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# A structure-preserving iteration method of model updating based on matrix approximation theory*

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**Abstract.** Some theories and a method are discussed on updating a generalized centrosymmetric model. It gives a generalized centrosymmetric modified solution with partial prescribed least square spectra constraints. The emphasis is given on exploiting structure-preserving algorithm based on matrix approximation theory. A perturbation theory for the modified solution is established. The convergence of an iterative solution is investigated. Illustrative examples are provided.

**Mathematical subject classification:** 15A29, 15A90, 41A29, 65F15, 65L70.

**Key words:** structure-preserving algorithm, generalized centrosymmetric matrix, model updating, perturbation analysis.

## 1 Introduction

Structure design with prescribed natural frequencies and main vibration modes is an important research topic in structural engineering [12]. Structure finite element model updating technology is the most common method in structure design. For example, an undamped free vibration model [14] is described by

$$M\ddot{x} + Kx = 0,$$

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where $K$ is a stiffness matrix and $M$ is a mass matrix. The corresponding eigenvalue problem is

$$(K - \lambda M)\varphi = 0,$$

where $\lambda$ is an eigenvalue and $\varphi$ is a vibration mode. If $M = I$, it is a standard eigenvalue problem. In practice, a portion of eigenvalues and vibration modes can be identified and these data are credible, whereas the stiffness or mass matrix is always unknown and usually estimated by finite element method. Finite element model is not very close to the real structure because of some simplified hypothesis and treatment of boundary conditions. There often exists a discrepancy between the eigenvalues of the approximate model and the identified one. To modify the approximate model to minimize the difference becomes a must.

A correction of structural stiffness or mass matrix using vibration tests was solved by nonlinear optimization techniques [2, 3]. But these methods do not guarantee the existence and the uniqueness of the solution and the solution is not doomed to be the best one. To this end, we present a method to correct an approximate model based on structured inverse eigenproblem with two constraints-the spectral constraint referring to the prescribed spectral data, and the structural constraint referring to the desirable structure. They can be formulated as to find $A$ such that $AX = X\Lambda$, where $A$ is some desirable structure matrix, $X$ and $\Lambda$ are given identified modes and eigenvalue matrices, and to find the best approximate matrix $\hat{A}$ to minimize the Frobenius norm of $C - A$ for any estimate matrix $C$. But the determinations of eigenvalues and modes from vibration test data involve numerous sources of discrepancies or errors. Thus we consider its least-square problem.

Here desirable structure of matrices is generalized centrosymmetric. For the convenience of description, we translate the model updating problem into the following problems.

**Problem I.** Given a real $n \times m$ matrix $X = (x_1, x_2, \dots, x_m)$ and a diagonal matrix $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_m)$, find all generalized centrosymmetric matrices $A$ such that

$$\|AX - X\Lambda\| = \min. \tag{1}$$

**Problem II.** Given a real $n \times n$ matrix $C$, find $\hat{A} \in \mathbb{S}_{\mathbb{E}}$ such that

$$\|C - \hat{A}\| = \min_{A \in \mathbb{S}_{\mathbb{E}}} \|C - A\|, \tag{2}$$

where $\mathbb{S}_{\mathbb{E}}$ is the solution set of Problem I and $\|\cdot\|$ is the Frobenius norm. $X$, $\Lambda$, $C$ are consistent with above description.

Problem I is a structural inverse eigenproblem and Problem II is the best approximate problem with assigned least square spectra constraints. They arise in many areas of important applications [7, 8, 10-12]. Indeed, partial inverse eigenpair problems are used to modify some models [5, 13, 14]. Depending on the applications, inverse eigenproblems may be described in several different forms. Therefore inverse problems are different for different classes of matrices. Problem I and II were studied for some classes of structured matrices. We refer the reader to [1, 16, 19-22] and references therein. For example, Zhou et al. [22] and Zhang et al. [21] considered the problems for the case of centrosymmetric matrices and Hermitian-generalized Hamiltonian matrices, respectively. They established existence theorems for the solutions and derived expressions of the general solutions. Abdalla et al. [1] and Moreno et al. [16] discussed them in the case of the symmetric positive definite eigenproblem and quadratic inverse eigenvalue problem using some projection method, respectively. In this paper we investigate them for the set of all real generalized centrosymmetric matrices defined by the following definition. The solution to corresponding Problem II is the first modified solution. Generally a structure matrix is sparse. The sparse structure of the first modified solution may be destroyed. In this paper we will present a structure-preserving iteration algorithm and analysis a perturbation of the modified solution. We not only give an expression of the solutions but also provide a structure-preserving iterative algorithm of finite element model updating, based on the theory of inverse eigenpair problem. We also study a perturbation of the modified solution, which was not done in [20-22]. Next we introduce the definition of generalized centrosymmetric matrices.

**Definition 1.** Assume that $E$, $F$ are real $k \times k$ matrices, $u$ and $v$ are $k$-dimensional real vector, $P$ is some orthogonal $k \times k$ matrix and $\alpha$ is a real number. If

$$A_{2k} = \begin{pmatrix} E & FP \\ P^T F & P^T E P \end{pmatrix}, \tag{3}$$

$$A_{2k+1} = \begin{pmatrix} E & u & FP \\ v^T & \alpha & v^T P \\ P^T F & P^T u & P^T E P \end{pmatrix}, \tag{4}$$

then $A_{2k}$ and $A_{2k+1}$ are called generalized centrosymmetric matrices.

The centrosymmetric matrices have wide applications in many fields (see [4, 9-11]). If $P = (e_k, e_{k-1}, \ldots, e_1)$ and $e_i$ is the $i$th column of identity matrix $I_k$, the inverse eigenpairs problem of centrosymmetric matrices becomes a special case of this paper.

In this paper, we denote by $\mathbb{R}^{n\times m}$ the set of all real $n \times m$ matrices. The set of all $n \times n$ orthogonal matrices is represented by $\mathbb{O}^{n\times n}$. $A^+$ stands for the Moore-Penrose generalized inverse and $P_A = AA^+$ is the orthogonal projection onto $\mathbb{R}(A)$, the range of $A$. We define inner product $(A, B) = \text{trace}(B^T A)$ in $\mathbb{R}^{n\times m}$. Thus $\mathbb{R}^{n\times m}$ is a Hilbert space and the induced norm is the Frobenius norm $\|A\| = (\sum_{i,j} a_{ij}^2)^{1/2}$. For perturbation analysis, we also use the 2-norm $\|A\|_2$ for matrix $A \in \mathbb{R}^{n\times n}$.

This paper is organized as follows. In Section 2, we first give the expressions of the solutions to Problem I and II. And then we provide a structure-preserving iteration algorithm of model updating problem. In section 3, we study a perturbation bound of the modified solution and analyze the convergence of iteration solutions. Finally, some conclusions are presented in Section 4.

## 2 The basic theory and a numerical method

### *2.1 Expressions of the solutions to Problem I and II*

Many stiffness or mass matrices from vibration model are not only structured but also large scale. We can reduce their scale to half because generalized centrosymmetric matrices are similar to a block diagonal matrices. Firstly, we consider some properties of generalized centrosymmetric matrices.

Throughout this paper, $P$ is the same as in Definition 1. Let

$$k = \left[\frac{n}{2}\right], \ [x] \text{ is the maximum integer number that is not greater than } x. \tag{5}$$

When $n = 2k$, we take

$$D_{2k} = \frac{1}{\sqrt{2}} \begin{pmatrix} I_k & I_k \\ -P^T & P^T \end{pmatrix}, \tag{6}$$

when $n = 2k + 1$, we take

$$D_{2k+1} = \frac{1}{\sqrt{2}} \begin{pmatrix} I_k & 0 & I_k \\ 0^T & \sqrt{2} & 0^T \\ -P^T & 0 & P^T \end{pmatrix}. \tag{7}$$

Then $D_{2k}$ and $D_{2k+1}$ are orthogonal and $A_n$ in Definition 1 has the following formula

$$A_n = D \begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix} D^T, \tag{8}$$

where $G_1 \in \mathbb{R}^{k\times k}$, $G_2 \in \mathbb{R}^{(n-k)\times(n-k)}$, $D$ is the same as (6) or (7) when $n = 2k$ or $n = 2k + 1$.

Next our goal is to give an expression of the set $\mathbb{S}_{\mathbb{E}}$. First, we introduce the following lemma.

**Lemma 1 ([15]).** *Suppose $X, B \in \mathbb{R}^{n\times m}$. Then a matrix $A$ to satisfy $\|AX - B\| = \min$ is*

$$A = BX^+ + Z(I - XX^+), \forall Z \in \mathbb{R}^{n\times n}. \tag{9}$$

**Theorem 1.** *Given $X \in \mathbb{R}^{n\times m}$, $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m)$, assume $D^T X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$, where $X_1 \in \mathbb{R}^{k\times m}$, $X_2 \in \mathbb{R}^{(n-k)\times m}$, $k = \left[\frac{n}{2}\right]$. Then there is a generalized centrosymmetric matrix $A$ such that $\|AX - X\Lambda\| = \min$ and*

$$A = D \begin{pmatrix} X_1\Lambda X_1^+ + Z_1(I_k - X_1X_1^+) & 0 \\ 0 & X_2\Lambda X_2^+ + Z_2(I_{n-k} - X_2X_2^+) \end{pmatrix} D^T, \tag{10}$$
$$\forall Z_1 \in \mathbb{R}^{k\times k}, \quad \forall Z_2 \in \mathbb{R}^{(n-k)\times(n-k)}$$

*and $D$ is the same as* (6) *or* (7).

**Proof.** By (8),

$$A = D \begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix} D^T, \tag{11}$$

where $G_1 \in \mathbb{R}^{k\times k}$ and $G_2 \in \mathbb{R}^{(n-k)\times(n-k)}$. Because

$$\|AX - X\Lambda\|^2 = \|G_1X_1 - X_1\Lambda\|^2 + \|G_2X_2 - X_2\Lambda\|^2, \tag{12}$$

minimization of $\|AX - X\Lambda\|$ is equivalent to

$$\|G_1X_1 - X_1\Lambda\| = \min_{G_1\in\mathbb{R}^{k\times k}}, \tag{13}$$

$$\|G_2X_2 - X_2\Lambda\| = \min_{G_2\in\mathbb{R}^{(n-k)\times(n-k)}}. \tag{14}$$

From Lemma 1 we know that (13) and (14) are solvable and the solutions are

$$G_1 = X_1\Lambda X_1^+ + Z_1(I_k - X_1X_1^+), \quad Z_1 \in \mathbb{R}^{k\times k}, \tag{15}$$

$$G_2 = X_2\Lambda X_2^+ + Z_2(I_{n-k} - X_2X_2^+), \quad Z_2 \in \mathbb{R}^{(n-k)\times(n-k)}. \tag{16}$$

Substituting (15) and (16) into (11) we have (10). □

Next we discuss Problem II.

**Theorem 2.** *Given $C \in \mathbb{R}^{n\times n}$, the notation of $X$, $\Lambda$ and the conditions are the same as those in Theorem* 1. *Then there is a unique $\hat{A} \in \mathbb{S}_{\mathbb{E}}$ to Problem II and*

$$\hat{A} = D\begin{pmatrix} X_1\Lambda X_1^+ + \hat{Z}_1(I_k - X_1X_1^+) & 0 \\ 0 & X_2\Lambda X_2^+ + \hat{Z}_2(I_{n-k} - X_2X_2^+)\end{pmatrix}D^T, \tag{17}$$

*where if $n = 2k$, $D$ is the same as* (6) *and*

$$\hat{Z}_1 = \frac{1}{2}\begin{pmatrix} I_k & -P\end{pmatrix} C \begin{pmatrix} I_k \\ -P^T\end{pmatrix}, \qquad \hat{Z}_2 = \frac{1}{2}\begin{pmatrix} I_k & P\end{pmatrix} C \begin{pmatrix} I_k \\ P^T\end{pmatrix}, \tag{18}$$

*if $n = 2k+1$, $D$ is the same as* (7) *and*

$$\hat{Z}_1 = \frac{1}{2}\begin{pmatrix} I_k & 0 & -P\end{pmatrix} C \begin{pmatrix} I_k \\ 0^T \\ -P^T\end{pmatrix}, \qquad \hat{Z}_2 = \frac{1}{2}\begin{pmatrix} 0^T & \sqrt{2} & 0^T \\ I_k & 0 & P\end{pmatrix} C \begin{pmatrix} 0 & I_k \\ \sqrt{2} & 0^T \\ 0 & P^T\end{pmatrix}. \tag{19}$$

**Proof.** It is easy to verify that $\mathbb{S}_{\mathbb{E}}$ is a closed convex set. Therefore there exists a unique solution to Problem II [6, p. 22]. According to (10) any $A \in \mathbb{S}_{\mathbb{E}}$ can be represented as

$$A = D \begin{pmatrix} G_1 & 0 \\ 0 & G_2 \end{pmatrix} D^T,$$

where

$$G_1 = X_1 \Lambda X_1^+ + Z_1(I_k - X_1 X_1^+), \tag{20}$$

$$G_2 = X_2 \Lambda X_2^+ + Z_2(I_{n-k} - X_2 X_2^+). \tag{21}$$

Let

$$D^T C D = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}. \tag{22}$$

Since

$$\begin{aligned} \|A - C\|^2 &= \left\| D \begin{pmatrix} G_1 & O \\ O & G_2 \end{pmatrix} D^T - C \right\|^2 \\ &= \|G_1 - C_{11}\|^2 + \|C_{12}\|^2 + \|C_{21}\|^2 + \|G_2 - C_{22}\|^2, \end{aligned}$$

then $\|A - C\| = \min$, where $A$ is taken over all $n \times n$ generalized centrosymmetric matrices, is equivalent to

$$\|G_1 - C_{11}\| = \min, \tag{23}$$

and

$$\|G_2 - C_{22}\| = \min. \tag{24}$$

Equations (23) and (24) are equivalent to

$$\begin{aligned} &\|X_1 \Lambda X_1^+ + Z_1\big(I_k - X_1 X_1^+\big) - C_{11}\| = \min, && \forall Z_1 \in \mathbb{R}^{k \times k}, \\ &\|X_2 \Lambda X_2^+ + Z_2\big(I_{n-k} - X_2 X_2^+\big) - C_{22}\| = \min, && \forall Z_2 \in \mathbb{R}^{(n-k)\times(n-k)}. \end{aligned} \tag{25}$$

Suppose a singular value decomposition of $X_1$ is

$$X_1 = U \begin{pmatrix} \Sigma & O \\ O & O \end{pmatrix} V^T = U_1 \Sigma V_1^T,$$

where $U = (U_1, U_2) \in \mathbb{O}^{k\times k}$, $U_1 \in \mathbb{R}^{k\times r_1}$, $r_1 = \text{rank}(X_1)$, $V = (V_1, V_2) \in \mathbb{O}^{k\times k}$, $V_1 \in \mathbb{R}^{k\times r_1}$. Then it follows from orthogonal invariance of the Frobenius norm that

$$\begin{aligned} & \|X_1\Lambda X_1^+ + Z_1(I_k - X_1X_1^+) - C_{11}\|^2 \\ = \ & \|Z_1U_2U_2^TU - (C_{11} - X_1\Lambda X_1^+)U\|^2 \\ = \ & \|(C_{11} - X_1\Lambda X_1^+)U_1\|^2 + \|Z_1U_2 - C_{11}U_2\|^2. \end{aligned}$$

Therefore, (25) holds if and only if

$$Z_1U_2 = C_{11}U_2. \tag{26}$$

$Z_1 = \hat{Z}_1 = C_{11}$ is the solution of (26). Substituting all such $Z_1$ into (20) the solution of (23) is $\hat{G}_1 = X_1\Lambda X_1^+ + \hat{Z}_1(I_k - X_1X_1^+)$. Similarly, the solution of (24) is $\hat{G}_2 = X_2\Lambda X_2^+ + \hat{Z}_2(I_{n-k} - X_2X_2^+)$, where $\hat{Z}_2 = C_{22}$. By the definition of $D_{2n}$ or $D_{2n+1}$ we have, for $n = 2k$

$$\begin{aligned} C_{11} &= \frac{1}{2}\begin{pmatrix} I_k & -P \end{pmatrix} C \begin{pmatrix} I_k \\ -P^T \end{pmatrix}, \\ C_{22} &= \frac{1}{2}\begin{pmatrix} I_k & P \end{pmatrix} C \begin{pmatrix} I_k \\ P^T \end{pmatrix}, \end{aligned} \tag{27}$$

for $n = 2k + 1$

$$\begin{aligned} C_{11} &= \frac{1}{2}\begin{pmatrix} I_k & 0 & -P \end{pmatrix} C \begin{pmatrix} I_k \\ 0^T \\ -P^T \end{pmatrix}, \\ C_{22} &= \frac{1}{2}\begin{pmatrix} 0^T & \sqrt{2} & 0^T \\ I_k & 0 & P \end{pmatrix} C \begin{pmatrix} 0 & I_k \\ \sqrt{2} & 0^T \\ 0 & P^T \end{pmatrix}. \end{aligned} \tag{28}$$

Thus the unique generalized centrosymmetric matrix solution of Problem II is (17). □

### 2.2 *A structure-preserving iteration algorithm and numerical examples*

The structure constraint is usually imposed due to the realizability of the underlying physical system. In Theorem 2 $\hat{A}$ is the modified solution. Though $\hat{A}$

satisfies the spectra constraints and is the best approximation of $C$, it does not preserve desirable structure such as banded, sparse etc. Next we will modify $\hat{A}$ such that the corrected model preserves sparse structure. If $\hat{A} = (\hat{a}_{ij})$ in (17) is not sparse, we modify $\hat{A}$. Let $\tilde{A} = (\tilde{a}_{ij})$ where

$$\tilde{a}_{ij} = \begin{cases} \hat{a}_{ij} & c_{ij} \neq 0, \\ 0 & c_{ij} = 0, \end{cases} \tag{29}$$

$\tilde{A}$ is a projection of $\hat{A}$ in some sense [1, 16]. But $\tilde{A}$ is not the solution of Problem II. We modify it again by Theorem 2. To get a better numerical solution of Problem II we propose a structure-preserving iteration algorithm of model updating as follows.

**Algorithm 1:**

1) Input $\epsilon$, $C \in \mathbb{R}^{n\times n}$, $X \in \mathbb{R}^{n\times m}$ and $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathbb{R}^{n\times n}$. Take $k = [\frac{n}{2}]$.
2) If $C$ and the exact $A$ have the same zero elements structure we take an initial matrix $\tilde{A}_0 = C$ . Otherwise, get $\tilde{A}_0$ from $C$ according to (29).
3) Compute $X_1$ and $X_2$ by $D^T X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$.
4) Compute $X_1^+$ and $X_2^+$ by SVD.
5) $C$ is replaced by $\tilde{A}_0$. Compute $\hat{Z}_1$ and $\hat{Z}_2$ by (18) and (19) respectively.
6) Compute $\hat{A}_1$ by (17).
7) Get the projection $\tilde{A}_1$ of $\hat{A}_1$.
8) If $\|\tilde{A}_1 - \tilde{A}_0\|/\|C\| < \epsilon$, goto 9); otherwise, $\tilde{A}_0$ is replaced by $\tilde{A}_1$, go to 5).
9) Output $\tilde{A}_1$.
10) Stop.

We will prove that the matrix sequence $\{\tilde{A}_k\}$ generated by the algorithm converge to the exact $A$ in the next section. We first investigate its numerical results.

Guided by the algorithm many numerical experiments were carried out, and all of them were performed using Matlab 7.1. Next we report two numerical examples to illustrate our theory.

**Example 1.** For simulation in a vibrating system with 8 degrees of freedom, we assume an exact stiffness matrix to be

$$K = \begin{pmatrix} 1000 & -890 & 0 & 0 & 0 & 0 & 0 & 0 \\ -900 & 1210 & -1350 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1320 & 800 & -880 & 0 & 0 & 0 & 0 \\ 0 & 0 & -910 & 960 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 960 & -910 & 0 & 0 \\ 0 & 0 & 0 & 0 & -880 & 800 & -1320 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1350 & 1210 & 900 \\ 0 & 0 & 0 & 0 & 0 & 0 & 890 & 1000 \end{pmatrix}.$$

It is a tridiagonal generalized centrosymmetric matrix. In Definition 1

$$P = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The min and max eigenvalue is, respectively, $\lambda_{\min} = -797.075893$, $\lambda_{\max} = 2797.123691$, and associated modes matrix is

$$X = \begin{pmatrix} 0.2888 & -0.3389 \\ 0.5830 & 0.6843 \\ 0.6743 & -0.5786 \\ 0.3492 & 0.2866 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

To illustrate our theory we choose the identified min and max eigenvalue, associated modes matrix to be in accordance with their exact values respectively.

**Step 1.** Input $\epsilon = 10^{-7}$, $X$, $\Lambda = \text{diag}(-797.075893, 2797.123691)$ and an initial estimate stiffness matrix $C$ is

$$C = \begin{pmatrix} 990 & -880 & 0 & 0 & 0 & 0 & 0 & 0 \\ -890 & 1200 & -1320 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1340 & 815 & -900 & 0 & 0 & 0 & 0 \\ 0 & 0 & -890 & 950 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 950 & -890 & 0 & 0 \\ 0 & 0 & 0 & 0 & -900 & 815 & -1340 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1320 & 1200 & 890 \\ 0 & 0 & 0 & 0 & 0 & 0 & 880 & 990 \end{pmatrix}.$$

**Step 2.** $\tilde{K}_0 = C$. Its min and max eigenvalue is respectively

$$\lambda'_{\min} = -788.400516, \quad \lambda'_{\max} = 2787.197119.$$

There are big errors between the initial eigenvalues and the exact ones. Therefore modifying $C$ is necessary.

**Step 3.**

$$X_1 = \begin{pmatrix} 0.2042 & -0.2396 \\ 0.4123 & 0.4839 \\ 0.4768 & -0.4091 \\ 0.2469 & 0.2026 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0.2042 & -0.2396 \\ 0.4123 & 0.4839 \\ 0.4768 & -0.4091 \\ 0.2469 & 0.2026 \end{pmatrix}$$

**Step 4.**

$$X_1^{+} = \begin{pmatrix} 0.4137 & 0.8139 & 0.9628 & 0.4895 \\ -0.4838 & 0.9587 & -0.8289 & 0.3999 \end{pmatrix},$$

$$X_2^{+} = \begin{pmatrix} 0.4137 & 0.8139 & 0.9628 & 0.4895 \\ -0.4838 & 0.9587 & -0.8289 & 0.3999 \end{pmatrix}.$$

**Step 5.**

$$\hat{Z}_1 = 10^3 \begin{pmatrix} 0.9900 & -0.8800 & 0 & 0 \\ -0.8900 & 1.2000 & -1.3200 & 0 \\ 0 & -1.3400 & 0.8150 & -0.9000 \\ 0 & 0 & -0.8900 & 0.9500 \end{pmatrix},$$

$$\hat{Z}_2 = 10^3 \begin{pmatrix} 0.9900 & -0.8800 & 0 & 0 \\ -0.8900 & 1.2000 & -1.3200 & 0 \\ 0 & -1.3400 & 0.8150 & -0.9000 \\ 0 & 0 & -0.8900 & 0.9500 \end{pmatrix}.$$

**Step 6.**

$$\hat{K}_1 = 10^3 \begin{pmatrix} 0.9926 & -0.8886 & 0.0040 & -0.0039 & 0 & 0 & 0 & 0 \\ -0.9045 & 1.2088 & -1.3479 & 0.0018 & 0 & 0 & 0 & 0 \\ -0.0071 & -1.3160 & 0.8043 & -0.8891 & 0 & 0 & 0 & 0 \\ -0.0079 & 0.0040 & -0.9053 & 0.9506 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.9506 & -0.9053 & 0.0040 & 0.0079 \\ 0 & 0 & 0 & 0 & -0.8891 & 0.8043 & -1.3160 & 0.0071 \\ 0 & 0 & 0 & 0 & 0.0018 & -1.3479 & 1.2088 & 0.9045 \\ 0 & 0 & 0 & 0 & 0.0039 & -0.0040 & 0.8886 & 0.9926 \end{pmatrix}.$$

**Step 7.** $\hat{K}_1$ is closer to $K$ than $C$ ($\|\hat{K}_1 - K\| = 30.7597$, $\|C - K\| = 75.1665$). Though $\hat{K}_1$ satisfies the spectra constraints and is the best approximation of $C$, it is not the structured matrix just as $K$ and $C$. We get $\tilde{K}_1$ from $\hat{K}_1$ according to (29). Thus

$$\tilde{K}_1 = 10^3 \begin{pmatrix} 0.9926 & -0.8886 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.9045 & 1.2088 & -1.3479 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1.3160 & 0.8043 & -0.8891 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.9053 & 0.9506 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.9506 & -0.9053 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.8891 & 0.8043 & -1.3160 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1.3479 & 1.2088 & 0.9045 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.8886 & 0.9926 \end{pmatrix}.$$

**Step 8.** $\|\tilde{A}_1 - \tilde{A}_0\|/\|C\| = 0.01430621168271 > \epsilon$. $\tilde{K}_0$ is replaced by $\tilde{K}_1$. We repeat above steps. We can find desirable $\tilde{K}_m$ that is close to $K$ after finite iterations. Here after the 47-th modified solution is

$$\tilde{K}_{47}=10^3\begin{pmatrix} 1.0000 & -0.8900 & 0 & 0 & 0 & 0 & 0 & 0\\ -0.9045 & 1.2097 & -1.3478 & 0 & 0 & 0 & 0 & 0\\ 0 & -1.3155 & 0.8008 & -0.8891 & 0 & 0 & 0 & 0\\ 0 & 0 & -0.9100 & 0.9600 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0.9600 & -0.9100 & 0 & 0\\ 0 & 0 & 0 & 0 & -0.8891 & 0.8008 & -1.3155 & 0\\ 0 & 0 & 0 & 0 & 0 & -1.3478 & 1.2097 & 0.9045\\ 0 & 0 & 0 & 0 & 0 & 0 & 0.8900 & 1.0000 \end{pmatrix}.$$

It is nearer to $K$ than $\hat{K}_1$ ($\|\tilde{K}_{47} - K\| = 16.1148$). After 47 iterations the modified model $\tilde{K}_{47}$ and its min and max eigenvalues are close to their exact values. The min and max eigenvalues of updated model are in the following table.

| eigenvalue | $\lambda_{\min}^{\text{iter}}$ | $\lambda_{\max}^{\text{iter}}$ | $\frac{\lvert\lambda_{\min}-\lambda_{\min}^{\text{iter}}\rvert}{\lvert\lambda_{\min}\rvert}$ | $\frac{\lvert\lambda_{\max}-\lambda_{\max}^{\text{iter}}\rvert}{\lvert\lambda_{\max}\rvert}$ | $\frac{\|\tilde{K}_{\text{iter}}-K\|}{\|K\|}$ |
|---|---|---|---|---|---|
| initial | $-788.400516$ | 2787.197119 | 0.010884 | 0.003549 | 0.01617685 |
| iter=5 | $-797.920091$ | 2796.270209 | 0.001059 | $3.051287e-04$ | 0.00388285 |
| iter=20 | $-797.109866$ | 2797.089900 | $4.262220e-05$ | $1.208095e-05$ | 0.00346881 |
| iter=30 | $-797.079833$ | 2797.119772 | $4.943275e-06$ | $1.401223e-06$ | 0.00346813 |
| iter=40 | $-797.076350$ | 2797.123237 | $5.733447e-07$ | $1.625139e-07$ | 0.00346812 |
| iter=47 | $-797.075994$ | 2797.123591 | $1.269107e-07$ | $3.597137e-08$ | 0.00346812 |
| exact | $-797.075893$ | 2797.123691 | | | |

Table 1 – The eigenvalues of updated model.

where $iter$ is iteration number.

Next we see an example in the case of large scale matrices.

**Example 2.** Denote by magic(k) the Magic matrix of order $k$. For example,

$$magic(4) = \begin{pmatrix} 16 & 2 & 3 & 13\\ 5 & 11 & 10 & 8\\ 9 & 7 & 6 & 12\\ 4 & 14 & 15 & 1 \end{pmatrix}.$$

Assume $B = magic(k)$ with $k = 400$, $n = 800$. The elements of matrix $E$ is defined by

$$\begin{cases} E(i,i) = 0.001 * B(i,i), & i = 1,\ldots,k, \\ E(j,j+1) = 0.001 * B(j,j+1), & j = 1,\ldots,k-1, \\ E(j+1,j) = 0.001 * B(j+1,j), & j = 1,\ldots,k-1, \\ E(i,j) = 0, & else \end{cases}$$

In Definition 1, $P = (e_k,\ e_{k-1},\ \ldots,\ -e_1)$ and $e_i$ is the $i$th column of identity matrix $I_k$. If

$$A = \begin{pmatrix} E & 0 \\ 0 & P^T E P \end{pmatrix},$$

then $A$ is a triangular generalized centrosymmetric matrix. The condition number of $A$ is $\text{cond}(A) = 3928.726516440267$. Assume $\lambda_i, x_i$ are eigenpairs of $A$. The eigenvalues of the minimum and maximum modulus are $\lambda_{\min} = 0.124120226980$ and $\lambda_{\max} = 318.959325051064$, respectively. To illustrate our theory we choose the elements of $\Lambda$ and the columns of $X$ in Problem I and II to be a part of eigenvalues and associated eigenvectors of $A$. Let the elements of matrix $R$ be

$$\begin{cases} R(j,j+1) = -30, & j = 1,\ldots,n-1, \\ R(j+1,j) = -20, & j = 1,\ldots,n-1, \\ R(i,j) = 0, else. \end{cases}$$

An initial estimate matrix $C = A + R$ is a triangular matrix. Its eigenvalues of the minimal and the maximum modulus are respectively

$$\lambda'_{\min} = -0.441127715889, \quad \lambda'_{\max} = 311.2137757218917.$$

There are big errors between the initial eigenvalues $\lambda'_{\max}$, $\lambda'_{\min}$ and the exact eigenvalues $\lambda_{\max}$, $\lambda_{\min}$. Therefore modifying $C$ is necessary. Suppose $X_1$ consists of the eigenvectors associated with $\lambda_{\max}$ and $\lambda_{\min}$;

$$X_2 = (x_{56}, x_{57}, \ldots, x_{599}) \quad \text{and} \quad \Lambda_2 = \text{diag}(\lambda_{56}, \lambda_{57}, \ldots, \lambda_{599});$$

$$X_3 = (x_1, x_{57}, \ldots, x_{600}) \quad \text{and} \quad \Lambda_3 = \text{diag}(\lambda_1, \lambda_{57}, \ldots, \lambda_{600});$$

$$X_4 = (x_1, x_2, \ldots, x_{700}) \quad \text{and} \quad \Lambda_4 = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{700}).$$

The diagonal elements of $\Lambda_1$, $\Lambda_2$, $\Lambda_3$ and $\Lambda_4$ contain $\lambda_{\min}$ and $\lambda_{\max}$. By the algorithm we obtain $\tilde{A}_{\text{iter}}$. It approaches the exact $A$. It is triangular generalized centrosymmetric in structure and satisfies the spectra constraints. The updated eigenvalues of the minimum and maximum modulus are provided by the following table.

| X | $\lambda_{\max}^{\text{iter}}$ | $\lambda_{\min}^{\text{iter}}$ | $\frac{\lvert\lambda_{\max}-\lambda_{\max}^{\text{iter}}\rvert}{\lvert\lambda_{\max}\rvert}$ | $\frac{\lvert\lambda_{\min}-\lambda_{\min}^{\text{iter}}\rvert}{\lvert\lambda_{\min}\rvert}$ | $\frac{\|\tilde{A}_{\text{iter}}-A\|}{\|A\|}$ | iter |
|---|---|---|---|---|---|---|
| initial | 311.21378 | −0.44113 | 0.02428 | 4.55404 | 0.22535 | 0 |
| X1 | 326.12921 | 0.07922 | 0.02248 | 1.63825 | 0.20280 | 200 |
| X2 | 317.71539 | 0.09186 | 0.00390 | 0.25990 | 0.00400 | 150 |
| X3 | 318.95933 | 0.12412 | $4.99003 \cdot 10^{-15}$ | $2.67783 \cdot 10^{-13}$ | $5.55517 \cdot 10^{-15}$ | 2 |
| X4 | 318.95933 | 0.12412 | $4.99003 \cdot 10^{-15}$ | $2.67783 \cdot 10^{-13}$ | $5.55517 \cdot 10^{-15}$ | 2 |

Table 2 – The updated values of eigenvalues $\lambda_{\max}$ and $\lambda_{\min}$.

The eigenvalues $\lambda_{\min}^{\text{iter}}$, $\lambda_{\max}^{\text{iter}}$ approach their exact values and their relative errors decrease as the size of $X$ increases. In addition, because the pseudo inverse is computed by stable singular value decomposition and from (51) in next section it is easy to see that our algorithm is stable. Numerical examples show that the method is reliable and effective.

## 3 A perturbation and convergence

In this section, we will study a perturbation of the modified solution $\hat{A}$ in Theorem 2 and the convergence of the iteration method in Section 2.

**Theorem 3.** *Let $\tilde{X}$, $\tilde{\Lambda}$, $\tilde{X}_1$, $\tilde{X}_2$, and $\tilde{C}$ be perturbed counterparts of $X$, $\Lambda$, $X_1$, $X_2$ and $C$ in Theorem 2, respectively. And $\hat{A}$ and $\tilde{A}$ are the solutions of corresponding Problem II. Then*

$$\begin{aligned} \|\tilde{A} - \hat{A}\| \le a\|\tilde{X}_1 - X_1\| + \|X_1\|_2\|\tilde{X}_1^{+}\|_2\|\tilde{\Lambda} - \Lambda\| + \|\tilde{Z}_1 - \hat{Z}_1\| \\ + b\|\tilde{X}_2 - X_2\| + \|X_2\|_2\|\tilde{X}_2^{+}\|_2\|\tilde{\Lambda} - \Lambda\| + \|\tilde{Z}_2 - \hat{Z}_2\|, \end{aligned} \tag{30}$$

*where*

$$\begin{aligned} a &= \left(\|\tilde{\Lambda} - \Lambda\|_2 + \|\Lambda\|_2\right)\|\tilde{X}_1^{+}\|_2 + \sqrt{2}\|X_1\|_2\|\Lambda\|_2 \max\left\{\|X_1^{+}\|_2^2, \|\tilde{X}_1^{+}\|_2^2\right\} \\ &\quad + \sqrt{\|X_1^{+}\|_2^2 + \|\tilde{X}_1^{+}\|_2^2}\,\|\hat{Z}_1\|_2, \end{aligned} \tag{31}$$

$$\begin{aligned} b &= \left(\|\tilde{\Lambda} - \Lambda\|_2 + \|\Lambda\|_2\right)\|\tilde{X}_2^+\|_2 + \sqrt{2}\|X_2\|_2\|\Lambda\|_2 \max\left\{\|X_2^+\|_2^2, \|\tilde{X}_2^+\|_2^2\right\} \\ &\quad + \sqrt{\|X_2^+\|_2^2 + \|\tilde{X}_2^+\|_2^2}\,\|\hat{Z}_2\|_2. \end{aligned} \tag{32}$$

**Proof.** Because

$$\hat{A} = D\begin{pmatrix} X_1\Lambda X_1^+ + \hat{Z}_1(I_k - X_1X_1^+) & 0 \\ 0 & X_2\Lambda X_2^+ + \hat{Z}_2(I_{n-k} - X_2X_2^+) \end{pmatrix} D^T \tag{33}$$

and

$$\tilde{A} = D\begin{pmatrix} \tilde{X}_1\tilde{\Lambda}\tilde{X}_1^+ + \tilde{Z}_1(I_k - \tilde{X}_1\tilde{X}_1^+) & 0 \\ 0 & \tilde{X}_2\tilde{\Lambda}\tilde{X}_2^+ + \tilde{Z}_2(I_{n-k} - \tilde{X}_2\tilde{X}_2^+) \end{pmatrix} D^T, \tag{34}$$

where for $n = 2k$

$$\hat{Z}_1 = \frac{1}{2}\begin{pmatrix} I_k & -P \end{pmatrix} C \begin{pmatrix} I_k \\ -P^T \end{pmatrix}, \quad \hat{Z}_2 = \frac{1}{2}\begin{pmatrix} I_k & P \end{pmatrix} C \begin{pmatrix} I_k \\ P^T \end{pmatrix}, \tag{35}$$

and

$$\tilde{Z}_1 = \frac{1}{2}\begin{pmatrix} I_k & -P \end{pmatrix} \tilde{C} \begin{pmatrix} I_k \\ -P^T \end{pmatrix}, \quad \tilde{Z}_2 = \frac{1}{2}\begin{pmatrix} I_k & P \end{pmatrix} \tilde{C} \begin{pmatrix} I_k \\ P^T \end{pmatrix}, \tag{36}$$

for $n = 2k + 1$

$$\begin{aligned} \hat{Z}_1 &= \frac{1}{2}\begin{pmatrix} I_k & 0 & -P \end{pmatrix} C \begin{pmatrix} I_k \\ 0^T \\ -P^T \end{pmatrix}, \\ \hat{Z}_2 &= \frac{1}{2}\begin{pmatrix} 0^T & \sqrt{2} & 0^T \\ I_k & 0 & P \end{pmatrix} C \begin{pmatrix} 0 & I_k \\ \sqrt{2} & 0^T \\ 0 & P^T \end{pmatrix}, \end{aligned} \tag{37}$$

and

$$\begin{aligned} \tilde{Z}_1 &= \frac{1}{2}\begin{pmatrix} I_k & 0 & -P \end{pmatrix} \tilde{C} \begin{pmatrix} I_k \\ 0^T \\ -P^T \end{pmatrix}, \\ \tilde{Z}_2 &= \frac{1}{2}\begin{pmatrix} 0^T & \sqrt{2} & 0^T \\ I_k & 0 & P \end{pmatrix} \tilde{C} \begin{pmatrix} 0 & I_k \\ \sqrt{2} & 0^T \\ 0 & P^T \end{pmatrix}, \end{aligned} \tag{38}$$

we have

$$\|\tilde{A}-\hat{A}\|^2 = \|\tilde{X}_1\tilde{\Lambda}\tilde{X}_1^+ - X_1\Lambda X_1^+ + \tilde{Z}_1(I_k - P_{\tilde{X}_1}) - \hat{Z}_1(I_k - P_{X_1})\|^2 \\ + \|\tilde{X}_2\tilde{\Lambda}\tilde{X}_2^+ - X_2\Lambda X_2^+ + \tilde{Z}_2(I_{n-k} - P_{\tilde{X}_2}) - \hat{Z}_2(I_{n-k} - P_{X_2})\|^2.$$

It follows that

$$\begin{aligned}\|\tilde{A}-\hat{A}\| &\le \|\tilde{\Lambda}-\Lambda\|_2 + \|\Lambda\|_2)\|\tilde{X}_1^+\|_2\|\tilde{X}_1 - X_1\| + \|X_1\|_2\|\tilde{X}_1^+\|_2\|\tilde{\Lambda}-\Lambda\| \\ &+ \|X_1\|_2\|\Lambda\|_2\|\tilde{X}_1^+ - X_1^+\| + \|\tilde{Z}_1 - \hat{Z}_1\| + \|\hat{Z}_1\|_2\|P_{X_1} - P_{\tilde{X}_1}\| \\ &+ (\|\tilde{\Lambda}-\Lambda\|_2 + \|\Lambda\|_2)\|\tilde{X}_2^+\|_2\|\tilde{X}_2 - X_2\| \\ &+ \|X_2\|_2\|\tilde{X}_2^+\|_2\|\tilde{\Lambda}-\Lambda\| + \|X_2\|_2\|\Lambda\|_2\|\tilde{X}_2^+ - X_2^+\| \\ &+ \|(\tilde{Z}_2 - \hat{Z}_2)\| + \|\hat{Z}_2\|_2\|P_{X_2} - P_{\tilde{X}_2}\|.\end{aligned} \tag{39}$$

By [18],

$$\begin{aligned}\|\tilde{X}_1^+ - X_1^+\| &\le \sqrt{2}\max\left\{\|X_1^+\|_2^2, \|\tilde{X}_1^+\|_2^2\right\}\|\tilde{X}_1 - X_1\|, \\ \|\tilde{X}_2^+ - X_2^+\| &\le \sqrt{2}\max\left\{\|X_2^+\|_2^2, \|\tilde{X}_2^+\|_2^2\right\}\|\tilde{X}_2 - X_2\|.\end{aligned} \tag{40}$$

By [17],

$$\begin{aligned}\|P_{\tilde{X}_1} - P_{X_1}\| &\le \sqrt{\|X_1^+\|_2^2 + \|\tilde{X}_1^+\|_2^2}\|\tilde{X}_1 - X_1\|, \\ \|P_{\tilde{X}_2} - P_{X_2}\| &\le \sqrt{\|X_2^+\|_2^2 + \|\tilde{X}_2^+\|_2^2}\|\tilde{X}_2 - X_2\|.\end{aligned} \tag{41}$$

Substituting (40) and (41) into (39) we obtain (30)-(32). □

Even though $\|\tilde{X}_1 - X_1\|_2$ and $\|\tilde{X}_2 - X_2\|_2$ are small, $\|\tilde{X}_1^+ - X_1^+\|_2$ and $\|\tilde{X}_2^+ - X_2^+\|_2$ may be very large. It is because $\|\tilde{X}_1^+\|_2$ and $\|\tilde{X}_2^+\|_2$ may infinitely increase as $\|\tilde{X}_1 - X_1\|_2$ and $\|\tilde{X}_2 - X_2\|_2$ approach zero respectively. Therefore we provide the following conditions that $\tilde{X}_1^+$ and $\tilde{X}_2^+$ continuously change.

**Theorem 4.** *The notations are the same as those in Theorem* 3 *and assume*

$$\begin{aligned}&\operatorname{rank}(\tilde{X}_1) = \operatorname{rank}(X_1),\ \operatorname{rank}(\tilde{X}_2) = \operatorname{rank}(X_2), \\ &\|X_1^+\|_2\|\tilde{X}_1 - X_1\|_2 < 1,\ \|X_2^+\|_2\|\tilde{X}_2 - X_2\|_2 < 1,\end{aligned} \tag{42}$$

$$\begin{aligned}\kappa(X_1) &= \|X_1\|_2\|X_1^+\|_2,\ \gamma(X_1,\tilde{X}_1) = 1 - \|X_1^+\|_2\|\tilde{X}_1 - X_1\|_2, \\ \kappa(X_2) &= \|X_2\|_2\|X_2^+\|_2,\ \gamma(X_2,\tilde{X}_2) = 1 - \|X_2^+\|_2\|\tilde{X}_2 - X_2\|_2.\end{aligned} \tag{43}$$

*Then*

$$\begin{aligned} \|\tilde{A}-\hat{A}\| \;\leq\; & a\|X_1^{+}\|_2\|\tilde{X}_1-X_1\| + \frac{\kappa(X_1)}{\gamma(X_1,\tilde{X}_1)}\|\tilde{\Lambda}-\Lambda\| + \|\tilde{Z}_1-\hat{Z}_1\| \\ & + b\|X_2^{+}\|_2\|\tilde{X}_2-X_2\| + \frac{\kappa(X_2)}{\gamma(X_2,\tilde{X}_2)}\|\tilde{\Lambda}-\Lambda\| + \|\tilde{Z}_2-\hat{Z}_2\|, \end{aligned} \tag{44}$$

*where*

$$\begin{aligned} a &= \frac{\|\tilde{\Lambda}-\Lambda\|_2 + \|\Lambda\|_2 + \mu_1\kappa(X_1)\|\Lambda\|_2}{\gamma(X_1,\tilde{X}_1)} + \sqrt{2}\|\hat{Z}_1\|_2, \\ b &= \frac{\|\tilde{\Lambda}-\Lambda\|_2 + \|\Lambda\|_2 + \mu_2\kappa(X_2)\|\Lambda\|_2}{\gamma(X_2,\tilde{X}_2)} + \sqrt{2}\|\hat{Z}_2\|_2 \end{aligned} \tag{45}$$

*and*

$$\begin{aligned} \mu_1 &= \begin{cases} \sqrt{2}, & \operatorname{rank}(X_1) < \min(k,m), \\ 1, & \operatorname{rank}(X_1) = \min\limits_{k\neq m}(k,m), \end{cases} \\ \mu_2 &= \begin{cases} \sqrt{2}, & \operatorname{rank}(X_2) < \min(n-k,m), \\ 1, & \operatorname{rank}(X_2) = \min\limits_{n-k\neq m}(n-k,m). \end{cases} \end{aligned} \tag{46}$$

**Proof.** If the conditions (42) are satisfied, we have [17, 18]

$$\|\tilde{X}_1^{+}\|_2 \leq \|X_1^{+}\|_2/\gamma(X_1,\tilde{X}_1), \quad \|\tilde{X}_2^{+}\|_2 \leq \|X_2^{+}\|_2/\gamma(X_2,\tilde{X}_2) \tag{47}$$

and

$$\begin{aligned} \|\tilde{X}_1^{+}-X_1^{+}\| &\leq \mu_1\|X_1^{+}\|_2^2\|\tilde{X}_1-X_1\|/\gamma(X_1,\tilde{X}_1), \\ \|\tilde{X}_2^{+}-X_2^{+}\| &\leq \mu_2\|X_2^{+}\|_2^2\|\tilde{X}_2-X_2\|/\gamma(X_2,\tilde{X}_2), \end{aligned} \tag{48}$$

$$\begin{aligned} \|P_{\tilde{X}_1}-P_{X_1}\| &\leq \sqrt{2}\min\{\|X_1^{+}\|_2, \|\tilde{X}_1^{+}\|_2\}\|\tilde{X}_1-X_1\|, \\ \|P_{\tilde{X}_2}-P_{X_2}\| &\leq \sqrt{2}\min\{\|X_2^{+}\|_2, \|\tilde{X}_2^{+}\|_2\}\|\tilde{X}_2-X_2\|, \end{aligned} \tag{49}$$

where

$$\mu_1 = \begin{cases} \sqrt{2}, & \operatorname{rank}(X_1) < \min(k,m), \\ 1, & \operatorname{rank}(X_1) = \min\limits_{k\neq m}(k,m), \end{cases}$$

$$\mu_2 = \begin{cases} \sqrt{2}, & \operatorname{rank}(X_2) < \min(n-k,m), \\ 1, & \operatorname{rank}(X_2) = \min\limits_{n-k\neq m}(n-k,m). \end{cases}$$

Substituting (47)-(49) into (39) we obtain (44)-(45). □

In Theorem 3 if $\tilde{X} = X$, $\tilde{\Lambda} = \Lambda$, then

$$\|\tilde{A} - \hat{A}\| \leq \alpha \|\tilde{C} - C\|, \tag{50}$$

where $\alpha < 1$.

In fact,

$$\begin{aligned} \|\tilde{A} - \hat{A}\| &= \left\| \begin{pmatrix} (\tilde{Z}_1 - \hat{Z}_1)(I_k - P_{X_1}) & 0 \\ 0 & (\tilde{Z}_2 - \hat{Z}_2)(I_{n-k} - P_{X_2}) \end{pmatrix} \right\| \\ &\leq \left\| \begin{pmatrix} I_k - P_{X_1} & 0 \\ 0 & I_{n-k} - P_{X_2} \end{pmatrix} \right\|_2 \left\| \begin{pmatrix} \tilde{Z}_1 - \hat{Z}_1 & 0 \\ 0 & \tilde{Z}_2 - \hat{Z}_2 \end{pmatrix} \right\|. \end{aligned}$$

And $\hat{Z}_1 = C_{11}$, $\tilde{Z}_1 = \tilde{C}_{11}$, $\hat{Z}_2 = C_{22}$, $\tilde{Z}_2 = \tilde{C}_{22}$, where

$$D^T C D = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}, \quad D^T \tilde{C} D = \begin{pmatrix} \tilde{C}_{11} & \tilde{C}_{12} \\ \tilde{C}_{21} & \tilde{C}_{22} \end{pmatrix}.$$

Let

$$\alpha = \left\| \begin{pmatrix} I_k - P_{X_1} & 0 \\ 0 & I_{n-k} - P_{X_2} \end{pmatrix} \right\|_2. \tag{51}$$

Then $\alpha < 1$ for $X \neq 0$. Thus

$$\begin{aligned} \|\tilde{A} - \hat{A}\| &\leq \alpha \left\| \begin{pmatrix} \tilde{Z}_1 - \hat{Z}_1 & 0 \\ 0 & \tilde{Z}_2 - \hat{Z}_2 \end{pmatrix} \right\| \\ &\leq \alpha \left\| \begin{pmatrix} \tilde{C}_{11} - C_{11} & \tilde{C}_{12} - C_{12} \\ \tilde{C}_{21} - C_{21} & \tilde{C}_{22} - C_{22} \end{pmatrix} \right\| \\ &= \alpha \|D^T(\tilde{C} - C)D\| = \alpha \|\tilde{C} - C\|, \end{aligned}$$

**Theorem 5.** *The matrices sequences $\{\tilde{A}_m\}$ and $\{\hat{A}_m\}$ generated by Algorithm 1 is convergent to its exact matrix $A$.*

**Proof.** $C$ and $\tilde{C}$ are replaced by $\tilde{A}_0$ and $\tilde{A}_1$ in (50) respectively. It follows from (50) that

$$\|\hat{A}_2 - \hat{A}_1\| \leq \alpha \|\tilde{A}_1 - \tilde{A}_0\|.$$

By (29) and Algorithm 1, the nonzero elements of $\tilde{A}_2 - \tilde{A}_1$ are the same as those of $\hat{A}_2 - \hat{A}_1$ and the number of nonzero elements of $\tilde{A}_2 - \tilde{A}_1$ is fewer than that ones of $\hat{A}_2 - \hat{A}_1$. It means that

$$\|\tilde{A}_2 - \tilde{A}_1\| \leq \|\hat{A}_2 - \hat{A}_1\| \leq \alpha \|\tilde{A}_1 - \tilde{A}_0\|.$$

If $C$ is taken as the exact $A$ then $\hat{A} = A$. If $C$ and $\tilde{C}$ are placed by $A$ and $\tilde{A}_{m-1}$ in (50) respectively. We have

$$\|\hat{A}_m - A\| \leq \alpha \|\tilde{A}_{m-1} - A\|.$$

Analogously, the nonzero elements of $\tilde{A}_m - A$ are the same as those of $\hat{A}_m - A$ and the number of nonzero elements of $\tilde{A}_m - A$ is fewer than that ones of $\hat{A}_m - A$. Thus

$$\|\tilde{A}_m - A\| \leq \|\hat{A}_m - A\| \leq \alpha \|\tilde{A}_{m-1} - A\| \leq \cdots \leq \alpha^m \|\tilde{A}_0 - A\|.$$

Thus both $\{\tilde{A}_m\}$and $\{\hat{A}_m\}$ converge to $A$ because of $\alpha < 1$.

## 4 Conclusion

In this paper, we have investigated some theories and a numerical method on modifying a generalized centrosymmetric model. These include the structure-preserving algorithm for solving the model updating problem based on matrix approximation with least squares spectra constrains and perturbation analysis of the modified solution. We can draw the following items.

1. Perturbation theory of the modified solution is given.

2. The algorithm is suitable for both sparse and dense matrix $C$. In particular, if all elements of $C$ are not zero, the iteration number is "iter=1".

3. Convergence speed depends on $\alpha$. But $\alpha$ is determined by (51). If the ranks of $X_1$ and $X_2$ are nearer to $k$ and $(n-k)$ respectively, generally $X$ is a matrix of full column rank and $\alpha$ is close to 0. The iteration number and relative errors of the modified solution decrease as the size of $X$ increases. Thus the modified solutions sequence more quickly approaches the true model if more eigenvalues and modes are provided.

4. In addition, because the pseudo inverse is computed by stable singular value decomposition the algorithm is stable.

5. In this paper we not only give theory but also provide a structure-preserving iterative algorithm on updating a generalized centrosymmetric model, based on the theory of inverse eigenpair problem. The conclusions are correct and the method is very reliable and effective.

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