}^{\infty}\frac{(-1)^{\nu-1}}{\nu}\left(\frac{\beta - \alpha}{n}\right)^{\nu}\frac{1}{(1 - \beta/n)^{\nu}}$$

$$= \sum_{\nu=1}^{\infty}\frac{(-1)^{\nu-1}(\beta - \alpha)^{\nu}}{\nu n^{\nu}}\sum_{\mu=0}^{\infty}\binom{\nu + \mu - 1}{\nu - 1}\left(\frac{\beta}{n}\right)^{\mu}$$

$$= -\sum_{d=1}^{\infty}\left(\sum_{\nu=1}^{d}\binom{d - 1}{\nu - 1}\frac{(\alpha - \beta)^{\nu}\beta^{d-\nu}}{\nu}\right)\frac{1}{n^d} \quad \text{(with } d = \nu + \mu)$$

$$= -\sum_{d=1}^{\infty}\frac{\alpha^d - \beta^d}{dn^d}.$$

Replacing α by $\alpha - 1$ and β by $\beta - 1$, we get

$$\ln \frac{n - (\alpha - 1)}{n - (\beta - 1)} = -\sum_{d=1}^{\infty} \frac{(\alpha - 1)^d - (\beta - 1)^d}{dn^d} \tag{11}$$

and

$$\ln \frac{n}{n+1} = \sum_{d=1}^{\infty} \frac{(-1)^d}{dn^d}. \tag{12}$$

Finally, using

$$\frac{1}{n+1} = \sum_{d=1}^{\infty} \frac{(-1)^{d-1}}{n^d}, \tag{13}$$

it follows from (10)-(13):

$$M_n - M_{n+1} = \sum_{d=2}^{\infty}$$

$$\times \left((-1)^d (d-1) + \sum_{j=1}^{k} \left((\alpha_j - 1)^d - \alpha_j^d \right) \left((\beta_j - 1)^d - \beta_j^d \right) \right) \frac{1}{dn^d}. \tag{14}$$

For every d the corresponding term in brackets on the right-hand side of (14) is a polynomial in $2k$ variables, which is symmetric in $\alpha_1, \alpha_2, \ldots, \alpha_k$ and in $\beta_1, \beta_2, \ldots, \beta_k$. Therefore, the proof of the main theorem on symmetric polynomials involves an algorithm to express the polynomials (14) in terms of the elementary symmetric polynomials in $\alpha_1, \alpha_2, \ldots, \alpha_k$ and in $\beta_1, \beta_2, \ldots, \beta_k$. So, one obtains equations in terms of the coefficients $a_0, \ldots, a_k$ and $b_0, \ldots, b_k$.

But, a second method to deduce the rational coefficients $a_0, \ldots, a_k$ and $b_0, \ldots, b_k$ from the (complex) solution of the system

$$(-1)^d (d-1) + \sum_{j=1}^{k}$$

$$\times \left((\alpha_j - 1)^d - \alpha_j^d \right) \left((\beta_j - 1)^d - \beta_j^d \right) = 0 \quad (2 \leq d \leq 2k + 1) \tag{15}$$

is based on the numerical continued fraction algorithm. Provided that the coefficients $a_0, \ldots, a_{k-1}$ and $b_0, \ldots, b_{k-1}$ are rationals, the following pure numerical method works without using any computer algebra software in order to obtain the coefficients. Here, we first solve the system (15) numerically,

which for large k is much simpler than to compute the coefficients $a_0, \ldots, a_k$ and $b_0, \ldots, b_k$ by a computer algebra system. We demonstrate the method by computing the rationals a_0, a_1, a_2 and b_0, b_1, b_2 for $k = 3$. The system

$$0 = 1 + \sum_{j=1}^{3} \left((\alpha_j - 1)^2 - \alpha_j^2 \right) \left((\beta_j - 1)^2 - \beta_j^2 \right)$$

$$0 = -2 + \sum_{j=1}^{3} \left((\alpha_j - 1)^3 - \alpha_j^3 \right) \left((\beta_j - 1)^3 - \beta_j^3 \right)$$

$$0 = 3 + \sum_{j=1}^{3} \left((\alpha_j - 1)^4 - \alpha_j^4 \right) \left((\beta_j - 1)^4 - \beta_j^4 \right)$$

$$0 = -4 + \sum_{j=1}^{3} \left((\alpha_j - 1)^5 - \alpha_j^5 \right) \left((\beta_j - 1)^5 - \beta_j^5 \right)$$

$$0 = 5 + \sum_{j=1}^{3} \left((\alpha_j - 1)^6 - \alpha_j^6 \right) \left((\beta_j - 1)^6 - \beta_j^6 \right)$$

$$0 = -6 + \sum_{j=1}^{3} \left((\alpha_j - 1)^7 - \alpha_j^7 \right) \left((\beta_j - 1)^7 - \beta_j^7 \right)$$

has (the unique) solution

$\alpha_1 = -0.2581587158791677070704309274485346644896433 0026005\ldots$
$\quad\quad +i \cdot 0.48397242106377239253881674751055982149144184289297\ldots$

$\alpha_2 = 0.03820146532461197107162439313624308217173138 1749928\ldots$

$\alpha_3 = -0.2581587158791677070704309274485346644896433 0026005\ldots$
$\quad\quad -i \cdot 0.48397242106377239253881674751055982149144184289297\ldots$

$\beta_1 = -0.38406133637441025553569425604004004944633805438883\ldots$

$\beta_2 = -0.29702731502965659376677160286039309868060858219067\ldots$
$\quad\quad -i \cdot \cdot 0.49547669307316290426997439537218740680230881850428\ldots$

$\beta_3 = -0.29702731502965659376677160286039309868060858219067\ldots$
$\quad\quad +i \cdot 0.49547669307316290426997439537218740680230881850428\ldots$

It follows that

$$-a_2 = \alpha_1 + \alpha_2 + \alpha_3$$
$$= -0.4781159664337234430692374617608262468075552187709\!7...$$

$$a_1 = \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1$$
$$= 0.2811511444689014782472726397960042819855903968054\!4...$$

$$-a_0 = \alpha_1\alpha_2\alpha_3$$
$$= 0.0114938745487810908371656035207437076595955100627\!67...$$

$$-b_2 = \beta_1 + \beta_2 + \beta_3$$
$$= -0.9781159664337234430692374617608262468075552187708\!1...$$

$$b_1 = \beta_1\beta_2 + \beta_2\beta_3 + \beta_3\beta_1$$
$$= 0.5618757943524298664485580373430840720560346728575\!0...$$

$$-b_0 = \beta_1\beta_2\beta_3$$
$$= -0.1281698629537414584143556106172928602835144891220\!2...$$

Next, the continued fraction algorithm applied to the above numerical values, gives

$$a_2 = -<-1, 1, 1, 10, 1, 12, 7, 1, 1, 5, 1, 4> = \frac{68143}{142524}$$

$$a_1 = <0, 3, 1, 1, 3, 1, 9, 10, 3, 1, 3, 1, 10> = \frac{186997}{665112}$$

$$a_0 = -<0, 87, 349, 4, 1, 5, 3, 2> = -\frac{74381}{6471360}$$

$$b_2 = -<-1, 45, 1, 2, 3, 1, 1, 7, 2, 2, 3> = \frac{139405}{142524}$$

$$b_1 = <0, 1, 1, 3, 1, 1, 5, 1, 2, 3, 1, 1, 8, 1, 21, 1, 3> = \frac{1121131}{1995336}$$

$$b_0 = -<-1, 1, 6, 1, 4, 18, 2, 3, 1, 1, 4, 1, 8, 1, 5, 3, 1, 1, 2, 3> = \frac{30689033}{239440320}.$$

Then we have

$$\lim_{n\to\infty} n^7 M_n = \frac{\theta}{7}$$

with

$$\theta = \frac{1}{8} \left(7 + \sum_{j=1}^{3} \left((\alpha_j - 1)^8 - \alpha_j^8 \right) \left((\beta_j - 1)^8 - \beta_j^8 \right) \right)$$
$$= -0.0183277286604680788438143924471373415644789 8226906...$$
$$= < -1, 1, 53, 1, 1, 3, 1, 1, 10, 7, 1, 1, 1, 3, 2, 3, 4, 1, 2, 1, 36, 1, 8, 1, 3, 3 >$$
$$= -\frac{10833071983}{591075532800}.$$

It is to be noticed that these results were rediscovered by us using Lemma 1 presented in the first part of this paper. We omit the proof for sake of simplicity.

Finally, remark that if the polynomials P and Q of k-th degree are already determined, say using the previous numerical method, then the problem of verifying the speed of convergence of the corresponding sequence M_n using Lemma 1 becomes a much easier task.

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