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Space-time finite element approximation and numerical solution of hereditary linear viscoelasticity problems

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Abstract. In this paper we suggest a fast numerical approach to treat problems of the hereditary linear viscoelasticity, which results in the system of elliptic partial differential equations in space variables, who's coefficients are Volterra integral operators of the second kind in time. We propose to approximate the relaxation kernels by the product of purely time- and space-dependent terms, which is achieved by their piecewise-polynomial space-interpolation. A priori error estimate was obtained and it was shown, that such approximation does not decrease the convergence order, when an interpolation polynomial is chosen of the same order as the shape functions for the spatial finite element approximation, while the computational effort is significantly reduced.

Mathematical subject classification: 22E46, 53C35, 57S20.

Key words: hereditary viscoelasticity; kern approximation by interpolation; space-time finite element approximation, stability and a priori estimate.

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1 Introduction

Our task is to develop an approach for the numerical treatment of mathematical problems, which arise from considering the behavior of hereditary viscoelastic solids. These result in a system of elliptic partial differential equations in space variables, whose coefficients are Volterra integral operators of the second kind in time, which allow for weak-singular kernels, i.e.,

$$\frac{\partial}{\partial x}\left(a_0(x,t)\frac{\partial u(x,t)}{\partial x}\right)+\int_0^t \frac{\partial}{\partial x}\left(a(x,t,\tau)\frac{\partial u(x,\tau)}{\partial x}\right)d\tau = f(x,t). \tag{1.1}$$

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In Sections 2 and 3 a general mathematical model of the boundary value problem of the inhomogeneous hereditary ageing viscoelasticity is given in classical and weak formulations. The solvability of such a Volterra integral equation in Sobolev spaces and the stability of the solution with respect to the right-hand side is recalled here in Theorem 3.2 and proven in [3], [6], [5]. It will be used for the proof of Lemmas 6.1, 6.2, therefore is recalled here.

Our basic idea for the numerical solution of such a problem, was to treat the space and time dependence of the solution separately, with Finite Elements technique in x and with spline collocation in t, τ. Similar idea is presented in [13] with only difference that we allow for the non-convolutional and weakly singular relaxation kernels, i.e., our kernels must be only integrable and continuous after the integration and not necessary essentially bounded in the integrating time-variable as in [13].

The separate numerical space-time treatment of the problem can be performed trivially, if the time and space dependence in the instantaneous elastic coefficients $a_0(x, t)$, relaxation kernels $a(x, t, \tau)$ and the external forces $f(x, t)$ can be separated in a straightforward manner.

In Section 5 we suggest to approximate the integral kernels, out-of-integral terms and right-hand side in space by polynomial interpolation, possibly with the same shape functions that we use in the FE approximation of the solution, thus representing them by a product of purely time- and space-dependent terms. The idea of the kernel approximation by interpolation allows us to reduce the calculation time. It is widely used in spatial boundary integral equations of the kind

$$\lambda u(x) + \int_{\Gamma} k(x, y) u(y) dy = f(x), \quad \forall x \in \Gamma \subset \mathbb{R}^n,$$

see, e.g., works of Hackbusch [21], [20].

Our next innovation is employing suitable quadrature formulas for the weakly singular kernel approximation in the well-known [7] collocation method for Volterra integral equations. Like in [25], where finite dimensional integral equations with weakly singular kernels are treated, we suggested to approximate the kernels by interpolation in singular points with the Lagrange canonical polynomials. This allows to avoid the singularity and analytically pre-calculate the weights in quadrature formulas.

We also use the fine property of the Volterra integral here, that if split the full integration interval on many subintervals and start the solution from the first subinterval, we are able to solve the problem on each subinterval separately and get a recurrent formula containing solution on all previous subintervals in the right-hand side.

$$\begin{aligned}
&\frac{\partial}{\partial x}\left(a_0(x,t)\frac{\partial u_n(x,t)}{\partial x}\right)+\int_{t_{n_1-1}}^{t}\frac{\partial}{\partial x}\left(a(x,t,\tau)\frac{\partial u_n(x,\tau)}{\partial x}\right)d\tau \\
&\quad = \hat{f}(x,t,\{u_i\}_{i=1,\dots,n_1-1}),\ \hat{f}(x,t,\{u_i\}_{i=1,\dots,n_1-1}) \\
&\quad := f(x,t)-\sum_{i=1}^{n_1-1}\int_{t_{i-1}}^{t_i}\frac{\partial}{\partial x}\left(a(x,t,\tau)\frac{\partial u_i(x,\tau)}{\partial x}\right)d\tau.
\end{aligned} \tag{1.2}$$

In Section 6 the errors, introduced by the approximation of kernels, out-of-integral coefficients and external loads as well as the total error due to the numerical treatment are estimated. It is shown, that choosing an interpolation polynomial of the same or even one order less compared to the shape functions in the finite element approximation of the solution, we do not decrease the convergence order. Therefore we suggest the analyst to use the kernel approximation method, even though it requires more effort in preliminary work.

For the software realization of our numerical method we have chosen ANSYS as the basic simulation tool due to its extensive modeling capabilities and convenient user interface. The operations, that are not standard for ANSYS, like, for example, spline collocation in time or kernel approximation in space, were coded in separate procedures and integrated into the ANSYS environment as User Predefined Routines. A corresponding numerical example is considered in Section 7.

2 Definition of the problem

We consider a linear viscoelastic and aging (of the non-convolutional integral type) body, which is subject to some external loading. We denote the volume occupied by the body by Ω, which is assumed to be a Lipschitz domain.

We are going to consider the equilibrium equations for such a solid. Note that a viscoelastic solid is still a solid and therefore its deformation is slow and we restrict ourselves to the quasi-static case description, i.e., classical for the solid mechanics statement of problem, without the inertial term. A summation from 1 to n over repeating indices is assumed in all the present work, unless the opposite is stated.

$$\frac{\partial}{\partial x_h}\left(\left({a_{ij}^{hk}}_0(x,t)+a_{ij}^{hk}(x)\star\right)\frac{\partial u_j(x,t)}{\partial x_k}\right)=-f_{i0}(x,t),\quad x\in\Omega \qquad i,j,h,k=1,2,\dots,n \tag{2.1}$$

with boundary conditions:

$$u_i(x,t)=\psi_i(x,t),\qquad x\in\partial\Omega_u, \tag{2.2}$$

$$\left(\left({a_{ij}^{hk}}_0(x,t)+a_{ij}^{hk}(x)\star\right)\frac{\partial u_j(x,t)}{\partial x_k}\right)n_h(x)=\phi_i(x,t),\quad x\in\partial\Omega_\sigma, \tag{2.3}$$

$i=1,2,3$, $\forall x\in\Omega$ holding for any $t\in[0,T]$.

$$(a_{ij}^{hk}(x)\star e_k^j)(t):=\int_0^t a_{ij}^{hk}(x,t,\tau)\cdot e_k^j(x,\tau)d\tau \tag{2.4}$$

are Volterra integral operators with kernels $a_{ij}^{hk}(x,t,\tau)$; ${a_{ij}^{hk}}_0(x,t)$ are instantaneous elastic coefficients (out-of-integral terms) and

$$\underline{a}_{ij}^{hk}(x):={a_{ij}^{hk}}_0(x,t)+a_{ij}^{hk}(x)\star\,;$$

f_{i0} are components of a vector of external forces; $\phi_i(x,t)$ are components of a vector of boundary traction on the part $\partial\Omega_\sigma$ of the external boundary; $\psi_i(x,t)$ are components of the displacement vector on the rest part $\partial\Omega_u$ of the boundary, n_h are components of the outer unit normal vector to the boundary of Ω. All functions are supposed to be continuous w.r.t. $t\in[0,T]$ and sufficiently smooth w.r.t. x in domain Ω (for performing a partial integration).

The whole viscoelastic operator tensor $\left(\underline{a}_{ij}^{hk}(x)\right)_{n\times n}^{n\times n}$ is assumed to be symmetric at each point $x\in\Omega$:

$$\underline{a}_{ij}^{hk}(x)=\underline{a}_{ji}^{kh}(x)=\underline{a}_{hj}^{ik}(x)=\underline{a}_{ik}^{hj}(x). \tag{2.5}$$

The tensor $\left({a_{ij}^{hk}}_0(x,t)\right)_{n\times n}^{n\times n}$ is additionally positive-definite, with elements bounded at each point $x\in\Omega$

$$c_0\eta_k^j\eta_k^j\le {a_{ij}^{hk}}_0(x,t)\eta_h^i\eta_k^j\le C_0\eta_k^j\eta_k^j, \tag{2.6}$$

for all $\eta_k^j = \eta_j^k \in \mathbb{R}$ and $t \in [0, T]$, where the constants $0 < c_0 \leq C_0 < \infty$ are independent of x and t.

For isotropic materials we get:

$$\underline{a}_{ij}^{hk} = \underline{\lambda}\delta_{hi}\delta_{kj} + \underline{\mu}\delta_{ij}\delta_{hk} + \underline{\mu}\delta_{ik}\delta_{hj}. \tag{2.7}$$

Example 2.1.

(i) Often, the kernels $a_{ij}^{hk}(x, t, \tau)$ are of the convolution type and are taken in the exponential form:

$$a_{ij}^{hk}(x, t, \tau) = \begin{cases} \sum_{p=1}^{m} \alpha_{ij\;p}^{hk}(x) e^{-\beta_p(x)(t-\tau)}, & \text{if } t \geq \tau \\ 0 & \text{if } t < \tau; \end{cases} \tag{2.8}$$

where the $\beta_p(x)$ and the $\alpha_{ij\;p}^{hk}(x)$ are piecewise continuous functions for $x \in \Omega$, often β_p are just constants;

(ii) The $a_{ij}^{hk}(x, t, \tau)$ may also be kernels of the Abel type (e.g., relaxation kernels of concrete, rocks [1], polymers [2]):

$$a_{ij}^{hk}(x, t, \tau) = \begin{cases} A_{ij\;1}^{hk}(x, t, \tau)(t-\tau)^{-\alpha} & \\ \quad + A_{ij\;2}^{hk}(x, t, \tau)(\tau)^{-\beta} & \\ \quad + A_{ij\;3}^{hk}(x, t, \tau)t^{-\gamma}, & \text{if } t \geq \tau \\ 0 & \text{otherwise}; \end{cases} \tag{2.9}$$

with $0 \leq \alpha, \beta, \gamma < 1$. The $A_{ij\;p}^{hk}$, $p = 1, 2, 3$, are continuous in t and τ, and piece-wise continuous in $x \in \Omega$.

3 Weak problem formulation and results on solvability and stability estimate

In this section we derive a weak problem formulation from the classical one, given by (2.1)-(2.3) in the previous section, by partial integration, and then assume general functional classes for its coefficients and right-hand sides.

In order to obtain the variational formulation, we multiply equation (2.1) by test functions $v_i(x) \in H_0^1(\Omega, \partial\Omega_u)$, $i = 1, \ldots, n$, where $H_0^1(\Omega, \partial\Omega_u) := \{v \in$

$H^1(\Omega)\colon\, v(x) = 0,\ x \in \partial\Omega_u\}$, and integrate over the whole domain Ω. Integrating by parts and taking into account boundary condition (2.2), we obtain:

$$\int_{\Omega} -\underline{a}_{ij}^{hk} \frac{\partial u_j}{\partial x_k} \frac{\partial v_i}{\partial x_h} dx + \int_{\partial\Omega_\sigma} \left(\underline{a}_{ij}^{hk} \frac{\partial u_j}{\partial x_k} \right) v_i n_h ds = - \int_{\Omega} f_{i0} v_i dx. \tag{3.1}$$

If for equation (3.1) we take into consideration the boundary condition, we will obtain the following weak problem:

Find $u_j \in H^1(\Omega)$, $j = 1, \dots, n$, satisfying (2.2) and equation:

$$\begin{aligned} \int_{\Omega} \underline{a}_{ij}^{hk} \frac{\partial u_j}{\partial x_k} \frac{\partial v_i}{\partial x_h} dx &= l(v), \\ l(v) := \int_{\Omega} f_{i0} v_i dx &+ \int_{\partial\Omega_\sigma} \phi_i v_i ds \end{aligned} \tag{3.2}$$

$\forall v_i \in H_0^1(\Omega, \partial\Omega_u)$, $i = 1, \dots, n$.

Definition 3.1 (General weak formulation). *Consider the matrix of instantaneous elastic coefficients* $\left(a_{ij\,0}^{hk}\right)_{n\times n} \in C([0, T]; \mathbf{L}^\infty(\Omega))$*, the relaxation operators* $\left(a_{ij}^{hk} \star\right)_{n\times n}$*, such that*

$$\begin{aligned} &a_{ij}^{hk}(t, \tau) = 0 \quad \forall \tau > t, \quad \textit{and} \\ &a_{ij}^{hk} \in C\left([0, T]; L^1([0, T], L^\infty(\Omega))\right), \quad \textit{and} \\ &f_0 := (f_{i0})_n \in C([0, T]; \mathbf{H^{-1}}(\Omega)), \end{aligned}$$

the boundary tractions $\phi := (\phi_i)_n \in C([0, T]; \mathbf{H^{-1/2}}(\partial\Omega_\sigma))$ *and boundary displacements* $\psi := (\psi_i)_n \in C([0, T]; \mathbf{H^{1/2}}(\partial\Omega_u))$.

We define a weak solution of problem (2.1)-(2.3) *as a vector-valued function* $u \in C([0, T]; \mathbf{H^1}(\Omega))$*, which can be represented in the form* $u = \hat{u} + \tilde{\psi}$*, where* $\tilde{\psi} \in C([0, T]; \mathbf{H^1}(\Omega))$ *satisfies* $(\tilde{\psi}|_{\partial\Omega_u}) = \psi$ *and* $\hat{u}_i \in C([0, T]; H_0^1(\Omega, \partial\Omega_u))$*,* $i = 1, \dots, n$*, satisfies the integral identity*

$$\left[\underline{a}(\hat{u}, v)\right](t) := \int_{\Omega} \left[\underline{a}_{ij}^{hk} \frac{\partial \hat{u}_j}{\partial x_k} \right](t) \frac{\partial v_i}{\partial x_h} dx = \hat{l}(v)(t) \quad \forall t \in [0, T], \tag{3.3}$$

for any $v_i \in H_0^1(\Omega, \partial\Omega_u)$*. The right hand-side of* (3.3) *is, for all* $t \in [0, T]$*, a linear functional on the* $\mathbf{H_0^1}(\Omega, \partial\Omega_u)$

$$\hat{l}(v)(t) := \int_{\Omega} \left(f_{i0}(t) v_i - \left[\underline{a}_{ij}^{hk} \frac{\partial \tilde{\psi}_j}{\partial x_k} \right](t) \frac{\partial v_i}{\partial x_h} \right) dx + \int_{\partial\Omega_\sigma} \phi_i(t) v_i ds. \tag{3.4}$$

The space of linear bounded functionals on $\mathbf{H_0^1}(\Omega, \partial\Omega_u)$ is denoted by $H^{-1}(\Omega)$.

We denote further

$$
\begin{aligned}
a_0(\hat{u}, v)(t) &:= \int_\Omega a_{ij\ 0}^{hk}(x,t) \frac{\partial \hat{u}_j(x,t)}{\partial x_k} \frac{\partial v_i(x)}{\partial x_h} dx \qquad \text{and} \\
a(\hat{u}, v)(t,\tau) &:= \int_\Omega a_{ij}^{hk}(x,t,\tau) \frac{\partial \hat{u}_j(x,\tau)}{\partial x_k} \frac{\partial v_i(x)}{\partial x_h} dx.
\end{aligned} \tag{3.5}
$$

Obviously,

$$
\big[\underline{a}(\hat{u}, v)\big](t) = a_0(\hat{u}, v)(t) + \int_0^t a(\hat{u}, v)(t,\tau) d\tau.
$$

Note that $a_0(\hat{u}, v)$ and $a(\hat{u}, v)$ are bilinear forms on $\mathbf{H_0^1}(\Omega, \partial\Omega_u)$ for every t and almost every τ. We can rewrite the weak formulation (3.3) as follows

$$
a_0(\hat{u}, v)(t) + \int_0^t a(\hat{u}, v)(t,\tau) d\tau = \hat{l}(v)(t), \quad t \in [0, T] \tag{3.6}
$$

Now let us rewrite equation (3.6) in the operator form. For this purpose we introduce the following notations:

$$
A_{0x}\hat{u} := a_0(\hat{u}, \cdot), \qquad A_x\hat{u} := a(\hat{u}, \cdot), \qquad F(t) := \hat{l}(v)(t). \tag{3.7}
$$

$A_{0x}(t), A_x(t,\tau) : \mathbf{H_0^1}(\Omega, \partial\Omega_u) \to H^{-1}(\Omega)$ for all fixed t and almost all $\tau \in [0, T]$. Now we can represent equation (3.3) in the form

$$
\underline{A}\hat{u} = F(t),
$$

where $\underline{A}\cdot$ is an infinite dimensional integro-differential operator of the following form:

$$
\underline{A} = A_{0x} \cdot + A_x \star
$$

and the weak problem formulation (3.3) takes the form:

$$
A_{0x}(t)\hat{u}(t) + \big[A_x \star \hat{u}\big](t) = F(t). \tag{3.8}
$$

Equation (3.8) provides the most general form of the time-space integro-differential dependencies of the considered problem. The following theorem is used as an auxiliary result for showing the solvability of such equations.

Theorem 3.2 (Data stability). *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain and $\partial\Omega_u \subseteq \partial\Omega$, let $A_{0x} \in C([0,T]; \mathcal{L}(\mathbf{H_0^1}(\Omega, \partial\Omega_u), H^{-1}))$, and let $A_{0x}(t)$ be boundedly-invertible uniformly in $[0,T]$, $A_x(t,\tau) = 0\ \forall\tau > t$, $A_x \in C\big([0,T]; L^1\big([0,T], \mathcal{L}(\mathbf{H_0^1}(\Omega, \partial\Omega_u), H^{-1})\big)\big)$, and $F \in C([0,T]; H^{-1})$. Then there exists a unique global solution u of the problem*

$$A_{0x}(t)u(t) + [A_x \star u](t) = F(t) \tag{3.9}$$

in $C([0,T]; \mathbf{H_0^1}(\Omega, \partial\Omega_u))$, which depends continuously on F, that is

$$\|u\|_{C([0,T];H^1(\Omega))} \leq C_1 \|F\|_{C([0,T];H^{-1})} \tag{3.10}$$

where the constant C_1 is independent of F, and if $\|A_{0x}^{-1}(t)\|_{\mathcal{L}(H^{-1},H^1)} \leq \frac{1}{c_0}$, then

$$C_1 \leq \tilde{C}\left(\frac{1}{c_0} \max_{i,j,h,k} \|a_{ij}^{hk}(t,\tau)\|_{C([0,T],L^1([0,T],L^\infty(\Omega)))}\right) \tag{3.11}$$

where $\tilde{C}$ is some real-valued function, independent of f.

The proof of this theorem can be found in [3], [6].

Lemma 3.3. *Let Ω be a Lipschitz domain in $\mathbb{R}^n$, the instantaneous elastic (out-of-integral) coefficients $a_{ij\,0}^{hk} \in C([0,T]; L^\infty(\Omega))$ satisfy the symmetry* (2.5) *and positivity condition (the first of* (2.6)*) with a constant c_0, and the relaxation kernels $a_{ij}^{hk} \in C([0,T], L^1([0,T], L^\infty(\Omega)))$ also satisfy the symmetry condition* (2.5). *Then*

(i) *A_{0x} belongs to $C([0,T]; \mathcal{L}(\mathbf{H_0^1}(\Omega, \partial\Omega_u), H^{-1}))$, and $A_{0x}(t)$ has an inverse operator $A_{0x}^{-1}(t)\ \forall t \in [0,T]$. This inverse operator is uniformly bounded in $[0,T]$, that is, the following estimate*

$$\|A_{0x}^{-1}(t)\|_{\mathcal{L}(H^{-1},H^1)} \leq \frac{1}{c_0}, \tag{3.12}$$

holds for any $t \in [0,T]$, and c_0 is independent from t.

(ii) *$A_x(t,\tau)$ satisfies the following estimate*

$$\|A_x(t,\tau)\|_{\mathcal{L}(H^1(\Omega),H^{-1})} \leq \max_{i,j,h,k} \|a_{ij}^{hk}(t,\tau)\|_{L^\infty(\Omega)}, \tag{3.13}$$

$\forall t$ and almost all $\tau \in [0, T]$. Furthermore, the condition

$$a_{ij}^{hk} \in C([0, T], L^1([0, T], L^\infty(\Omega)))$$

implies that $A_x \in C([0, T], L^1([0, T], \mathcal{L}(\mathbf{H_0^1}(\Omega, \partial\Omega_u), H^{-1})))$.

See [3], [6] for proof.

Lemma 3.2. *In both cases of Example* 2.1*,*

$$a_{ij}^{hk} \in C([0, T], L^1([0, T], L^\infty(\Omega))).$$

See [3], [6] for proof.

Remark 3.5. Consider the weak problem given by Definition 3.1. The functional defined by (3.4) is a continuous functional, satisfying the following estimate

$$\begin{aligned} \|\hat{l}(t)\|_{C([0,T],H^{-1})} \leq C \Big(&||f_0||_{C([0,T],L^2(\Omega))} + ||\phi||_{C([0,T],H^{-1/2}(\partial\Omega_\sigma))} \\ &+ ||\psi||_{C([0,T],H^{1/2}(\partial\Omega_u))}\Big), \end{aligned} \tag{3.14}$$

where C depends on Ω, $\partial\Omega_\sigma$, $\max_{i,j,h,k} \|a_{ij}^{hk}(t, \tau)\|_{C([0,T],L^1([0,T],L^\infty(\Omega)))}$.

See [3], [6], [10] for proof.

4 Preliminaries w.r.t. approximation and interpolation

We construct on domain $\bar{\Omega}$ a quasi-uniform triangular/tetrahedral space mesh of N_{el} elements and denote each element j of Ω by Ω_j. We denote $h :=$ $\max_j(\text{diam } \Omega_j)$. Then, we perform the semi-discrete (spatial) FE approximation of the solution in space by:

$$\begin{gathered} u_h(x, t) := P_r u = \sum_{j=1}^{n_{nodes}} N_j(x) U_j(t), \\ t \in [0, T], \quad x \in \Omega_k, \quad k = 1, \dots, N_{el}, \end{gathered} \tag{4.1}$$

where $N_j(x)$ are shape functions, n_{nodes} denotes the number of nodes in each finite element and P_r is a projection on the space of polynomials of degree at most r.

The following standard (see, e.g. [17] or (32), Th. 5 in [13]) error estimate

$$\|u - u_h\|_{L^\infty([0,T],H^1(\Omega))} \leq Ch\|u\|_{L^\infty([0,T],H^2(\Omega))}, \quad u \in H^2(\Omega), \tag{4.2}$$

or

$$\|u - u_h\|_{L^\infty([0,T],L^2(\Omega))} \leq Ch^2\|u\|_{L^\infty([0,T],H^2(\Omega))} \tag{4.3}$$

is known for the solution u of problem (2.1)–(2.3), and its semi-discrete projection u_h defined by (4.1), if the triangulation of Ω is quasi-uniform. It is known from the standard theory of elliptic second order problems [12], that if Ω is $C^{1,1}$ convex domain in $\mathbb{R}^n$, $\partial\Omega_u$ is non-empty and the coefficients are piece-wise smooth in the space variable then the solution of the problem (2.1)–(2.3) is regular in H^2 and, according to [12], Chap. 8, Th. 8.12, using (3.14) (assuming ϕ, $\psi = 0$),

$$\|u\|_{C([0,T];H^2(\Omega))} \leq C\| \|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}\|_{C([0,T])}. \tag{4.4}$$

Furthermore, if Ω has a C^{r+1}-boundary, then the problem (2.1)–(2.3) is H^{r+1}-regular, and,

$$\|u\|_{C([0,T];H^{r+1}(\Omega))} \leq C\| \|u\|_{H^{r-1}(\Omega)} + \|f\|_{H^{r-1}(\Omega)}\|_{C([0,T])}. \tag{4.5}$$

Remark 4.1. Note only that this is difficult to reach for r higher value than 1, since C^3-regular boundary may arise some problems with a suitable triangulation at the boundaries. Therefore the best choice for the FE-approximation are linear shape functions.

Using (4.2) and the stability estimate (4.4), (3.10), (3.14), we estimate the error of the spatial semi-discrete FE-discretization:

$$\begin{aligned} \|\theta_r\|_{L^\infty([0,T],H^1(\Omega))} &:= \|u - P_r u\|_{L^\infty([0,T],H^1(\Omega))} \\ &\leq Ch^r\|f\|_{C([0,T],L^2(\Omega))}, \quad 0 \leq r \leq 1, \end{aligned} \tag{4.6}$$

or

$$\|\theta_r\|_{L^\infty([0,T],H^1(\Omega))} \leq Ch^r\|f\|_{C([0,T],H^{r-1}(\Omega))}, \quad r > 1, \tag{4.7}$$

Then we can refer to Theorem 2 from [13] or Theorem 6 from [15] for discrete data stability estimate

$$\|u_h\|_{L^\infty([0,T);H^1(\Omega))} \leq C_S\|F\|_{L^\infty([0,T);H^{-1})}. \tag{4.8}$$

We should only replace in its proof the estimate of the history term by the following estimate

$$\begin{aligned}&\sum_{m=1}^{i-1}\int_{t_{m-1}}^{t_m}\|a_{ij}^{hk}(t,\tau)\|_{L^\infty(\Omega)}\|u_h(\tau)\|_{H^1(\Omega)}d\tau\\ &\qquad\le \max_{i,j,h,k}\|a_{ij}^{hk}\|_{C(([0,T],L^1([0,T],L^\infty(\Omega)))}\|u_h\|_{C([0,t_i);H^1(\Omega))}\end{aligned}\tag{4.9}$$

Further, we divide the time interval $[0,T]$ by (n_1+1) points $0=t_0<t_1<\dots<t_{n_1}=T$ and denote $q:=\max_i|t_i-t_{i-1}|$. In the next step, we perform fully discrete polynomial approximation of the n-dimensional vectors $U_j=(U_{j\ 1},\dots,U_{j\ n})$, $j=1,\dots,n_{nodes}$, in time:

$$\begin{gathered}\pi_p U_j(t):=\sum_{k=0}^{n_1} Poly_k^p(t)U_{j,k},\ t\in[t_i,t_{i+1}]\subset[0,T],\\ i=0,\dots,n_1-1,\end{gathered}\tag{4.10}$$

where $Poly_k^p$ are polynomials of the power $p\in[0,\infty)$ on intervals $[t_i,t_{i+1}]\subset[0,T]$. Thus, we obtain the complete space-time approximation $\pi_p P_r u$ of the exact solution $u(x,t)$.

According to Lemma 12 from [14],

$$\begin{aligned}\|\rho_p\|_{L^\infty([0,T],B)} &:= \|P_r u-\pi_p P_r u\|_{L^\infty([0,T],B)}\\ &\le C_\pi||u_h||_{C^{\hat p}([0,T],B)}q^{\hat p}\overset{(4.8)}{\le} C_1||f||_{C^{\hat p}([0,T],B}q^{\hat p}\end{aligned}$$

where $B=\{H^1(\Omega),L^2(\Omega)\}$, $\hat p=p+1>0$ and $q:=\max_i|t_i-t_{i-1}|$, $i=1,\dots,n_1$. The same result is presented in [8] (Theorem 1.1) and [21] (Theorems 4.4.7 and 4.3.15) for spline collocation approximation (4.10) of finite dimensional functions with collocation parameters $\{c_k\}$ chosen as equidistant points in [0,1].

Remark 4.2. In general, for $p>1$, the stability constant C_π depends upon the numerical solution and discretization $||\pi_p||$. But for piece-wise constant or linear interpolation, C_π is an independent constant (see Sec. 4.4 in [21] or Lem. 12 in [14]).

We restrict ourselves on the case $p\le 1$ in this paper.

Remark 4.3. Estimate (4.11) again requires a higher regularity (now w.r.t. time) as it is assumed in Section 3. Nevertheless, looking through the proofs (see [1]) of statements recalled in Section 3, one can see that all estimates could be justified also for right-hand side functions, integral kernels (after integration) and instantaneous elastic coefficient from $C^1([0, T])$.

In the next step we analyze the space-time dependencies in the out-of-integral (instantaneous elastic) coefficients, the integral kernels and the right-hand side of the problem (2.1)–(2.3). In our following considerations we will represent the integral kernels in the form:

$$a_{ij}^{hk}(x, t, \tau) = \Big(\alpha(x, t, \tau)\beta(t, \tau)\Big)_{ij}^{hk}, \tag{4.11}$$

where $\alpha_{ij}^{hk}(x, t, \tau)$ are the well-behaved parts of the kernel, normally piece-wise-smooth and bounded, and $\beta_{ij}^{hk}(t, \tau)$ are singular parts.

If it is possible to separate the space and time dependence in the non-singular part of the kernels, as well as in the out-of-integral coefficients and the right-hand side functions, and use their global stiffness matrix (matrices) as the constant coefficient matrix in constructing the system of finite dimensional integral equations later on.

If such separation is not applicable, we are forced to discretize the time interval and carry out the series of FEM analyzes for large number of values of t_m and τ_l. The stiffness matrices, which we obtain, are used afterwards for approximating the coefficient matrices of the resulting system of integral equations.

It is obvious, that for the second, more complicated case, the time interval discretization points must be chosen very carefully in order to minimize the number of spatial analyzes on the one hand (note that even a single run of the FEM package on a non-trivial geometry can appear extremely time- and resource-consuming), and on the other hand, to capture all the non-smoothness properties of the kernel, like oscillations, jumps etc.

In the sections below we suggest an approach to the numerical treatment of the equations with relaxation kernels $\alpha_{ij}^{hk}(x, t, \tau)$, the instantaneous elastic coefficients (out-of-integral terms) $a_{ij\,0}^{hk}(x, t)$, components of vectors of external forces f_{i0} and boundary traction $\phi_i(x, t)$, with inseparable time and space dependencies. Let us recall the following error estimate for multidimensional interpolation from [22].

Theorem 4.4. *Let $\Omega \subset \mathbb{R}^n$, $f \in C^{r+1}(\Omega)$ and $S^r f(\cdot)$ be its unique interpolation by Lagrange polynomials of degree $\leq r$ taken on the triangular/tetrahedral discretization $(\Omega_j)_{j=1,\ldots,N_{el}}$ of domain Ω. Let further $N = N(r) = \dim S^r$ be the number of interpolation points in each Ω_j, $j = 1, \ldots, N_{el}$. Then the following interpolation estimate*

$$\left\| f - S^r f \right\|_{C(\Omega)} \leq C M_{r+1} h^{r+1} \tag{4.12}$$

is valid, where $C = C(r, n)$ and

$$M_{r+1} = \sup_{x\in\Omega} ||D^{r+1} f(x)|| < +\infty,$$

where the notation $D^r f = \left\{\partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n} f\right\}$, $\alpha_1 + \cdots + \alpha_n = r$ is used.

Furthermore, if $f \in H^{r+1}(\Omega)$, and $S^r \in \mathcal{L}(H^{r+1}(\Omega), H^m(\Omega))$

$$\left\| f - S^r f \right\|_{H^m(\Omega)} \leq C||D^{r+1} f||_{L^2(\Omega)} h^{r+1-m} \qquad \forall\, 0 \leq m \leq r+1. \tag{4.13}$$

These estimates can be generalized for the essentially bounded (like in [13], [15]) or continuous on a closed interval, i.e., bounded (like in our case) in time semi-discrete (spatial) approximations.

5 Spatial approximation via interpolation and subsequent FE/collocation methods

By the space-time approximations (4.1) and (4.10) we reduce the infinite dimensional system of integral equations (1.1) in Hilbert spaces to the system of linear algebraic equations and the viscoelastic problem with memory (1.2) to the system of recurrent pure elastic problems in this section.

Suppose that $\psi \equiv 0$, and functions α_{ij}^{hk}, $a_{ij\ 0}^{hk}$, f_{i0} and ϕ_i are $m+1$ time differentiable in Ω. We suggest to approximate all these terms with respect to $x \in \Omega$ by continuous functions that are piecewise-polynomial on the finite elements Ω_p, $p = 1, \ldots, N_{el}$, i.e.

$$\begin{aligned} \alpha_{ij}^{hk}(x,t,\tau) &\approx S^m \alpha_{ij}^{hk}(\cdot,t,\tau) \\ &:= \textstyle\sum_{p=1}^{N_{el}} \sum_{q:x_q\in\Omega_p} \alpha_{ij}^{hk}(x_q,t,\tau) P_q^m(x)\chi_p(x), \qquad x\in\Omega \end{aligned} \tag{5.1}$$

where N_{el} is the number of elements in the model, x_q is the node of the p-th element, χ_p is a characteristic function on the elements, i.e.:

$$\chi_p(x) = \begin{cases} 1 & \text{if } x \text{ belongs to the } p\text{-th element} \\ 0 & \text{otherwise} \end{cases}$$

and $P_q^m(x)$ is a Lagrange polynomial of power m in the q-th node of the p-th element.

Performing FE-approximation (4.1), we obtain entries of the global stiffness matrix

$$\begin{aligned} \Lambda^{(u,l)}_{nd,\ ij}(t, [U_{d\ j}]) \ := \sum_{\delta : x_\delta \in \Omega_p} q_d \Phi^{hk}_{nd\ ij}(x_\delta, t, [U_{d\ j}]) \\ \times \int_{\Omega_p} P^m_\delta(x) \frac{\partial N_u(x)}{\partial x_k} \frac{\partial N_l(x)}{\partial x_h} dx, \end{aligned} \tag{5.2}$$

where $i, j = 1, \ldots, n$, stand for displacement vector components, u, l are global numbers of the nodes belonging to element p, and $\Phi_{nd,\gamma}(x_q, t, [u_d])$ are integrals

$$\Phi_{nd,\gamma}(x_q, t, [u_d]) := \begin{cases} \displaystyle\int_0^1 (\alpha(x_q, t_n, t_d + sq_d)\beta(t_n, t_d + sq_d))_\gamma \\ \quad \times u_d(t_d + sq_d) ds, \ 0 \le d \le n-1; \\ \displaystyle\int_0^t (\alpha(x_q, t_n, t_n + sq_n)\beta(t_n, t_n + sq_n))_\gamma \\ \quad \times u_n(t_n + sq_n) ds, \ d = n. \end{cases} \tag{5.3}$$

with $n = 0, \ldots n_1$ and γ standing for a generic 4−index.

Now, let us apply approximation (5.1) to the instantaneous elastic coefficients $a^{hk}_{ij\ 0}(x, t)$ and then apply the Finite Element approximation (4.1) to the out-of-integral term of operator equation (3.6). Thus we eliminate the space dependence, and obtain elements of the out-of-integral stiffness matrix

$$\begin{aligned} \int_{\Omega_p} a^{hk}_{ij\ 0}(x, t) \frac{\partial N_u(x)}{\partial x_k} \frac{\partial N_l(x)}{\partial x_h} dx \approx \sum_{q : x_q \in \Omega_p} a^{hk}_{ij\ 0}(x_q, t) \\ \times \int_{\Omega_p} P^m_q(x) \frac{\partial N_u(x)}{\partial x_k} \frac{\partial N_l(x)}{\partial x_h} dx, \\ \Lambda^{(u,l)}_{ij\ 0}(t) := \sum_{q : x_q \in \Omega_p} a^{hk}_{ij\ 0}(x_q, t) \int_{\Omega_p} P^m_q(x) \frac{\partial N_u(x)}{\partial x_k} \frac{\partial N_l(x)}{\partial x_h} dx, \\ i, j = 1, \ldots, n. \end{aligned} \tag{5.4}$$

Finally, we calculate the elements entries of the global load vector. For this reason we refer to (3.4), keeping in mind that we let for simplicity $\psi(x,t) \equiv 0$.

$$\begin{aligned}
&\int_{\Omega_p} f_{i0}(x,t) N_l(x) dx + \int_{\partial\Omega_\sigma \cap \Omega_p} \phi_i(s,t) N_l(s) ds \approx F_i^{(l)}(t) \\
&:= \sum_{q: x_q \in \Omega_p} f_{i0}(x_q, t) \int_{\Omega_p} P_q^m(x) N_l(x) dx \\
&\quad + \sum_{q: x_q \in \Omega_p \cap \partial\Omega_\sigma} \phi_i(x_q, t) \int_{\partial\Omega_\sigma \cap \Omega_p} P_q^m(s) N_l(s) ds.
\end{aligned} \tag{5.5}$$

Collecting these element stiffness matrices and load vectors to the global ones by the standard assembly procedure, we can rewrite the variational statement of the problem (3.1) in the form:

$$\begin{gathered}
\Lambda_{ij\ 0}(t) U_{n\ j}(t) + \Lambda_{nn,\ ij}[t, U_{n\ j}] = F_i(t) - \sum_{d=0}^{n-1} \Lambda_{nd,\ ij}[t_d, U_{d\ j}], \\
n = 0, \ldots n_1
\end{gathered} \tag{5.6}$$

Further, we look for the solution of (5.6) recurrently in each subinterval using (4.10), which can be done by standard collocation procedure.

Remark 5.1. Note that we can replace $P_q^m(x)$ in definitions (5.2), (5.4), (5.5) by $N_q(x)$, if the shape functions approximating the solution and the Lagrange shape functions $P_q^m(x)$, approximating the inhomogeneous coefficients and the external loading, all are of the same order.

In most applications the integrals (5.3) occurring in the collocation equation (5.6) cannot be evaluated analytically. Besides, the term $\beta(t,\tau)$ possesses a weak singularity in the end of every subinterval (see (2.9)). One is forced to resort the collocation algorithm given in [7] to employing suitable quadrature formulae for kernel approximation. Thus, we suggest here to use the Lagrange canonical polynomials

$$L_l(s) := \prod_{k=1, k\neq l}^{p+1} \frac{s - c_k}{c_l - c_k}$$

for collocation parameters $\{c_j\}$ with $0 \leq c_1 < \cdots < c_{p+1} \leq 1$ as weights for quadrature approximation of integrals (5.3), i.e., replace (5.3) by the sum:

$$\hat{\Phi}^{(j)}_{n,d}[u_d] := \sum_{l=1}^{p+1} w^{(j)}_{n,d,l}\alpha(t_{n,j}, t_{d,l})u_d(t_{i,l}) \quad \text{if} \quad 0 \leq d \leq n-1, \tag{5.7}$$

with the quadrature weights given by

$$w^{(j)}_{n,i,l} := \int_0^1 L_l(s)\beta(t_{n,j}, t_i + sq_i)ds. \tag{5.8}$$

A similar idea was also presented in [25].

Example 5.2. Let $\beta(t, s) \equiv 1$. Then we are in a framework of the standard collocation and

$$w^{(j)}_{n,i,l} := \int_0^1 L_l(s)ds. \tag{5.9}$$

Example 5.3. Let $\beta(t, s) = (t - s)^{-\alpha}$ and let the discretization by t_n, $n = \{0, n_1\}$ be equidistant with step size q. Then

$$w^{(j)}_{n,i,l} := q^{-\alpha} \int_0^1 L_l(s)(n - i + c_j - s)^{-\alpha}ds$$

This integral could be analytically taken in the form of a hypergeometric series (see [23]). However it can be reduced to the finite expression through Euler's $\Gamma-$ or $\mathcal{B}-$ functions only for $n - i + c_j = 1$. Therefore, we prefer to avoid the weak singularity by partial integration

$$w^{(j)}_{n,i,l} \approx \begin{cases} 0, & \text{if } l \neq 1 \text{ and } l \neq p+1 \\ q^{-\alpha}L_1(0)\frac{(n-i+c_j)^{1-\alpha}}{1-\alpha}, & \text{if } l = 1 \\ -q^{-\alpha}L_{p+1}(1)\frac{(n-i+c_j-1)^{1-\alpha}}{1-\alpha}, & \text{if } l = p+1 \end{cases}$$

$$+\frac{q^{-\alpha}(n - i + c_j - c_l)^{1-\alpha}}{1-\alpha} \int_0^1 L'_l(s)ds$$

The detailed recurrent formulas for the weights can be found in [24].

6 A priori error estimates

Let us first suppose that the instantaneous elastic coefficients $a_{ij\,0}^{hk}$ as well as the external loads f_{i0} and ϕ_i are either space-independent functions or change homogeneously in space, i.e. they are represented as a product of purely space and time dependent functions respectively. Furthermore, we suppose, that $\beta_{ij}^{hk} \in L^1([0, T])$, i.e., the weak singular part of integral kernels is space independent, $\alpha_{ij}^{hk} \in C([0, T] \times [0, T], C^{m+1}(\Omega))$. We apply the suggested in Section 5 approximation in space by interpolation to the relaxation kernels only. Let

$$A_x u := \int_\Omega \big(\alpha(x, t, \tau)\beta(t, \tau)\big)_{ij}^{hk} \frac{\partial u_j(x, \tau)}{\partial x_k} \frac{\partial \cdot_i (x)}{\partial x_h} dx, \tag{6.1}$$

$$A_{xS} u_S := \int_\Omega \big(S^m \alpha(\cdot, t, \tau)\beta(t, \tau)\big)_{ij}^{hk} \frac{\partial u_{Sj}(x, \tau)}{\partial x_k} \frac{\partial \cdot_i (x)}{\partial x_h} dx, \tag{6.2}$$

where $S^m \alpha_{ij}^{hk}(x, t, \tau)$ is the approximation of the non-singular part of the kernels, defined by (5.1), and u_S denotes the solution of our problem (3.9) corresponding to this approximation. We introduce the error in the solution caused by the kernel approximation by interpolation as:

$$\epsilon := u - u_S. \tag{6.3}$$

Lemma 6.1. *The error in the solution caused by kernel approximation* (5.1) *is of the same order as the error of kernel approximation itself, i.e.*

$$\|\epsilon(t)\|_{H^1(\Omega)} := \|u(t) - u_S(t)\|_{H^1(\Omega)} \leq C h^{m+1} \tag{6.4}$$

for any $t \in [0, T]$, *where*

$$\begin{aligned} C &= \frac{\tilde{C}\left(\frac{1}{c_0} a \star\right)}{c_0^2} \|f\|_{C([0,T],L^2(\Omega))} \\ &\quad \times \max_{i,j,h,k} \left(\left\| D^{m+1} \alpha_{ij}^{hk} \right\|_{C([0,T]\times[0,T]\times\Omega)} \left\| \beta_{ij}^{hk} \right\|_{C([0,T],L^1([0,T]))} \right) \end{aligned}$$

Proof. To begin with, let us represent equation (3.9) in the form:

$$u(t) + \int_0^t A_{0_x}^{-1}(t) A_x(t, \tau) u(\tau) d\tau = A_{0_x}^{-1}(t) F(t) \tag{6.5}$$

Similarly, we can write for $u_S(t)$:

$$u_S(t) + \int_0^t A_{0_x}^{-1}(t) A_{xS}(t,\tau) u_S(\tau) d\tau = A_{0_x}^{-1}(t) F(t) \tag{6.6}$$

The difference $u - u_S$ should then satisfy equation

$$u(t) - u_S(t) + \int_0^t A_{0_x}^{-1}(t) A_x(t,\tau)(u(\tau) - u_S(\tau)) d\tau = A_{0_x}^{-1}(t) F^*(t),$$

$$F^*(t) := \int_0^t \Big(A_{xS}(t,\tau) - A_x(t,\tau) \Big) u_S(\tau) d\tau$$

Owing to Theorem 3.2, (4.8), the error (6.3) can then be estimated as follows:

$$\begin{aligned}
& \|u(t) - u_S(t)\|_{C([0,T],H^1(\Omega))} \le \frac{1}{c_0} \tilde{C}\Big(\frac{1}{c_0} a \star\Big) \Big\| F^* \Big\|_{C([0,T],H^{-1}(\Omega))} \\
= \;& \frac{1}{c_0} \tilde{C}\Big(\frac{1}{c_0} a \star\Big) \Big\| \int_0^t \Big(A_{xS}(t,\tau) - A_x(t,\tau) \Big) u_S(\tau) d\tau \Big\|_{C([0,T],H^{-1}(\Omega))} \\
\le \;& \frac{1}{c_0} \tilde{C}\Big(\frac{1}{c_0} a \star\Big) \|u_S\|_{C([0,T],H^1(\Omega))} \\
& \times \Big\| \int_0^t \Big\| A_{xS}(t,\tau) - A_x(t,\tau) \Big\|_{\mathcal{L}(H^1(\Omega))} d\tau \Big\|_{C([0,T])} \\
\overset{(3.13)}{\le} \;& \frac{1}{c_0} \tilde{C}\Big(\frac{1}{c_0} a \star\Big) \|u_S\|_{C([0,T],H^1(\Omega))} \\
& \times \max_{i,j,k,h} \Big(\Big\| S^m \alpha_{ij}^{hk} - \alpha_{ij}^{hk} \Big\|_{C([0,T]\times[0,T],L^\infty(\Omega))} \Big\| \beta_{ij}^{hk} \Big\|_{C([0,T],L^1([0,T]))} \Big) \\
\overset{\text{Theorem 3.2, (4.8)}}{\le} \;& \frac{1}{c_0^2} \tilde{C}\Big(\frac{1}{c_0} \max(a, a_S) \star\Big) \Big\| F \Big\|_{C([0,T],H^{-1}(\Omega))} \\
& \times \max_{i,j,k,h} \Big(\Big\| S^m \alpha_{ij}^{hk} - \alpha_{ij}^{hk} \Big\|_{C([0,T]\times[0,T],L^\infty(\Omega))} \Big\| \beta_{ij}^{hk} \Big\|_{C([0,T],L^1([0,T]))} \Big) \\
\overset{\text{Theorem 4.4}}{\le} \;& \frac{1}{c_0^2} \tilde{C}\Big(\frac{1}{c_0} \max(a, a_S) \star\Big) \Big\| f \Big\|_{C([0,T],L^2(\Omega))} \\
& \times \max_{i,j,k,h} \Big(\Big\| D^{m+1} \alpha_{ij}^{hk} \Big\|_{C([0,T]\times[0,T]\times\Omega))} \Big\| \beta_{ij}^{hk} \Big\|_{C([0,T],L^1([0,T]))} \Big) h^{m+1}
\end{aligned}$$

□

Let now the instantaneous elastic coefficients $a_{ij\,0}^{hk}$ and the external loads f_{i0} and ϕ_i also possess inseparable time-space dependencies and $a_{ij\,0}^{hk}$, f_{i0} and ϕ_i belong to $C([0,T], C^{m+1}(\Omega))$. So, we apply the approximation S^m, defined by (5.1), to these terms too. We define, additionally to (6.1), (6.2),

$$A_{0xS}(t) u_S(t) := \int_\Omega S^m a_{ij\,0}^{hk}(\cdot, t) \frac{\partial u_{Sj}(x,t)}{\partial x_k} \frac{\partial \cdot_i (x)}{\partial x_h} dx, \tag{6.7}$$

$$F_S(t) := \int_\Omega S^m f_{i0}(\cdot,t) u_{Si}(x,t)dx + \int_{\partial\Omega_\sigma} S^{m+1}\phi_i(\cdot,t)u_{Si}(s,t)ds, \tag{6.8}$$

where S^m is the approximation defined by (5.1), and u_S denotes the solution of (3.9) corresponding to this approximation.

Lemma 6.2. *The error in the solution of* (3.9) *caused by approximations* $S^m a^{hk}_{ij\,0}$, $S^m \alpha^{hk}_{ij}$, $S^m f_{i0}$ *and* $S^{m+1}\phi_i$ *is again of the order* $O(h^{m+1})$, *i.e.,*

$$\|\epsilon(t)\|_{H^1(\Omega)} := \|u(t) - u_S(t)\|_{H^1(\Omega)} \le Ch^{m+1} \tag{6.9}$$

for any $t \in [0, T]$.

Proof. Consider again equations (6.5) and

$$u_S(t) + \int_0^t A^{-1}_{0xS}(t) A_{xS}(t,\tau) u_S(\tau) d\tau = A^{-1}_{0xS}(t) F_S(t) \tag{6.10}$$

We estimate the error on the same way as in previous lemma just extending the proof by the triangle inequality:

$$\begin{aligned}
&\|u(t) - u_S(t)\|_{H^1(\Omega)} \\
\le\ & \frac{1}{c_0}\tilde{C}\Big(\frac{1}{c_0}a\star\Big)\Big(\|u_S\|_{C([0,T],H^1(\Omega))} \\
& \times \Big\| \int_0^t \Big(A^{-1}_{0xS}(t)A_{xS}(t,\tau) - {A_{0xS}}^{-1}(t)A_x(t,\tau)\Big)d\tau \\
& + \int_0^t \Big(A^{-1}_{0xS}(t)A_x(t,\tau) - {A_{0x}}^{-1}(t)A_x(t,\tau)\Big)d\tau \Big\|_{\mathcal{L}(H^1(\Omega),H^1(\Omega))} \\
& + \Big\| A^{-1}_{0x}(t)F(t) - A^{-1}_{0x}(t)F_S(t) + A^{-1}_{0x}(t)F_S(t) - A^{-1}_{0xS}(t)F_S(t)\Big\|_{H^1(\Omega)}\Big) \\
\overset{\text{Sec. 3}}{\le}\ & \frac{1}{c_0^2}\tilde{C}\Big(\frac{1}{c_0}a\star\Big)\|F_S\|_{C([0,T],H^{-1}(\Omega))} \qquad (6.11) \\
& \times \Bigg[\frac{1}{c_0}\max_{i,j,k,h}\Big(\|\beta^{hk}_{ij}\|_{C([0,T],L^1([0,T]))} \|S^m\alpha^{hk}_{ij} - \alpha^{hk}_{ij}\|_{C([0,T]\times[0,T],L^\infty(\Omega))}\Big) \\
& + \max_{i,j,k,h}\Big(\Big\|\alpha^{hk}_{ij}\Big\|_{C([0,T]\times[0,T],L^\infty(\Omega))}\Big\|\beta^{hk}_{ij}\Big\|_{C([0,T],L^1([0,T]))}\Big)
\end{aligned}$$

$$\left. \times \left\| A_{0xS}^{-1} - A_{0x}^{-1} \right\|_{C([0,T],\mathcal{L}(H^{-1},H^1(\Omega)))} \right]$$
$$+ \frac{\tilde{C}}{c_0^2} \left(\left\| f_0 - S^m f_0 \right\|_{C([0,T],L^2(\Omega))} + \left\| \phi - S^{m+1}\phi \right\|_{C([0,T],H^1(\Omega))} \right)$$
$$+ \frac{\tilde{C}}{c_0} \left\| F_S \right\|_{C([0,T],H^{-1}(\Omega))} \left\| A_{0xS}^{-1} - A_{0x}^{-1} \right\|_{C([0,T],\mathcal{L}(H^{-1},H^1(\Omega)))}.$$

First we estimate

$$\|F_S\|_{C([0,T],H^{-1}(\Omega))} \leq \|F_S - F\|_{C([0,T],H^{-1}(\Omega))} + \|F\|_{C([0,T],H^{-1}(\Omega))}, \tag{6.12}$$

and, according to Remark (3.5)

$$\|F\|_{C([0,T],H^{-1}(\Omega))} \leq C \left(||f_0||_{C([0,T]\times\Omega)} + ||\phi||_{C([0,T],H^{-1/2}(\partial\Omega_\sigma))} \right).$$

Let us estimate the term $\left\| A_{0xS}^{-1} - A_{0x}^{-1} \right\|_{C([0,T],\mathcal{L}(H^{-1},H^1(\Omega)))}$ now:

$$\begin{aligned}
&\left\| A_{0xS}^{-1} - A_{0x}^{-1} \right\|_{C([0,T],\mathcal{L}(H^{-1},H^1(\Omega)))} \\
&\quad = \left\| A_{0xS}^{-1}(A_{0x} - A_{0xS})A_{0x}^{-1} \right\|_{C([0,T],\mathcal{L}(H^{-1},H^1(\Omega)))} \\
&\quad \leq \frac{1}{c_0^2} \left\| A_{0x} - A_{0xS} \right\|_{C([0,T],\mathcal{L}(H^1(\Omega),H^{-1}))} \\
&\quad \leq \frac{1}{c_0^2} \max_{i,j,k,h} \left\| a_{ij\ 0}^{hk} - S^m a_{ij\ 0}^{hk} \right\|_{C([0,T]\times\Omega)}.
\end{aligned} \tag{6.13}$$

Application of Theorem 4.4 to each term of (6.11), to (6.12) and to (6.13) completes the proof. □

The following lemma gives the a priori estimate for the total solution error.

Theorem 6.3 (A priori error estimate). *The error estimate satisfies:*

$$\left\| u - \pi_p P_r u_S \right\|_{L^\infty([0,T],H^1(\Omega))} \leq C_S h^{m+1} + C h^r + C_\pi q^{\hat{p}} \tag{6.14}$$

Proof. Using triangle inequality, we can see that:

$$\begin{aligned}
\left\| u - \pi_p P_r u_S \right\|_{L^\infty([0,T],H^1(\Omega))} &\leq \left\| u(t) - u_S(t) \right\|_{L^\infty([0,T],H^1(\Omega))} \\
&+ \left\| u_S(t) - P_r u_S(t) \right\|_{L^\infty([0,T],H^1(\Omega))}
\end{aligned}$$

$$+ \left\| P_r u_S(t) - \pi_p P_r u_S(t) \right\|_{L^\infty([0,T], H^1(\Omega))}$$

The estimates for the first term in the right-hand side are given by (6.4), (6.9), for the second by (4.2) and (4.6) for L^2- and H^1-norms respectively, and for the third term by (4.11). □

Thus, we can conclude that the error, introduced by spatial approximation of relaxation kernels, instantaneous elastic coefficients and external loads, does not increase the total error, if $m \geq r$. On the other hand, it significantly simplifies the calculation procedure, allowing us to significantly reduce the computations of the spatial finite element analysis.

7 Numerical example

Consider the homogeneous isotropic viscoelastic prismatic rod of length ℓ, as shown in Figure 1.

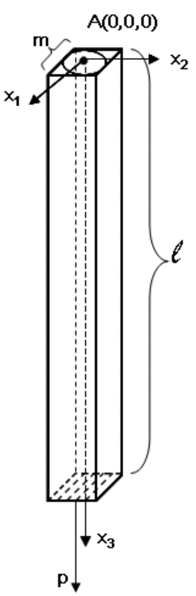

Figure 1 – Test problem: isotropic viscoelastic rod.

It is subject to stretching under its own weight and under an external tension pressure p, homogeneously distributed over its lower end. The rod is rigidly fixed in the middle point $A = (0, 0, 0)$ on its upper face. Besides, the zero displacement constraints in vertical direction are applied to the circle of diameter m, which completely belongs to the upper face of the rod.

Consider the system of equilibrium equations (2.1)–(2.3) under isotropy condition (2.7), and switch over to Poisson's ratio and Young's modulus through the relation:

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \qquad \mu = \frac{E}{2(1+\nu)}. \tag{7.1}$$

We assume further that the Poisson's ratio ν is time-independent and that the Young's modulus $\underline{E}$ is taken in the form of the following Volterra integral operator

$$\underline{E} = E_0 + E\star; \qquad E\star = -E_0 \int_0^t k(t-\tau)\cdot d\tau. \tag{7.2}$$

Now, the system of equilibrium equations (2.1)–(2.3) can be rewritten as follows:

$$\frac{\underline{E}}{2(1+\nu)(1-2\nu)} \frac{\partial}{\partial x_i} \operatorname{div} \mathbf{u}(x,t) + \frac{\underline{E}}{2(1+\nu)} \Delta u_i(x,t) = -f_i(x,t), \tag{7.3}$$

with boundary conditions:

$$u_i\big|_A = 0, \quad i = 1, 2, 3, \tag{7.4}$$

$$u_3\big|_{\substack{x_3=0,\\ x_1^2+x_2^2=(\frac{m}{2})^2}} = 0, \tag{7.5}$$

$$\sigma_3\big|_{x_3=\ell} = p f_2(t), \tag{7.6}$$

$$\sigma_{ij} n_j\big|_{\text{on the rest of boundary}} = 0, \quad i, j = 1, 2, 3, \tag{7.7}$$

holding for any $t \in [0, T]$. Here

$$\mathbf{f}^{\mathbf{T}}(x,t) = (f_i(x,t))_n = (0, 0, \rho g f_2(t)), \tag{7.8}$$

where ρ is the material density, and we choose

$$k(s) = \left\{ \frac{1}{\sqrt{s}}, \quad \text{or} \quad \frac{s^2}{2} e^{-s} \right., \tag{7.9}$$

$$f_2(t) = \left\{ t + 1 - \left(2\sqrt{t} + \frac{4}{3}\sqrt{t^3} \right), \quad \text{or} \quad (t^2 + t + 1)e^{-t}. \right. \tag{7.10}$$

Since the right-hand side of our system can be represented as a product of a purely space- and time-dependent function, and the kernel (7.9) of operator (7.2) is space-independent, the solution of the problem (7.3)–(7.7) will be of the form:

$$u_i(x,t) := u_{i1}(x)u_2(t). \tag{7.11}$$

We rewrite our system as:

$$\begin{aligned}\left(\frac{1}{2(1+\nu)(1-2\nu)}\frac{\partial}{\partial x_i}\operatorname{div}\ u_1(x)+\frac{1}{2(1+\nu)}\Delta u_{i1}(x)\right)u_2(t)\\ = -\left[(\underline{E})^{-1}f_i\right](t).\end{aligned} \tag{7.12}$$

Then the pair of equations, consisting of the purely spatial one:

$$\begin{aligned}\frac{E_0}{2(1+\nu)(1-2\nu)}\frac{\partial}{\partial x_i}\operatorname{div}\ \mathbf{u_1}(x)+\frac{E_0}{2(1+\nu)}\Delta u_{i1}(x)\\ = -\begin{cases}0, & i=1,2;\\ \rho g, & i=3\end{cases}\end{aligned} \tag{7.13}$$

and the temporal one:

$$u_2(t) = (t+1)-\left(2\sqrt{t}+\frac{4}{3}\sqrt{t^3}\right)+\int_0^t \frac{u_2(\tau)}{\sqrt{t-\tau}}\,d\tau \quad \text{or} \tag{7.14}$$

$$u_2(t) = (t^2+t+1)e^{-t}+\int_0^t \frac{(t-\tau)^2}{2}e^{\tau-t}u_2(\tau)d\tau \tag{7.15}$$

can be solved analytically to yield the desired solution:

$$u_1(x) = \frac{1}{E_0}\left(\begin{array}{c} -\rho g\nu x_1x_3-\nu(\rho g\ell-p)x_1,\\ -\rho g\nu x_2x_3-\nu(\rho g\ell-p)x_2,\\ \dfrac{\rho g x_3^2}{2}+(\rho g\ell-p)x_3+\dfrac{\rho g\nu}{2}\left(x_1^2+x_2^2\right)-\dfrac{\rho g\nu m^2}{8}\end{array}\right)$$

for the spatial part of the solution, and

$$u_2(t) = 1+t \quad \text{or} \tag{7.16}$$

$$u_2(t) = 1+\frac{1}{3}\left(1-e^{-\frac{3}{2}t}\left[\cos\left(\frac{\sqrt{3}}{2}t\right)+\sqrt{3}\sin\left(\frac{\sqrt{3}}{2}t\right)\right]\right) \tag{7.17}$$

For a numerical simulation of the system's space-dependent part we used the ANSYS Finite Element package. Since the ANSYS element library does not contain elements supporting materials with weakly singular kernels, we combined ANSYS with the collocation method implemented in an own fortran code based on the algorithm from [8]. The latter algorithm was modified to allow for weak-singular kernel parts as well as a coefficient matrix in front of the out-of-integral term.

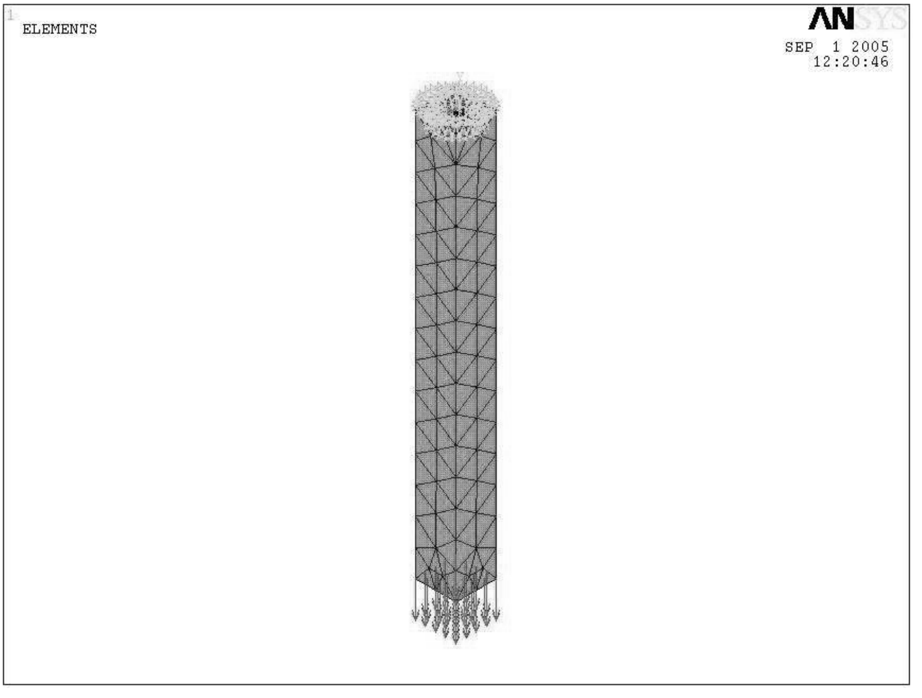

Figure 2 – FE-mesh and boundary conditions.

The FEM discretization of $\frac{\partial}{\partial x}\left(\alpha(x, t_m, \tau_l)\frac{\partial u}{\partial x}\right)$ w.r.t. the space variable is obtained as the global stiffness matrix of the problem with appropriate material properties, and the one for the right-hand side as a global load vector. For the extraction of the global stiffness matrix from ANSYS in a text format, we modified **rdsubs.F** code, taken from the ANSYS distribution medium and incorporated into original ANSYS as a User Predefined Routine (UPF). Thus, we reduced the infinite-dimensional system of the Volterra integral equations to a finite-dimensional one. Then, the obtained integral equation system was solved

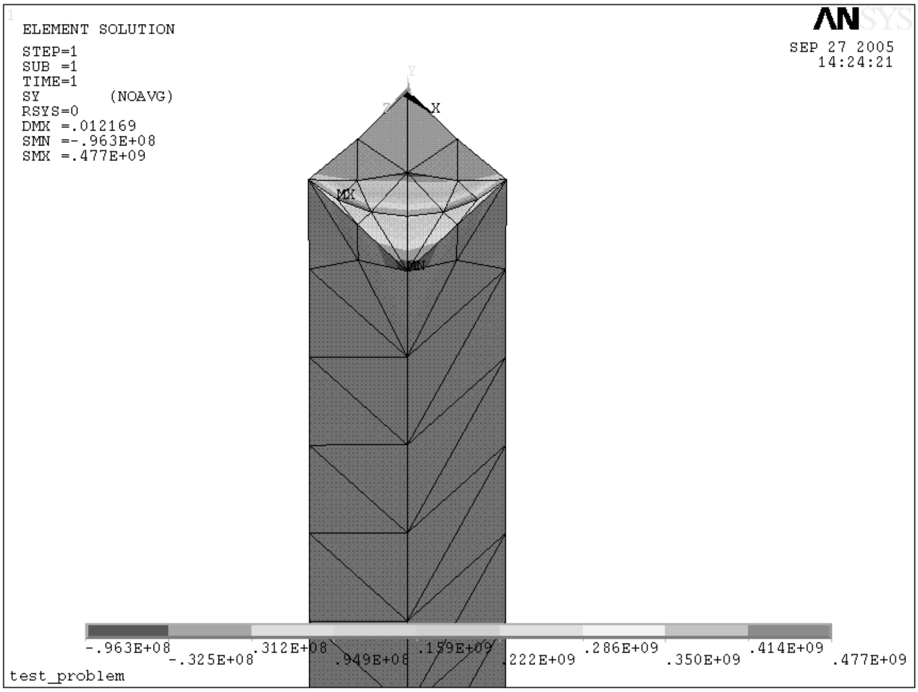

Figure 3 – FE-mesh of the quarter of rod and σ_{22} at the time-step $t = 0$.

numerically with the spline collocation method described in Section 5. Finally the numerical solution for the system of integral equations was obtained by iteratively solving a system of linear algebraic equations with help of the conjugate gradient method, as $u_{ij} = u_i(t_j)$, $j = 0, \ldots, n_1$, $i = 1, \ldots, 3N$, where n_1 is the number of discretization points for the time interval and N is the number of nodes in the finite element discretization of the body.

7.1 Convergence in time

To test the temporal convergence of the collocation method, we solve both problems, (7.14) and (7.15), on the unit time interval with five collocation points on a unit subinterval and compare the analytical $u_2(t)$ and numerical $U_2(t)$ solutions in the arbitrary chosen time point $T = \frac{1}{3}$. The summary of numerical performance is presented in Table 1.

Time, equation	$u_2(t)$	$U_2(t)$	$\lvert u_2(t) - U_2(t)\rvert$
T = 1/3, (7.14)	0.1333371E+01	0.132760E+01	0.577E-02
T = 1/3, (7.15)	0.13983189911	0.13983187675	0.224E-07

Table 1 – Comparison of analytical ($u_2(t)$) and numerical ($U_2(t)$) solutions of equations (7.14) and (7.15).

7.2 *Spatial and full convergence*

The following parameter set: $E_0 = 8.5 \cdot 10^{10}$ Pa, $\nu = 0.16$, $\rho = 1.2770$ kg/m^3, $p = -10 \cdot 10^7$ Pa, $m = 1$ cm, $l = 10$ cm was used for a numerical example. To perform the numerical simulation with ANSYS, we took advantage of the symmetry of the modeled body and therefore considered only a quarter of it (see Fig. 3). The body has been discretized with 244 elements, 620 nodes. The summary of successive spatial and temporal numerical performances of ANSYS and the collocation fortran-routine is presented in Tables 2 and 3.

Time	$\lVert u_1(t)\rVert_{L_\infty(\Omega)}$	$\lVert U_1(t)\rVert_{L_\infty(\Omega)}$	$\lVert u_1(t) - U_1(t)\rVert_{L_\infty(\Omega)}$
T = 0	0.946305E-04	0.94118E-04	0.512456E-06

Table 2 – Comparison of analytical ($u_1(t)$) and numerical ($U_1(t)$) solution.

Time	$\lVert u_3(t)\rVert_{L_\infty(\Omega)}$	$\lVert U_3(t)\rVert_{L_\infty(\Omega)}$	$\lVert u_3(t) - U_3(t)\rVert_{L_\infty(\Omega)}$
T = 0	0.11812E-01	0.12168E-01	0.35534E-03
T = 1/3, (7.17)	0.157497E-01	0.161542E-01	0.54400936E-03
T = 1/3, (7.18)	0.1651694E-02	0.1701474E-02	0.496881E-04

Table 3 – Comparison of analytical ($u_3(t)$) and numerical ($U_3(t)$) solution.

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