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A sharp observability inequality for Kirchhoff plate systems with potentials*

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Abstract. In this paper, we derive a sharp observability inequality for Kirchhoff plate equations with lower order terms. More precisely, for any $T > 0$ and suitable boundary observation domains (satisfying the geometric conditions that the multiplier method imposes), we prove an estimate with an explicit observability constant for Kirchhoff plate systems with an arbitrary finite number of components and in any space dimension with lower order bounded potentials.

Mathematical subject classification: Primary: 93B07; Secondary: 93B05, 35B37.

Key words: Kirchhoff plate system, observability constant, Carleman inequalities, potential, Meshkov's construction.

1 Introduction

Let $n \geq 1$ and $N \geq 1$ be two integers. Let Ω be a bounded domain in $\mathbb{R}^n$ with C^4 boundary Γ, Γ_0 be a nonempty open subset of Γ, and $T > 0$ be given and sufficiently large. Put $Q \stackrel{\triangle}{=} (0, T) \times \Omega$, $\Sigma \stackrel{\triangle}{=} (0, T) \times \Gamma$ and $\Sigma_0 \stackrel{\triangle}{=} (0, T) \times \Gamma_0$. For simplicity, we will use the notation $y_i = \frac{\partial y}{\partial x_i}$, where x_i is the i-th coordinate

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of a generic point $x = (x_1, \cdots, x_n)$ in $\mathbb{R}^n$. Throughout this paper, we will use $C = C(T, \Omega, \Gamma_0)$ and $C^* = C^*(\Omega, \Gamma_0)$ to denote generic positive constants depending on their arguments which may vary from line to line.

Set

$$Y \stackrel{\triangle}{=} \left\{ y \in H^3(\Omega) \Big| \; y|_\Gamma = \Delta y|_\Gamma = 0 \right\}.$$

We consider the following $\mathbb{R}^N$-valued plate system with a potential $a \in L^\infty(0, T; L^p(\Omega; \mathbb{R}^{N\times N}))$ for some $p \in [n/3, \infty]$:

$$\begin{cases} y_{tt} + \Delta^2 y - \Delta y_{tt} + ay = 0 & \text{in } Q, \\ y = \Delta y = 0 & \text{on } \Sigma, \\ y(0) = y^0, \quad y_t(0) = y^1 & \text{in } \Omega, \end{cases} \tag{1}$$

where $y = (y_1, \cdots, y_N)^\top$, and the initial datum (y^0, y^1) is supposed to belong to $\mathcal{H} \stackrel{\triangle}{=} \left\{ \varphi \in H^3(\Omega) \big| \; \varphi|_\Gamma = \Delta\varphi|_\Gamma = 0 \right\}^N \times \left(H^2(\Omega) \bigcap H_0^1(\Omega) \right)^N$, the state space of system (1). It is easy to show that system (1) admits one and only one weak solution $y \in C([0, T]; \mathcal{H})$.

In what follows, we shall denote by $|\cdot|$, $||\cdot||_p$ and $|||\cdot|||_p$ the (canonical) norms on $\mathbb{R}^N$, $L^\infty(0, T; L^p(\Omega; \mathbb{R}^{N\times N}))$ and $L^\infty(0, T; W^{1,p}(\Omega; \mathbb{R}^{N\times N}))$, respectively.

We shall study the observability constant $K(a)$ of system (1), defined as the smallest (possibly infinite) constant such that the following observability estimate for system (1) holds:

$$\begin{aligned} &||(y^0, y^1)||_{\mathcal{H}}^2 \stackrel{\triangle}{=} ||y^0||^2_{(H^3(\Omega))^N} + ||y^1||^2_{(H^2(\Omega))^N} \\ &\le K(a) \int_{\Sigma_0} \left(\left| \frac{\partial y}{\partial \nu} \right|^2 + \left| \frac{\partial y_t}{\partial \nu} \right|^2 + \left| \frac{\partial \Delta y}{\partial \nu} \right|^2 \right) dx dt, \qquad \forall \, (y^0, y^1) \in \mathcal{H}. \end{aligned} \tag{2}$$

This inequality, the so-called *observability inequality*, allows estimating the total energy of solutions in terms of the energy localized in the observation subdomain Γ_0. It is relevant for control problems. In particular, in this linear setting, this (observability) inequality is equivalent to the so-called exact controllability property, i.e., that of driving solutions to rest by means of control forces localized in Σ_0 (see [6, 11]). This type of inequality, with explicit estimates on the observability constant, is also relevant for the control of semilinear problems ([10]). Similar inequalities are also useful for solving a variety of Inverse Problems ([9]). We remark that, as for the wave equations, (2) holds for the Kirchhoff plate only if (Ω, Γ_0, T) satisfies suitable conditions, i. e. Γ_0 needs to satisfy certain geometric conditions and T needs to be large enough.

Obviously the observability constant $K(a)$ in (2) not only depends on the potential a, but also on the domains Ω and Γ_0 and on the time T. The main purpose of this paper is to analyze only its explicit and sharp dependence on the potential a.

The main tools to derive the explicit observability estimates are the so-called *Carleman inequalities*. Here we have chosen to work in the space $\mathcal{H}$ in which Carleman inequalities can be applied more naturally. But some other choices of the state space are possible. For example, one may consider similar problems in state spaces of the form $(H_0^1(\Omega))^N \times (L^2(\Omega))^N$ or $(H^2(\Omega) \cap H_0^1(\Omega))^N \times (H_0^1(\Omega))^N$ where the Kirchhoff plate system is also well posed. But the corresponding analysis on the observability constants, in turn, is technically more involved.

One of the key points to derive inequality (2) for system (1) is the possibility of decomposing the Kirchhoff plate operator $\partial_t^2 + \Delta^2 - \partial_t^2\Delta$ as follows:

$$\partial_t^2 + \Delta^2 - \partial_t^2\Delta = (\partial_{tt} - \Delta)(I - \Delta) + \Delta, \tag{3}$$

where I is the identity operator. Actually, we set

$$z = y - \Delta y, \tag{4}$$

where y is the solution of (1). By the first equation of (1) and noting (3), it follows that

$$-ay = y_{tt} + \Delta^2 y - \Delta y_{tt} = (\partial_{tt} - \Delta)(y - \Delta y) + \Delta y = z_{tt} - \Delta z + y - z.$$

Therefore the Kirchhoff plate system (1) can be written equivalently as the following coupled elliptic-wave system

$$\begin{cases} \Delta y + z - y = 0 & \text{in } Q, \\ z_{tt} - \Delta z + y - z + ay = 0 & \text{in } Q, \\ y = z = 0 & \text{on } \Sigma, \\ z(0) = y^0 - \Delta y^0, \quad z_t(0) = y^1 - \Delta y^1 & \text{in } \Omega. \end{cases} \tag{5}$$

Consequently, in order to derive the desired observability inequality for system (1), it is natural to proceed in cascade by applying the global Carleman estimates to the second order operators in the two equations in system (5). We refer to [2, 3] for related works on Carleman inequalities for other cascade systems of partial differential equations.

Similar (boundary and/or internal) observability problems (in suitable spaces) have been considered for the heat and wave equations in [1], and for the Euler-Bernoulli plate equations in [5]. According to [1] and [5], the sharp observability constants for the heat, wave and Euler-Bernoulli plate equations with bounded potentials a (i.e., $p = \infty$) contain respectively the product of the following two terms (Recall that $C^* = C^*(\Omega, \Gamma_0)$ and $C = C(T, \Omega, \Gamma_0)$)

$$H_1(T,\ a) = \exp(C^* T ||a||_\infty), \qquad H_2(T,\ a) = \exp\left(C^* ||a||_\infty^{2/3}\right),$$

$$W_1(T,\ a) = \exp\left(C ||a||_\infty^{1/2}\right), \qquad W_2(T,\ a) = \exp\left(C ||a||_\infty^{2/3}\right),$$

and

$$P_1(T,\ a) = \exp(C^* T |||a|||_\infty^{1/2}), \qquad P_2(T,\ a) = \exp\left(C^* ||a||_\infty^{1/3}\right).$$

As explained in [1, 5], the role that each of these constants plays in the observability inequality is of different nature: $H_1(T,\ a)$, $W_1(T,\ a)$ and $P_1(T,\ a)$ are the constants which arise when applying Gronwall's inequality to establish the energy estimates for solutions of evolution equations; while $H_2(T,\ a)$, $W_2(T,\ a)$ and $P_2(T,\ a)$ appear when using global Carleman estimates to derive the observability inequality by absorbing the undesired lower order terms.

It is shown in [1, Theorems 1.1 and 1.2] and [5, Theorem 3] that the above observability constants are optimal for the heat, wave and Euler-Bernoulli plate systems ($N \geq 2$) with bounded potentials, in even dimensions $n \geq 2$. The proof of this optimality result uses the following two key ingredients:

1) For the heat and Euler-Bernoulli plate equations, because of the infinite speed of propagation, one can choose T as small as one likes and henceforth $H_1(T,\ a)$ and $P_1(T,\ a)$ can be bounded above by $H_2(T,\ a)$ and $P_2(T,\ a)$, respectively for $T = O\left(||a||_\infty^{-1/3}\right)$ and $O\left(||a||_\infty^{-1/6}\right)$. On the other hand, for the wave equation, although one has to take T to be large

enough (because of the finite velocity of propagation), for any finite T, $W_1(T,\ a)$ can bounded by $W_2(T,\ a)$ because the power $1/2$ for $||a||_\infty$ in $W_1(T,\ a)$, given by the modified energy estimate, is smaller than $2/3$, the power for $||a||_\infty$ in $W_2(T,\ a)$, arising from the Carleman estimate. In this way, for any finite T large enough, one gets an upper bound on the observability constant (for the wave equation) of the order of $\exp\big(C||a||_\infty^{2/3}\big)$.

2) Based on the Meshkov's construction [8] which allows finding potentials and non-trivial solutions for elliptic systems decaying at infinity in a super-exponential way, one can construct a family of solutions (for the heat, wave and Euler-Bernoulli plate equations) with suitable localization properties showing that most of the energy is concentrated away from the observation domain. According to this, the observed energies grow exponentially as $\exp\big(-||a||_\infty^{2/3}\big)$ for the wave and heat systems and as $\exp\big(-||a||_\infty^{1/3}\big)$ for the Euler-Bernoulli plate ones.

Things are more complicated for the Kirchhoff plate systems under consideration. Indeed, on one hand, due to the finite speed of propagation, one has to choose the observability time T to be large enough. On the other hand, a modified energy estimate for the Kirchhoff plate systems (*see* (10) in Lemma 1 in Section 2) yields a power $1/2$ for $||a||_\infty$ which can not be absorbed by the one, $1/3$, arising from the Carleman estimate. To overcome this difficulty, the key observation in this paper is that, although T has to be taken to be large, one can manage to use the indispensable energy estimate only in a very short time interval when deriving the desired observability estimate. However, we do not know how to show the optimality of the observability constant at this moment. Indeed, when proving the optimality, the energy estimate has to be used in the whole time duration $[0,T]$ and this breaks down the concentration effect that Meshkov's construction guarantees, which is valid only for very small time durations for the Kirchhoff plate systems. Therefore, proving the optimality of the observability estimates obtained in this paper is an interesting open problem.

The rest of this paper is organized as follows. In Section 2 we give some preliminary energy estimate for Kirchhoff plate systems, and show some fundamental weighted pointwise estimates for the wave and elliptic operators. In Section 3 we present the sharp observability estimate for the Kirchhoff plate system. In Section 4 we explain more carefully the main difficulty to show the optimality of the observability constant for Kirchhoff plate systems by means of the above mentioned Meshkov's construction.

2 Preliminaries

In this section, we show some preliminary energy estimates for Kirchhoff plate systems, and weighted pointwise estimates for the wave and elliptic operators. The estimates for the Kirchhoff plate system will then be obtained by noting the equivalence between system (1) and the coupled wave-elliptic system (5).

2.1 Energy estimates for Kirchhoff plate systems

Denote the energy of system (1) by

$$E(t) = \frac{1}{2}\Big[|\Delta y_t(t,\cdot)|^2_{(L^2(\Omega))^N} + |y_t(t,\cdot)|^2_{(H^1_0(\Omega))^N} + |\Delta y(t,\cdot)|^2_{(H^1_0(\Omega))^N}\Big]. \tag{6}$$

Note that this energy is equivalent to the square of the norm in $\mathcal{H}$. For

$$s_0 = \frac{n}{3p}, \tag{7}$$

consider also the modified energy function:

$$\mathcal{E}(t) = E(t) + \frac{1}{2}||a||_p^{\frac{2}{2-s_0}} |y(t,\cdot)|^2_{(L^2(\Omega))^N}. \tag{8}$$

It is clear that both energies are equivalent. Indeed,

$$E(t) \le \mathcal{E}(t) \le C\left(1 + ||a||_p^{\frac{2}{2-s_0}}\right) E(t). \tag{9}$$

The following estimate holds for the modified energy:

Lemma 1. *Let $a \in L^\infty(0,T; L^p(\Omega;\mathbb{R}^{N\times N}))$ for some $p \in [n/3,\infty]$. Then there is a constant $C_0 = C_0(\Omega, p, n) > 0$, independent of T, such that*

$$\mathcal{E}(t) \le C_0 e^{C_0 ||a||_p^{\frac{1}{2-s_0}} |t-s|} \mathcal{E}(s), \qquad \forall\, t, s \in [0,T]. \tag{10}$$

Proof. For simplicity, we assume $N = 1$. The same proof applies to a system with any finite number of components N. Using (8) and noting system (1), it is easy to see that

$$\frac{d\mathcal{E}(t)}{dt} = -\int_\Omega ay\Delta y_t dx + ||a||_p^{\frac{2}{2-s_0}} \int_\Omega yy_t dx. \tag{11}$$

Put $p_1 = \frac{2}{s_0 - 2p^{-1}}$ and $p_2 = \frac{2}{1-s_0}$. Noting that

$$\frac{1}{p} + \frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{2} = 1 \quad \text{and} \quad \frac{1}{2s_0^{-1}} + \frac{1}{2(1-s_0)^{-1}} + \frac{1}{2} = 1,$$

by Hölder's inequality and Sobolev's embedding theorem, and recalling (7)–(8) and observing $s_0 p_1 = \frac{2s_0}{s_0 - 2p^{-1}} = \frac{2n}{n-6}$, we get

$$\begin{aligned}
&\left| -\int_\Omega ay\Delta y_t dx \right| \le \int_\Omega |a||y|^{s_0}|y|^{1-s_0}|\Delta y_t| dx \\
&\le ||a||_p \left|\left| |y(t,\cdot)|^{s_0} \right|\right|_{L^{p_1}(\Omega)} \left|\left| |y(t,\cdot)|^{1-s_0} \right|\right|_{L^{p_2}(\Omega)} ||\Delta y_t(t,\cdot)||_{L^2(\Omega)} \\
&= ||a||_p ||y(t,\cdot)||^{s_0}_{L^{s_0 p_1}(\Omega)} ||y(t,\cdot)||^{1-s_0}_{L^{(1-s_0)p_2}(\Omega)} ||\Delta y_t(t,\cdot)||_{L^2(\Omega)} \\
&= ||a||_p ||y(t,\cdot)||^{s_0}_{L^{\frac{2n}{n-6}}(\Omega)} ||y(t,\cdot)||^{1-s_0}_{L^2(\Omega)} ||\Delta y_t(t,\cdot)||_{L^2(\Omega)} \\
&= C||a||_p^{\frac{1}{2-s_0}} \underbrace{||y(t,\cdot)||^{s_0}_{L^{\frac{2n}{n-6}}(\Omega)}}_{\le \mathcal{E}(t)^{\frac{s_0}{2}}} \underbrace{\left(||a||_p^{\frac{1-s_0}{2-s_0}} ||y(t,\cdot)||^{1-s_0}_{L^2(\Omega)} \right)}_{\le \mathcal{E}(t)^{\frac{1-s_0}{2}}} \underbrace{||\Delta y_t(t,\cdot)||_{L^2(\Omega)}}_{\le \mathcal{E}(t)^{1/2}} \\
&\le C||a||_p^{\frac{1}{2-s_0}} \mathcal{E}(t).
\end{aligned} \tag{12}$$

Similarly,

$$\begin{aligned}
||a||_p^{\frac{2}{2-s_0}} \left| \int_\Omega yy_t dx \right| &\le \frac{||a||_p^{\frac{1}{2-s_0}}}{2} \int_\Omega \left(||a||_p^{\frac{2}{2-s_0}} |y|^2 + |y_t|^2 \right) dx \\
&\le C||a||_p^{\frac{1}{2-s_0}} \mathcal{E}(t).
\end{aligned} \tag{13}$$

Now, combining (11)–(13), and applying Gronwall's inequality, we conclude the desired estimate (10). □

2.2 Pointwise weighted estimates for the wave and elliptic operators

In this subsection, we present some pointwise weighted estimates for the wave and elliptic equations that will play a key role when deriving the sharp observability estimates for the Kirchhoff plate system.

First, we show a pointwise weighted estimate for the wave operator "$\partial_{tt} - \Delta$". For this, for any (large) $\lambda > 0$, any $x_0 \in \mathbb{R}^n$ and $c \in \mathbb{R}$, set

$$\ell(t, x) = \lambda\left[|x - x_0|^2 - c\left(t - \frac{T}{2}\right)^2\right]. \tag{14}$$

By taking $(a^{ij})_{n\times n} = I$, the identity matrix, and $\theta = e^{\ell}$ (with ℓ given by (14)) in [4, Corollary 4.1] (*see* also [7, Lemma 5.1]), one has the following pointwise weighted estimate for the wave operator.

Lemma 2. *For any $u = u(t, x) \in C^2(\mathbb{R}^{1+n})$, any $k \in \mathbb{R}$ and $v \overset{\triangle}{=} \theta u$, it holds*

$$\begin{aligned}
&\theta^2|u_{tt} - \Delta u|^2 + 2\Big[\ell_t(v_t^2 + |\nabla v|^2) - 2(\nabla\ell)\cdot(\nabla v)v_t - \Psi v v_t \\
&\qquad\qquad + (A + \Psi)\ell_t v^2\Big]_t \\
&\quad + 2\sum_{i=1}^{n}\Big\{2v_i(\nabla\ell)\cdot(\nabla v) - \ell_i|\nabla v|^2 + \Psi v v_i - 2\ell_t v_t v_i + \ell_i v_t^2 \\
&\qquad\qquad - (A + \Psi)\ell_i v^2\Big\}_i \\
&\quad \geq 2\lambda(1 - k)v_t^2 + 2\lambda(k + 3 - 4c)|\nabla v|^2 + Bv^2, \quad \forall\,(t, x) \in \mathbb{R}^{1+n},
\end{aligned} \tag{15}$$

where

$$\begin{cases}
\Psi \overset{\triangle}{=} \lambda(2n - 2c - 1 + k), \\
A = 4\lambda^2\left[c^2(t - T/2)^2 - |x - x_0|^2\right] + \lambda(4c + 1 - k), \\
B = 8\lambda^3\left[(4c + 5 - k)|x - x_0|^2 - (8c + 1 - k)c^2(t - T/2)^2\right] + O(\lambda^2).
\end{cases} \tag{16}$$

As a consequence of Lemma 2, we have the following pointwise weighted estimate for the elliptic operator.

Corollary 1. *Let* $p = p(t,x) \in C^2(\mathbb{R}^{1+n})$*, and set* $q = \theta p$*. Then*

$$\theta^2|\Delta p|^2 + 2\sum_{i=1}^{n}\Big\{2q_i(\nabla\ell)\cdot(\nabla q) - \ell_i|\nabla q|^2 + \widetilde{\Psi} q q_i - (\widetilde{A}+\widetilde{\Psi})\ell_i q^2\Big\}_i \geq 6\lambda|\nabla q|^2 + \widetilde{B}q^2, \qquad \forall\,(t,x)\in\mathbb{R}^{1+n}, \tag{17}$$

where

$$\begin{cases} \widetilde{\Psi} \stackrel{\triangle}{=} \lambda(2n-1), \quad \widetilde{A} = -4\lambda^2|x-x_0|^2 + \lambda, \\ \widetilde{B} = 40\lambda^3|x-x_0|^2 + O(\lambda^2), \quad \text{uniformly w.r.t. } t \in [0,T]. \end{cases} \tag{18}$$

Proof. We fix an arbitrary $t \in [0,T]$ and view the corresponding function which depends on x as a function of (x,s) with s being a fictitious time parameter. We then set

$$U(s,x) \equiv p(t,x), \quad V(s,x) = \Xi(x)U(s,x) \qquad \forall (s,x)\in\mathbb{R}^{1+n},$$

where $\Xi = e^L$ and $L = \lambda|x-x_0|^2$. Choosing $c = 0$ in (14), and applying Lemma 2 (with $k = 0$) in the variable (x,s) to the above U and V, we get

$$\Xi^2|\Delta U|^2 + 2\sum_{i=1}^{n}\Big\{2V_i(\nabla L)\cdot(\nabla V) - L_i|\nabla V|^2 + \widetilde{\Psi}VV_i - (\widetilde{A}+\widetilde{\Psi})L_iV^2\Big\}_i \geq 6\lambda|\nabla V|^2 + \widetilde{B}V^2, \tag{19}$$

with $\widetilde{\Psi}$, $\widetilde{A}$ and $\widetilde{B}$ given by (18). Now, for any $c \in \mathbb{R}$, multiplying both sides of (19) by $e^{-2c\lambda(t-\frac{T}{2})^2}$, noting

$$\theta = \Xi e^{-c\lambda(t-\frac{T}{2})^2}, \quad \ell = L - c\lambda\left(t-\frac{T}{2}\right)^2 \quad \text{and} \quad q = e^{-c\lambda(t-\frac{T}{2})^2}V,$$

the desired inequality (17) follows. □

Remark. The key point in Corollary 1 is that we choose the same weight θ in (17) as that in (15). This will play a key role in the sequel when we deduce the sharp observability estimate for Kirchhoff plates.

In the sequel, for simplicity, we assume $x_0 \in \mathbb{R}^n \setminus \overline{\Omega}$ (For the general case where, possibly, $x_0 \in \overline{\Omega}$, we can modify an argument in [7, Case 2 in the proof of Theorem 5.1] to derive the same result). Hence

$$0 < R_0 \stackrel{\triangle}{=} \min_{x\in\Omega}|x - x_0| < R_1 \stackrel{\triangle}{=} \max_{x\in\Omega}|x - x_0|. \tag{20}$$

Also, for any $\beta > 0$, we set

$$\Theta = \Theta(t) \stackrel{\triangle}{=} \exp\Big\{ -\frac{\beta R_1}{t} - \frac{\beta R_1}{T-t}\Big\}, \qquad 0 < t < T. \tag{21}$$

It is easy to see that $\Theta(t)$ decays rapidly to 0 as $t \to 0$ or $t \to T$. The desired pointwise Carleman-type estimate with singular weight Θ for the wave operator reads as follows:

Theorem 1. *Let $u \in C^2([0, T] \times \overline{\Omega})$ and $v = \theta u$. Then there exist four constants $T_0 > 0$, $\lambda_0 > 0$, $\beta_0 > 0$ and $c_0 > 0$, independent of u, such that for all $T \geq T_0$, $\beta \in (0, \beta_0)$ and $\lambda \geq \lambda_0$ it holds*

$$\begin{aligned}
&\theta^2\Theta|u_{tt} - \Delta u|^2 + 2\Big\{\Theta\Big[\ell_t(v_t^2 + |\nabla v|^2) - 2(\nabla\ell)\cdot(\nabla v)v_t - \Psi v v_t \\
&\qquad\qquad + (A+\Psi)\ell_t v^2\Big]\Big\}_t \\
&\quad + 2\Theta\sum_{i=1}^n\Big\{2v_i(\nabla\ell)\cdot(\nabla v) - \ell_i|\nabla v|^2 + \Psi v v_i - 2\ell_t v_t v_i + \ell_i v_t^2 \\
&\qquad\qquad -(A+\Psi)\ell_i v^2\Big\}_i \\
&\geq c_0\lambda\theta^2\Theta(u_t^2 + |\nabla u|^2 + \lambda^2 u^2),
\end{aligned} \tag{22}$$

with A and Ψ given by (16).

Remark. The main difference between the pointwise estimates (15) and (22) is that we introduce a singular "pointwise" weight in (22). As we will see later, this point plays a crucial role in the proof of Theorem 3 in the next section. Another difference between (15) and (22) is that T is arbitrary in the former estimate; while for the later one needs to take T_0, and hence T, to be large enough.

Proof of Theorem 1. The proof is divided into several steps.

Step 1. We multiply both sides of inequality (15) by Θ. Obviously, we have (recall (16) for A and Ψ)

$$\begin{aligned}
&\Theta\Big[\ell_t(v_t^2+|\nabla v|^2)-2(\nabla\ell)\cdot(\nabla v)v_t-\Psi vv_t+(A+\Psi)\ell_t v^2\Big]_t\\
&\quad=\Big\{\Theta\Big[\ell_t(v_t^2+|\nabla v|^2)-2(\nabla\ell)\cdot(\nabla v)v_t-\Psi vv_t+(A+\Psi)\ell_t v^2\Big]\Big\}_t\\
&\qquad-\beta TR_1t^{-2}(T-t)^{-2}(T-2t)\Theta\Big[\ell_t(v_t^2+|\nabla v|^2)\\
&\qquad-2(\nabla\ell)\cdot(\nabla v)v_t-\Psi vv_t+(A+\Psi)\ell_t v^2\Big].
\end{aligned}\tag{23}$$

Note that

$$\begin{aligned}
&\Big|-\beta TR_1t^{-2}(T-t)^{-2}(T-2t)\Theta\Big[-2(\nabla\ell)\cdot(\nabla v)v_t-\Psi vv_t\Big]\Big|\\
&\quad\le\beta TR_1t^{-2}(T-t)^{-2}|T-2t|\Theta\Big[2|(\nabla\ell)\cdot(\nabla v)v_t|+|\Psi v_t v|\Big]\\
&\le\beta TR_1t^{-2}(T-t)^{-2}|T-2t|\Theta\Big[(|\nabla\ell|+1)v_t^2+|\nabla\ell||\nabla v|^2+\frac{1}{4}\Psi^2v^2\Big].
\end{aligned}\tag{24}$$

Thus by (15), and using (23)–(24), we get

$$\begin{aligned}
&\theta^2\Theta|u_{tt}-\Delta u|^2+2\Big\{\Theta\Big[\ell_t(v_t^2+|\nabla v|^2)-2(\nabla\ell)\cdot(\nabla v)v_t-\Psi vv_t\\
&\qquad\qquad+(A+\Psi)\ell_t v^2\Big]\Big\}_t\\
&\qquad+2\Theta\sum_{i=1}^n\Big\{2v_i(\nabla\ell)\cdot(\nabla v)-\ell_i|\nabla v|^2+\Psi vv_i-2\ell_t v_t v_i+\ell_i v_t^2\\
&\qquad\qquad-(A+\Psi)\ell_i v^2\Big\}_i\\
&\quad\ge\ 2\Theta\lambda(1-k)v_t^2+2\Theta\lambda(k+3-4c)|\nabla v|^2\\
&\qquad+2\beta TR_1t^{-2}(T-t)^{-2}(T-2t)\ell_t\Theta(v_t^2+|\nabla v|^2)\\
&\qquad-2\beta TR_1t^{-2}(T-t)^{-2}|T-2t|\Theta\Big[(|\nabla\ell|+1)v_t^2+|\nabla\ell||\nabla v|^2\Big]\\
&\qquad+\Theta\Big[B+2\beta TR_1t^{-2}(T-t)^{-2}(T-2t)\ell_t(A+\Psi)\\
&\qquad-\beta TR_1t^{-2}(T-t)^{-2}\frac{|T-2t|}{2}\Psi^2\Big]v^2,
\end{aligned}\tag{25}$$

where B is given by (16).

Step 2. Recalling that ℓ and Ψ are given respectively by (14) and (16), we get

$$\begin{aligned}
&2\Theta\lambda(1-k)v_t^2 + 2\Theta\lambda(k+3-4c)|\nabla v|^2 \\
&\quad + 2\beta T R_1 t^{-2}(T-t)^{-2}(T-2t)\ell_t\Theta(v_t^2+|\nabla v|^2) \\
&\quad - 2\beta T R_1 t^{-2}(T-t)^{-2}|T-2t|\Theta\Big[(|\nabla\ell|+1)v_t^2 + |\nabla\ell||\nabla v|^2\Big] \\
&\quad + \Theta\Big[B + 2\beta T R_1 t^{-2}(T-t)^{-2}(T-2t)\ell_t(A+\Psi) \\
&\quad - \beta T R_1 t^{-2}(T-t)^{-2}\frac{|T-2t|}{2}\Psi^2\Big]v^2 \\
&= \lambda\Theta(F_1 v_t^2 + F_2|\nabla v|^2) + \lambda^3\Theta G v^2,
\end{aligned} \tag{26}$$

where

$$\begin{aligned}
F_1 \stackrel{\triangle}{=}\ & 2(1-k) + 2c\beta T R_1 t^{-2}(T-t)^{-2}(T-2t)^2 \\
& -2\beta T R_1 t^{-2}(T-t)^{-2}|T-2t|(2|x-x_0|+\lambda^{-1}),
\end{aligned} \tag{27}$$

$$\begin{aligned}
F_2 \stackrel{\triangle}{=}\ & 2(k+3-4c) + 2c\beta T R_1 t^{-2}(T-t)^{-2}(T-2t)^2 \\
& -4\beta T R_1 t^{-2}(T-t)^{-2}|T-2t||x-x_0|
\end{aligned} \tag{28}$$

and

$$\begin{aligned}
G \stackrel{\triangle}{=}\ & 8\left[(4c+5-k)|x-x_0|^2 - (8c+1-k)c^2(t-T/2)^2\right] \\
& + O(\lambda^{-1}) \\
& + 8c\beta T R_1 t^{-2}(T-t)^{-2}(T-2t)^2\Big[c^2(t-T/2)^2 \\
& -|x-x_0|^2 + O(\lambda^{-1})\Big] \\
& - \beta(2n-2c-1+k)^2 T R_1 t^{-2}(T-t)^{-2}|t-T/2|\lambda^{-1}.
\end{aligned} \tag{29}$$

Thus, by (25) and (26), we have

$$\begin{aligned}
&\theta^2\Theta|u_{tt}-\Delta u|^2 + 2\left\{\Theta\Big[\ell_t(v_t^2+|\nabla v|^2) - 2(\nabla\ell)\cdot(\nabla v)v_t - \Psi v v_t \right. \\
&\qquad \left. +(A+\Psi)\ell_t v^2\Big]\right\}_t \\
&\quad + 2\Theta\sum_{i=1}^n\Big\{2v_i(\nabla\ell)\cdot(\nabla v) - \ell_i|\nabla v|^2 + \Psi v v_i - 2\ell_t v_t v_i + \ell_i v_t^2 \\
&\qquad -(A+\Psi)\ell_i v^2\Big\}_i \\
&\geq \lambda\Theta(F_1 v_t^2 + F_2|\nabla v|^2) + \lambda^3\Theta G v^2.
\end{aligned} \tag{30}$$

Step 3. Let us show that F_1, F_2 and G are positive when λ is large enough. For this purpose, we choose $c \in (0, 1)$ sufficiently small so that

$$\frac{(4+5c)R_0^2}{9c} > R_1^2, \tag{31}$$

and $T(> 2R_1)$ sufficiently large such that

$$\frac{4(4+5c)R_0^2}{9c} > c^2T^2 > 4R_1^2. \tag{32}$$

Also, we choose

$$k = 1 - c. \tag{33}$$

By (33) and recalling that $c \in (0, 4/5)$, it is easy to see that the nonsingular part $F_1^0 \stackrel{\triangle}{=} 2(1-k)$ of F_1 (*resp.* $F_2^0 \stackrel{\triangle}{=} 2(k+3-4c)$ of F_1) is positive. Using (33) again, the nonsingular part of G reads

$$\begin{aligned} G^0 \stackrel{\triangle}{=}\; & 8\left[(4c+5-k)|x-x_0|^2 - (8c+1-k)c^2(t-T/2)^2\right] + O(\lambda^{-1}) \\ & \geq 2\left[4(4+5c)R_0^2 - 9c^3T^2\right] + O(\lambda^{-1}), \end{aligned}$$

which, via the first inequality in (32), is positive provided that λ is sufficiently large.

When t is near 0 and T, i.e., $t \in I_0 \stackrel{\triangle}{=} (0, \delta_0) \cup (T-\delta_0, T)$ for some sufficiently small $\delta_0 \in (0, T/2)$, the dominant terms in F_i ($i = 1, 2$) and G are the singular ones. For $t \in I_0$, the singular part of F_1 reads

$$\begin{aligned} F_1^1 \stackrel{\triangle}{=}\; & 2c\beta TR_1t^{-2}(T-t)^{-2}(T-2t)^2 \\ & - 2\beta TR_1t^{-2}(T-t)^{-2}|T-2t|(2|x-x_0| + \lambda^{-1}) \\ \geq\; & 2\beta TR_1t^{-2}(T-t)^{-2}|T-2t|[c(T-2\delta_0) - 2R_1 - \lambda^{-1})] \\ =\; & 2\beta TR_1t^{-2}(T-t)^{-2}|T-2t|(cT - 2R_1 - 2c\delta_0 - \lambda^{-1}), \end{aligned}$$

which, via the second inequality in (32), is positive provided that both δ_0 and λ^{-1} are sufficiently small. Similarly, for $t \in I_0$, the singular part of F_2,

$$\begin{aligned} F_2^1 \stackrel{\triangle}{=}\; & 2c\beta TR_1t^{-2}(T-t)^{-2}(T-2t)^2 \\ & - 4\beta TR_1t^{-2}(T-t)^{-2}|T-2t||x-x_0| \end{aligned}$$

is positive provided that δ_0 is sufficiently small. Also, for $t \in I_0$, the singular part of G reads

$$\begin{aligned} G^1 \overset{\triangle}{=} \;& 8\, c\beta T R_1 t^{-2}(T-t)^{-2}(T-2t)^2\Big[c^2(t-T/2)^2 - |x-x_0|^2 \\ & + O(\lambda^{-1})\Big] - \beta(2n-2c-1+k)^2 T R_1 t^{-2}(T-t)^{-2}|t-T/2|\lambda^{-1}. \end{aligned}$$

It is easy to see that, for $t \in I_0$, it holds

$$\begin{aligned} G^1 =\;& \beta T R_1 t^{-2}(T-t)^{-2}|T-2t| \\ & \Big\{8c|T-2t|\big[c^2(t-T/2)^2 - |x-x_0|^2 + O(\lambda^{-1})\big] \\ & - (2n-2c-1+k)^2(2\lambda)^{-1}\Big\} \\ =\;& \beta T R_1 t^{-2}(T-t)^{-2}|T-2t| \\ & \Big\{8c|T-2t|\big[c^2(t-T/2)^2 - |x-x_0|^2\big] + O(\lambda^{-1})\Big\} \\ \geq\;& \beta T R_1 t^{-2}(T-t)^{-2}|T-2t| \\ & \Big\{8c|T-2\delta_0|\big[c^2(\delta_0-T/2)^2 - R_1^2\big] + O(\lambda^{-1})\Big\} \\ \geq\;& \beta T R_1 t^{-2}(T-t)^{-2}|T-2t| \\ & \Big\{8c|T-2\delta_0|\big[c^2T^2/4 - R_1^2 + c^2\delta_0(\delta_0-T)\big] + O(\lambda^{-1})\Big\}, \end{aligned}$$

which, via the second inequality in (32), is positive provided that both δ_0 and λ^{-1} are sufficiently small.

By (27)–(29), we see that $F_1 = F_1^0 + F_1^1$, $F_2 = F_2^0 + F_2^1$ and $G = G^0 + G^1$. Since F_1^0, F_2^0 and G^0 are positive, by the above argument, we see that F_1, F_2 and G are positive for $t \in I_0$. For $t \in (0, T) \setminus I_0$, noting again the positivity of F_1^0, F_2^0 and G^0, one can choose $\beta > 0$ sufficiently small such that F_1^1, F_2^1 and G^1 are very small, hence so that F_1, F_2 and G are positive. Hence (30) yields the desired (22). This completes the proof of Theorem 1. □

Similar to Theorem 1, by multiplying both sides of (17) by Θ, we have

Theorem 2. *Let $p = p(t,x) \in C^2([0,T] \times \overline{\Omega})$, and set $q = \theta p$. Then there exist two constants $\lambda_0 > 0$ and $c_0 > 0$, independent of p, such that for all $T > 0$, $\beta > 0$ and $\lambda \geq \lambda_0$ it holds*

$$\theta^2\Theta|\Delta p|^2 + 2\Theta\sum_{i=1}^{n}\Big\{2q_i(\nabla\ell)\cdot(\nabla q) - \ell_i|\nabla q|^2 + \widetilde{\Psi}qq_i - (\widetilde{A}+\widetilde{\Psi})\ell_i q^2\Big\}_i \geq c_0\lambda\theta^2\Theta\left(|\nabla p|^2 + \lambda^2 p^2\right), \tag{34}$$

with $\widetilde{A}$ *and* $\widetilde{\Psi}$ *given by* (18).

3 Sharp observability estimate

In this section we establish a sharp observability estimate for system (1).

For this purpose, for any fixed $x_0 \in \mathbb{R}^n \setminus \overline{\Omega}$ (As mentioned before, we do not really need to assume that x_0 is out of $\overline{\Omega}$. Indeed, for the case where, possibly, $x_0 \in \overline{\Omega}$, we can modify an argument in [7, Case 2 in the proof of Theorem 5.1] to derive the same observability result in this section), we introduce the following set:

$$\Gamma_0 \stackrel{\triangle}{=} \Big\{x \in \Gamma \;\Big|\; (x - x_0)\cdot\nu(x) > 0\Big\}. \tag{35}$$

One of the main results in this paper is the following observability inequality with explicit dependence of the observability constant on the potential a for system (1):

Theorem 3. *Let* Γ_0 *be given by* (35) *and* $p \in [5n/2, \infty]$. *Then there is a constant* $C > 0$ *such that for any* $T > T_0$, *with* T_0 *as in Theorem 1, and any* $a \in L^\infty(0, T; L^p(\Omega; \mathbb{R}^{N\times N}))$, *the weak solution* y *of system* (1) *satisfies estimate* (2) *with the observability constant* $K(a) > 0$ *verifying*

$$K(a) \;\leq\; C\exp\left(C||a||_p^{\frac{1}{3-5n/2p}}\right). \tag{36}$$

We now sketch the main points in the proof of Theorem 3. The first ingredient consists in decomposing the Kirchhoff plate equation into a coupled system of wave and elliptic equations as in (5) and to apply the pointwise estimates of the previous section in cascade. First, we apply Theorem 1 to z. Integrating (22) in

Q, noting that $\Theta(t)$ decays rapidly to 0 as $t \to 0+$ or $t \to T-$, recalling that $z|_\Sigma = 0$ (and hence $\nabla z = \frac{\partial z}{\partial \nu}\nu$ and $z_i = \frac{\partial z}{\partial \nu}\nu_i$ on Σ), one may deduce that

$$\begin{aligned}
&\lambda \int_Q \theta^2\Theta(z_t^2 + |\nabla z|^2)dxdt + \lambda^3 \int_Q \theta^2\Theta z^2 dxdt \\
&\le C\left\{\int_Q \theta^2\Theta(z_{tt} - \Delta z)^2 dxdt + 4\lambda \int_\Sigma \theta^2\Theta\Big|\frac{\partial z}{\partial \nu}\Big|^2 (x - x_0)\cdot\nu(x)dxdt\right\} \\
&\le C\left\{\int_Q \theta^2\Theta(z_{tt} - \Delta z)^2 dxdt + e^{C\lambda}\int_{\Sigma_0} \Theta\Big|\frac{\partial z}{\partial \nu}\Big|^2 dxdt\right\} \\
&\le C\left\{\int_Q \theta^2\Theta[(ay)^2 + y^2 + z^2]dxdt + e^{C\lambda}\int_{\Sigma_0} \Theta\Big(\Big|\frac{\partial y}{\partial \nu}\Big|^2 + \Big|\frac{\partial \Delta y}{\partial \nu}\Big|^2\Big)dxdt\right\},
\end{aligned} \tag{37}$$

with $\Sigma_0 \overset{\triangle}{=} (0, T) \times \Gamma_0$ and Γ_0 being given in (35).

Similarly, applying Theorem 2 respectively to y and y_t, we deduce that

$$\begin{aligned}
&\lambda \int_Q \theta^2\Theta|\nabla y|^2 dxdt + \lambda^3 \int_Q \theta^2\Theta y^2 dxdt \\
&\quad \le C\Big\{\int_Q \theta^2\Theta(\Delta y)^2 dxdt + e^{C\lambda}\int_{\Sigma_0} \Theta\Big|\frac{\partial y}{\partial \nu}\Big|^2 dxdt\Big\} \\
&\quad \le C\Big\{\int_Q \theta^2\Theta(y^2 + z^2)dxdt + e^{C\lambda}\int_{\Sigma_0} \Theta\Big|\frac{\partial y}{\partial \nu}\Big|^2 dxdt\Big\},
\end{aligned} \tag{38}$$

and

$$\begin{aligned}
&\lambda \int_Q \theta^2\Theta|\nabla y_t|^2 dxdt + \lambda^3 \int_Q \theta^2\Theta y_t^2 dxdt \\
&\quad \le C\Big\{\int_Q \theta^2\Theta(\Delta y_t)^2 dxdt + e^{C\lambda}\int_{\Sigma_0} \Theta\Big|\frac{\partial y_t}{\partial \nu}\Big|^2 dxdt\Big\} \\
&\quad \le C\Big\{\int_Q \theta^2\Theta(y_t^2 + z_t^2)dxdt + e^{C\lambda}\int_{\Sigma_0} \Theta\Big|\frac{\partial y_t}{\partial \nu}\Big|^2 dxdt\Big\}.
\end{aligned} \tag{39}$$

It is easy to see that the term $C\int_Q \theta^2\Theta z^2 dxdt$ (*resp.* $C\int_Q \theta^2\Theta y^2 dxdt$ and $C\int_Q \theta^2\Theta y_t^2 dxdt$) in the right hand side of (37) (*resp.* (38) and (39)) can be absorbed by its left hand side. Hence, for any $\varepsilon_0 > 0$, (37) added first to $\varepsilon_0\lambda^3\times$(38), then to $\varepsilon_0\lambda\times$(39), makes

$$
\begin{aligned}
&\lambda \int_Q \theta^2 \Theta (z_t^2 + |\nabla z|^2) dxdt + \lambda^3 \int_Q \theta^2 \Theta z^2 dxdt \\
&\quad + \ \varepsilon_0 \lambda^2 \int_Q \theta^2 \Theta |\nabla y_t|^2 dxdt + \varepsilon_0 \lambda^4 \int_Q \theta^2 \Theta (y_t^2 + |\nabla y|^2) dxdt \\
&\quad + \ \varepsilon_0 \lambda^6 \int_Q \theta^2 \Theta y^2 dxdt \\
&\leq C \Big\{ ||\theta \sqrt{\Theta} a y||^2_{L^2(Q)} + \int_Q \theta^2 \Theta y^2 dxdt + \varepsilon_0 \int_Q \theta^2 \Theta (\lambda^3 z^2 + \lambda z_t^2) dxdt \\
&\qquad + \ e^{C\lambda} \int_{\Sigma_0} \Theta \Big(\Big| \frac{\partial y}{\partial \nu} \Big|^2 + \Big| \frac{\partial y_t}{\partial \nu} \Big|^2 + \Big| \frac{\partial \Delta y}{\partial \nu} \Big|^2 \Big) dxdt \Big\}.
\end{aligned}
\tag{40}
$$

By taking $\varepsilon_0 > 0$ sufficiently small (which is independent of λ), one can absorb the undesired terms $C\varepsilon_0 \int_Q \theta^2 \Theta (\lambda^3 z^2 + \lambda z_t^2) dxdt$ in the right hand side of (40) by its left hand side. Then, for this choice of ε_0 and taking taking $\lambda > 0$ sufficiently large, one can absorb further the undesired terms $C \int_Q \theta^2 \Theta y^2 dxdt$ in the right hand side of (40). Consequently, we arrive at

$$
\begin{aligned}
&\lambda \int_Q \theta^2 \Theta (z_t^2 + |\nabla z|^2) dxdt + \lambda^3 \int_Q \theta^2 \Theta z^2 dxdt \\
&\quad + \lambda^2 \int_Q \theta^2 \Theta |\nabla y_t|^2 dxdt + \lambda^4 \int_Q \theta^2 \Theta (y_t^2 + |\nabla y|^2) dxdt \\
&\quad + \lambda^6 \int_Q \theta^2 \Theta y^2 dxdt \\
&\leq C \Big\{ ||\theta \sqrt{\Theta} a y||^2_{L^2(Q)} + e^{C\lambda} \int_{\Sigma_0} \Theta \Big(\Big| \frac{\partial y}{\partial \nu} \Big|^2 + \Big| \frac{\partial y_t}{\partial \nu} \Big|^2 + \Big| \frac{\partial \Delta y}{\partial \nu} \Big|^2 \Big) dxdt \Big\}.
\end{aligned}
\tag{41}
$$

Recalling $z = y - \Delta y$, one has

$$
\begin{aligned}
&\lambda \int_Q \theta^2 \Theta (z_t^2 + |\nabla z|^2) dxdt + \lambda^3 \int_Q \theta^2 \Theta z^2 dxdt \\
&\geq \lambda \int_Q \theta^2 \Theta [(\Delta y_t - y_t)^2 + |\nabla \Delta y - \nabla y|^2] dxdt \\
&\quad + \lambda^3 \int_Q \theta^2 \Theta (\Delta y - y)^2 dxdt \\
&\geq \frac{\lambda}{2} \int_Q \theta^2 \Theta [(\Delta y_t)^2 + |\nabla \Delta y|^2] dxdt + \frac{\lambda^3}{2} \int_Q \theta^2 \Theta (\Delta y)^2 dxdt \\
&\quad - \lambda \int_Q \theta^2 \Theta (y_t^2 + |\nabla y|^2) dxdt - \lambda^3 \int_Q \theta^2 \Theta y^2 dxdt.
\end{aligned}
\tag{42}
$$

Combining (41) and (42), it follows

$$\begin{aligned}
&\lambda \int_Q \theta^2 \Theta[(\Delta y_t)^2 + |\nabla \Delta y|^2] dxdt + \lambda^2 \int_Q \theta^2 \Theta |\nabla y_t|^2 dxdt \\
&\quad + \lambda^3 \int_Q \theta^2 \Theta (\Delta y)^2 dxdt + \lambda^4 \int_Q \theta^2 \Theta |\nabla y|^2 dxdt \\
&\quad + \lambda^6 \int_Q \theta^2 \Theta y^2 dxdt \\
&\leq C\Big\{ ||\theta \sqrt{\Theta} a y||^2_{L^2(Q)} + e^{C\lambda} \int_{\Sigma_0} \Theta \Big(\Big|\frac{\partial y}{\partial \nu}\Big|^2 + \Big|\frac{\partial y_t}{\partial \nu}\Big|^2 + \Big|\frac{\partial \Delta y}{\partial \nu}\Big|^2 \Big) dxdt \Big\}.
\end{aligned} \tag{43}$$

Now we have to get rid of the term $||\theta a y||^2_{L^2(Q)}$.

By the proof of [1, Theorem 2.2], for any $\varepsilon > 0$, we have

$$\begin{aligned}
||\theta \sqrt{\Theta} a y||^2_{L^2(Q)} \;\leq\; & \varepsilon \lambda ||\theta \sqrt{\Theta} y||^2_{L^2(0,T;\, H_0^1(\Omega))} \\
& + \varepsilon^{-n/(p-n)} ||a||_p^{2p/(p-n)} \lambda^{-n/(p-n)} ||\theta \sqrt{\Theta} y||^2_{L^2(Q)}.
\end{aligned}$$

By taking ε small enough the first term $\varepsilon \lambda ||\theta \sqrt{\Theta} y||^2_{L^2(0,T;\, H_0^1(\Omega))}$ can be absorbed by the left hand side of (43). Then, for this choice of ε and taking λ sufficiently large, the term

$$C \varepsilon^{-n/(p-n)} ||a||_p^{2p/(p-n)} \lambda^{-n/(p-n)} ||\theta \sqrt{\Theta} y||^2_{L^2(Q)}$$

can be absorbed by $\lambda^6 \int_Q \theta^2 \Theta y^2 dxdt$. For this, we choose λ such that

$$C \varepsilon^{-n/(p-n)} ||a||_p^{2p/(p-n)} \lambda^{-n/(p-n)} \leq \frac{1}{2} \lambda^6,$$

which yields $\lambda \geq C ||a||_p^{2p/(6p-5n)} = C ||a||_p^{1/(3-5n/2p)}$.

Therefore, recalling the definition of $E(t)$ in (6), it follows from (43) that

$$\begin{aligned}
&\int_0^T \Theta E(t) dt \\
&\leq C \exp\left(C ||a||_p^{\frac{1}{3-5n/2p}} \right) \int_{\Sigma_0} \Big(\Big|\frac{\partial y}{\partial \nu}\Big|^2 + \Big|\frac{\partial y_t}{\partial \nu}\Big|^2 + \Big|\frac{\partial \Delta y}{\partial \nu}\Big|^2 \Big) dxdt.
\end{aligned} \tag{44}$$

Now, for $||a||_p$ sufficiently large, put

$$t_0 = \frac{1}{2} ||a||_p^{\frac{1}{3-5n/2p} - \frac{1}{2-n/3p}}. \tag{45}$$

Hence

$$2||a||_p^{\frac{1}{2-n/3p}} t_0 = ||a||_p^{\frac{1}{3-5n/2p}}. \tag{46}$$

Recall that $p > 5n/2$. Hence t_0 is small when $||a||_p$ is large. Therefore, by the definition of Θ in (21), it is obvious that

$$\int_0^T \Theta E(t)dt \geq \int_{t_0}^{2t_0} \Theta E(t)dt \geq \frac{1}{C} e^{-Ct_0^{-1}} \int_{t_0}^{2t_0} E(t)dt. \tag{47}$$

By (9) and Lemma 1 (with $s_0 = \frac{n}{3p}$), it follows that

$$\begin{aligned} \int_{t_0}^{2t_0} E(t)dt &\geq \frac{1}{1 + ||a||_p^{\frac{2}{2-n/3p}}} E(0) \int_{t_0}^{2t_0} e^{-C||a||_p^{\frac{1}{2-n/3p}} t} dt \\ &\geq \frac{t_0}{1 + ||a||_p^{\frac{2}{2-n/3p}}} E(0) e^{-2C||a||_p^{\frac{1}{2-n/3p}} t_0}. \end{aligned} \tag{48}$$

Combining (45)–(48), and noting (46), it follows

$$\int_0^T \Theta E(t)dt \geq E(0)\exp\left(-C||a||_p^{\frac{1}{3-5n/2p}}\right). \tag{49}$$

Finally, the desired estimates (2) and (36) follow from (44) and (49).

4 An open problem on the optimality of the observability constant for Kirchhoff plate systems

In [5, Theorem 3], it is shown that when $p = \infty$ the observability constant $P_2(T, a)$ for the Euler-Bernoulli plate systems with at least two equations in even space dimensions $n \geq 2$ is optimal in what concerns the dependence on the potential a. The main idea to prove this optimality result is the same as that in [1], which is based on a suitable construction of u and q satisfying the following bi-Laplacian equation:

$$\Delta^2 u = qu, \qquad \text{in } \mathbb{R}^n, \tag{50}$$

which decays at infinity sufficiently fast. More precisely, following Meshkov's construction [8, 5], we have the following result on u and q for (50):

Lemma 3. *Let $n \geq 2$ be even. Then there exist two nontrivial complex-valued functions:*

$$u \in C^\infty(\mathbb{R}^n;\ \mathbb{C}), \quad q \in C^\infty(\mathbb{R}^n;\ \mathbb{C}) \bigcap L^\infty(\mathbb{R}^n;\ \mathbb{C})$$

such that (50) *is satisfied, and for some constant C:*

$$|u(x)| + |\nabla u(x)| + |\nabla \Delta u(x)| \leq Ce^{-|x|^{4/3}}, \qquad \forall\, x \in \mathbb{R}^n. \tag{51}$$

One may expect that Lemma 3 can be applied to establish a similar optimality result for the observability constant $K(a)$ for the Kirchhoff plate systems as well. However, this is an open problem. We now explain why the above Meshkov's construction does not seem to suffice for Kirchhoff plate systems. Based on the construction of u and q in Lemma 3, by suitable scaling and localization arguments, one can find a family of rescaled potentials $a_R(x) = R^4 q(Rx)$ with an L^∞-norm of the order of R^4 and a family of solutions $u_R(x) = u(Rx)$ of the corresponding bi-harmonic problem, with a decay of the order of

$$|u_R(x)| \leq C\exp\Big(- R^{4/3}|x|^{4/3}\Big).$$

Without loss of generality we may assume that the boundary Γ (and therefore the observation subdomain Γ_0) is included in the region $|x| \geq 1$. This yields a sequence of solutions of the bi-Laplacian system $\Delta^2 u_R = a_R u_R$ in which the ratio between total energy and the energy concentrated in Γ_0 and the norm of the boundary traces is of the order of $\exp\Big(- R^{4/3}\Big)$. Taking into account that $||a_R||_\infty \sim R^4$, this ratio turns to be of the order of $\exp\Big(- ||a_R||_\infty^{1/3}\Big)$. These solutions of the above mentioned bi-Laplacian system can be regarded also as solutions of the Kirchhoff plate system for suitable initial data. However, they do not fulfill homogeneous boundary conditions. Therefore, one needs to compensate them by subtracting the solution taking their boundary data and zero initial ones. In turn, one has to show that these solutions are as small as $\exp\Big(- ||a_R||_\infty^{1/3}\Big)$ in the energy space $\mathcal{H}$. Due to the infinite speed of propagation, this can be easily done for the Euler-Bernoulli plate systems during a time interval of the order of $T \leq \mu ||a_R||_\infty^{-1/6}$ (because it suffices to use the energy estimate, which yields an exponential growth $\exp\Big(T ||a_R||_\infty^{1/2}\Big)$ for the energy evolution in a very short time). However, the same approach fails for Kirchhoff plate systems since, in order that the (boundary) observability estimate for these systems to hold, one needs to take the time T to be large enough. In fact, the key point is that, at this level the energy estimate yields an exponential growth $\exp\left(T||a_R||_\infty^{1/2}\right)$ for the energy evolution, and it has to be used in the whole time duration $[0, T]$. This breaks down the concentration effect that Meshkov's construction guarantees.

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