COMPUTATIONAL & APPLIED MATHEMATICS

Volume 28, N. 3, pp. 277–290, 2009

ISSN 0101-8205
www.scielo.br/cam

Recurrence relations between moments of order statistics from doubly truncated Makeham distribution

A.W. ABOUTAHOUN and N.M. AL-OTAIBI

Department of Mathematics, King Saud University, Riyadh, Kingdom of Saudi Arabia

E-mail: tahoun@ksu.edu.sa

Abstract. In this paper, we present recurrence relations between the single and the product moments for order statistics from doubly truncated Makeham distribution. Characterizations for the Makeham distribution are studied.

Mathematical subject classification: 62G30, 65C60.

Key words: moments of order statistics, Makeham distribution, doubly truncated distribution, recurrence relations.

1 Introduction

Many researchers have studies the moments of order statistics of several distributions. A number of recurrence relalations satisfied by these moments of order statistics are available in literature. Balakrishnan and Malik [2] derived some identities involving the density functions of order statistics. These identities are useful in checking the computation of the moments of order statistics. Balakrishnan and Malik [3] established some recurrence relations of order statistics from the liear-expoential distribution. Balakrishnan et al. [4] reviewed several recurrence relations and identities for the single and product moments of order statistics from some specific distributions. Mohie El-Din et al. [9, 10] presented recurrence relations for the single and product moments of order statistics from the doubly truncated parabolic and skewed distribution and linear-exponential distribution. Hendi et al. [1] developed recurrence relations for the single and

#CAM-20/08. Received: 18/IX/08. Accepted: 15/VI/09.

product moments of order statistics from doubly truncated Gompertz distribution. Khan et al. [7] established general result about recurrence relations between product moments of order statistics. They used that result to get the recurrence relations between product moments of some doubly truncated distributions (Weibull, expoential, Pareto, power function, and Cauchy). Several recurrence relations satisfied by these moments of order statistics are also available in Khan and Khan [5], [6].

The probability density function (pdf) of the Makeham distribution is given by

$$f_X(x) = \left(1 + \theta\left(1 - e^{-x}\right)\right) e^{-x-\theta\left(x+e^{-x}-1\right)}, \quad x \geq 0, \quad \theta \geq 0$$

The doubly truncated pdf of continuous rv is given by

$$f(x) = \frac{f_X(x)}{P - Q} = \frac{1}{P - Q}\left(1 + \theta\left(1 - e^{-x}\right)\right) e^{-x-\theta\left(x+e^{-x}-1\right)}, \quad Q_1 \leq x \leq P_1 \tag{1.1}$$

where

$$1 - P = e^{-P_1-\theta\left(P_1+e^{-P_1}-1\right)} \quad \text{and} \quad 1-Q=e^{-Q_1-\theta\left(Q_1+e^{-Q_1}-1\right)}$$

The cumulative distribution function *c.d.f.* is given by

$$1 - F(x) = \frac{f(x)}{1 + \theta\left(1 - e^{-x}\right)} - P_2 \tag{1.2}$$

where

$$P_2 = \frac{1 - P}{P - Q}$$

Let X be a continuous random variable having a *c.d.f.* (1.2) and *p.d.f.*. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from the Makeham distribution and $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ be the corresponding order statistics obtained from the doubly truncated Makeham distribution (1.1), then

$$f_{r:n}(x) = C_{r:n}\left[F(x)\right]^{r-1}\left[1 - F(x)\right]^{n-r} f(x) \tag{1.3}$$

where

$$C_{r:n} = \frac{n!}{(n-r)!\,(r-1)!}.$$

The expected value of any measurable function $h(x)$ can be obtained as follows:

$$\alpha_{r:n} = E\left[h\left(X_{r:n}\right)\right] = \tag{1.4}$$
$$C_{r:n}\int_{Q_1}^{P_1} h(x)\left[F(x)\right]^{r-1}\left[1-F(x)\right]^{n-r} f(x)\,dx, \quad 1 \le r \le n$$

and the expected value of any measurable joint function $h(x, y)$ can be calculated by

$$\alpha_{r,s:n} = E\left[h\left(X_{r:n}, X_{s:n}\right)\right] = \tag{1.5}$$
$$\int_{Q_1}^{P_1}\int_{x}^{P_1} h(x, y)\, f_{r,s:n}(x)\,dydx, \quad x \le y$$

where the joint density function of $X_{r:s}$ and $X_{s:n}$, $(1 \le r \le s \le n)$ is given by

$$f_{r,s:n}(x) =$$
$$C_{r,s:n}\left[F(x)\right]^{r-1}\left[F(y)-F(x)\right]^{s-r-1}\left[1-F(y)\right]^{n-s} f(x)\, f(y), \tag{1.6}$$
$$x \le y$$

where

$$C_{r,s:n} = \frac{n!}{(r-1)!\,(s-r-1)!\,(n-s)!}.$$

The rest of this paper is organized as follows: In Section 2 the recurrence relations for the single moments of order statistics from doubly truncated Makeham distribution is obtained. In Section 3 the recurrence relations for the product moments of order statistics from doubly truncated Makeham distribution is developed. Two results that characterize Maheham distribution are presented in Section 4. Some numerical results illustrating the developed recurrence relations are given in Section 5.

2 Recurrence relations for single moments of order statistics

Recurrence relations for the single moments of order statistics from the doubly truncated Maheham distribution are given by the following theorem.

Theorem 1. *Let $X_{i:n} \leq X_{i+1:n}$, $(1 \leq i \leq n)$ be an order statistics, $Q_1 \leq X_{r;n} \leq P_1$, $1 \leq r \leq n$, $n \geq 1$ and for any measurable function $h(x)$, then*

$$\alpha_{r;n} = -P_2\alpha_{r;n-1} + Q_2\alpha_{r-1;n-1} + \left(\frac{1}{n}\right) E\left(\frac{h'(X_{r:n})}{1+\theta\left(1-e^{-X_{r:n}}\right)}\right) \tag{2.1}$$

where $Q_2 = \dfrac{1-Q}{P-Q}$.

Proof. From (1.4), we find

$$\begin{aligned}\alpha_{r:n} - \alpha_{r-1:n-1} &= \frac{(n-1)!}{(n-r)!\,(r-1)!}\int_{Q_1}^{P_1} h(x)\,[F(x)]^{r-2}\,[1-F(x)]^{n-r} \\ &\quad \times \; [nF(x) - (r-1)]\, f(x)\, dx,\end{aligned}$$

By using integration by parts, we get

$$\alpha_{r:n} - \alpha_{r-1:n-1} =$$
$$\begin{pmatrix} n-1 \\ r-1 \end{pmatrix} \int_{Q_1}^{P_1} h'(x)\,[F(x)]^{r-1}\,[1-F(x)]^{n-r+1}\, dx$$

Using (1.2) in the previous equation, we obtain

$$\begin{aligned}&\alpha_{r:n} - \alpha_{r-1:n-1} = \\ &\begin{pmatrix} n-1 \\ r-1 \end{pmatrix} \int_{Q_1}^{P_1} h'(x)\,[F(x)]^{r-1}\,[1-F(x)]^{n-r}\left[\frac{f(x)}{1+\theta\left(1-e^{-x}\right)} - P_2\right] dx = \\ &-P_2 \begin{pmatrix} n-1 \\ r-1 \end{pmatrix} \int_{Q_1}^{P_1} h'(x)\,[F(x)]^{r-1}\,[1-F(x)]^{n-r}\, dx \\ &+ \begin{pmatrix} n-1 \\ r-1 \end{pmatrix} \int_{Q_1}^{P_1} h'(x)\,[F(x)]^{r-1}\,[1-F(x)]^{n-r}\left(\frac{1}{1+\theta\left(1-e^{-x}\right)}\right) f(x)\, dx\end{aligned} \tag{2.2}$$

Similarily, we can show that,

$$\begin{aligned}&\alpha_{r:n-1} - \alpha_{r-1:n-2} = \\ &\begin{pmatrix} n-2 \\ r-1 \end{pmatrix} \int_{Q_1}^{P_1} h'(x)\,[F(x)]^{r-1}\,[1-F(x)]^{n-r}\, dx\end{aligned} \tag{2.3}$$

From (2.2) and (2.3) , we obtain

$$\alpha_{r:n}-\alpha_{r-1:n-1}=$$
$$-\left(\frac{n-1}{n-r}\right)P_2\left(\alpha_{r:n-1}-\alpha_{r-1:n-2}\right)+\left(\frac{1}{n}\right)E\left(\frac{h^{'}\left(X_{r:n}\right)}{1+\theta\left(1-e^{-X_{r:n}}\right)}\right) \tag{2.4}$$

Since

$$(n-r)\,\alpha_{r-1:n-1}+(r-1)\,\alpha_{r:n-1}=(n-1)\,\alpha_{r-1:n-2}$$

Then

$$\alpha_{r-1:n-2}=\frac{(n-r)}{(n-1)}\alpha_{r-1:n-1}+\frac{(r-1)}{(n-1)}\alpha_{r:n-1}$$

By substituting for $\alpha_{r-1:n-2}$ from the previous equation into Equation (2.4) we get the relation (2.1) . □

Remark 1. Let $h(x)=x^k$ in Equation (2.1) , we obtain the single moments of the Makeham distribution

$$\mu_{r:n}^{(k)}=-P_2\mu_{r:n-1}^{(k)}+Q_2\mu_{r-1:n-1}^{(k)}+\left(\frac{k}{n}\right)E\left(\frac{X_{r:n}^{k-1}}{1+\theta\left(1-e^{-X_{r:n}}\right)}\right)$$

where $\mu_{r:n}^{(k)}=E\left(X_{r:n}^{k}\right)$.

Remark 2. For the special case $r=1$, $n=1$, we can find

$$\begin{aligned}\mu_{1:1} &= E\left(X_{1:1}\right)=-P_1P_2+Q_1Q_2+\frac{1}{P-Q}\int_{Q_1}^{P_1}e^{-x-\theta\left(x+e^{-x}-1\right)}dx\\ &= -P_1P_2+Q_1Q_2+E\left(\frac{1}{1+\theta\left(1-e^{-X_{r:n}}\right)}\right)\end{aligned}$$

where $\mu_{0:n}^{(k)}=Q_1^k$, and $\mu_{n:n-1}^{(k)}=P_1^k$

3 Recurrence relations for product moments of order statistics

Recurrence relations for the single moments of order statistics from the doubly truncated Maheham distribution are given by the following theorem.

Theorem 2. *Let $X_{r:n} \leq X_{r+1:n}, r = 1, 2, \ldots, n-1$ be an order statistics from a random sample of size n with pdf* (1.1) ,

$$\begin{aligned} \alpha_{r,s:n} &= \alpha_{r,s-1:n} - \frac{nP_2}{(n-s+1)} \left(\alpha_{r,s:n-1} - \alpha_{r,s-1:n-1}\right) \\ &\quad + \frac{1}{(n-s+1)} E\left(\frac{h'(X_{r:n}, X_{s:n})}{1+\theta\left(1-e^{-X_{r:n}}\right)}\right) \end{aligned} \tag{3.1}$$

where $h'(x, y) = \dfrac{\partial h(x, y)}{\partial y}$.

Proof. From (1.5)

$$\alpha_{r,s:n} - \alpha_{r,s-1:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s+1)!}$$

$$\times \int_{Q_1}^{P_1} \int_{x}^{P_1} h(x, y) [F(x)]^{r-1} [F(y) - F(x)]^{s-r-2} [1 - F(y)]^{n-s-1}$$

$$\times \left[(n-r) F(y) - (n+s-1) F(x) - (s-r-1)\right] f(x) f(y) \, dy dx$$

Suppose that

$$g(x, y) = -[F(y) - F(x)]^{s-r-1} [1 - F(y)]^{n-s+1},$$

then

$$\begin{aligned} \alpha_{r,s:n} - \alpha_{r,s-1:n} &= \frac{n!}{(r-1)!(s-r-1)!(n-s+1)!} \int_{Q_1}^{P_1} [F(x)]^{r-1} f(x) \\ &\quad \times \left\{\int_{x}^{P_1} \frac{\partial g(x, y)}{\partial y} h(x, y) \, dy\right\} dx \end{aligned}$$

By using integration by parts in the following integration

$$\int_{x}^{P_1} \frac{\partial g(x, y)}{\partial y} h(x, y) \, dy = [h(x, y) g(x, y)]_{x}^{P_1}$$

$$- \int_{x}^{P_1} h'(x, y) g(x, y) \, dy = - \int_{x}^{P_1} h'(x, y) g(x, y) \, dy$$

$$= \int_{x}^{P_1} h'(x, y) [F(y) - F(x)]^{s-r-1} [1 - F(y)]^{n-s+1} \, dy$$

Hence,

$$\alpha_{r,s:n} - \alpha_{r,s-1:n} = \frac{n!}{(r-1)!\,(s-r-1)!\,(n-s+1)!}$$
$$\times \int_{Q_1}^{P_1} \int_{x}^{P_1} h'(x,y)\,[F(x)]^{r-1}\,[F(y)-F(x)]^{s-r-1}\,[1-F(y)]^{n-s+1}$$
$$\times f(x)\,dydx, \quad 1 \le r \le s \le n-1$$

By using (1.2)

$$\alpha_{r,s:n} - \alpha_{r,s-1:n} = \frac{n!}{(r-1)!\,(s-r-1)!\,(n-s+1)!}$$
$$\times \int_{Q_1}^{P_1} \int_{x}^{P_1} h'(x,y)\,[F(x)]^{r-1}\,[F(y)-F(x)]^{s-r-1}\,[1-F(y)]^{n-s} \tag{3.2}$$
$$\times \left[\frac{f(y)}{1+\theta\,(1-e^{-y})} - P_2\right] f(x)\,dydx, \quad 1 \le r \le s \le n-1$$

Similarily, we can find that

$$\alpha_{r,s:n-1} - \alpha_{r,s-1:n-1} = \frac{(n-1)!}{(r-1)!\,(s-r-1)!\,(n-s)!}$$
$$\times \int_{Q_1}^{P_1} \int_{x}^{P_1} h'(x,y)\,[F(x)]^{r-1}\,[F(y)-F(x)]^{s-r-1}\,f(x)\,dydx,$$
$$1 \le r \le s \le n-1$$

Using the previous result in Equation (3.2)

$$\alpha_{r,s;n} - \alpha_{r,s-1;n} = \frac{-nP_2}{(n-s+1)}\left[\alpha_{r,s;n-1} - \alpha_{r,s-1;n-1}\right]$$
$$+\frac{n!}{(r-1)!\,(s-r-1)!\,(n-s+1)!}\int_{Q_1}^{P_1} \int_{x}^{P_1} h'(x,y)\,[F(x)]^{r-1}$$
$$\times [F(y)-F(x)]^{s-r-1}\,[1-F(y)]^{n-s}\left[\frac{f(y)}{1+\theta\,(1-e^{-y})}\right] f(x)\,dydx$$

then

$$\alpha_{r,s;n} - \alpha_{r,s-1;n} = \frac{-nP_2}{(n-s+1)}\left[\alpha_{r,s;n-1} - \alpha_{r,s-1;n-1}\right] + \frac{1}{(n-s+1)} E\left(\frac{h'\left(X_{r;n}, X_{s;n}\right)}{1+\theta\left(1-e^{-X_{r;n}}\right)}\right)$$

which completes the proof. □

Remark 3. If $h(x, y) = x^j y^k$, then (3.1) takes the form

$$\mu_{r,s:n}^{(j,k)} = \mu_{r,s-1:n}^{(j,k)} - \frac{nP_2}{n-s+1}\left[\mu_{r,s:n-1}^{(j,k)} - \mu_{r,s-1:n-1}^{(j,k)}\right] + \frac{k}{n-s+1} E\left(\frac{X_{r:n}^j X_{s:n}^{k-1}}{1+\theta\left(1-e^{-X_{r:n}}\right)}\right)$$

which represents the identities for the product moments for doubly truncated Makeham distribution.

Khan et al. [7, 8] established the following results

Remark 4. For $1 \leq r \leq s \leq n$ and $j > 0$

$$\mu_{r,s:n}^{(j,0)} = \mu_{r,s-1:n}^{(j,0)} = \cdots = \mu_{r,r+1:n}^{(j,0)} = \mu_{r:n}^{(j)}$$

$$\mu_{r,r:n}^{(j,k)} = \mu_{r:n}^{(j+k)}, \qquad 1 \leq r \leq n$$

$$\mu_{n-1,n:n-1}^{(j,k)} = P_1^k \mu_{n-1:n-1}^{(j)}$$

4 Characterization of Makeham distribution

We discuss in this section two theorems that characterize the truncated Makeham distribution using the properties of the order statistics.

The pdf of $(s-r)\,th$ order statistics of a sample of size $(n-r)$ is given by $(x \leq y)$

$$f\left(X_{s:n} \mid X_{r:n} = x\right) = \frac{(n-r)!\left[F(y) - F(x)\right]^{s-r-1}\left[1-F(y)\right]^{n-s} f(y)}{(n-s)!\,(s-r-1)!\left[1-F(x)\right]^{n-r}} \tag{4.1}$$

where, $f\left(X_{s:n}\left|X_{r:n}=x\right.\right)$ is the conditional density of $X_{s:n}$ given $X_{r:n} = x$ and the sample drawn from population with

$$pdf \frac{f(y)}{1-F(x)}, \quad cdf \frac{F(y)-F(x)}{1-F(x)} \quad \text{and} \quad x \leq y,$$

which is obtained from the truncated paraent distribution $F\,()$ at x.

In the case of the left truncation at x, we have

$$Q_1 = x, \quad P_1 = \infty, \quad P = 1, \quad Q = F(x), \quad P_2 = 0, \quad Q_2 = 1$$

and by putting $s = r + 1$, then (4.1) takes the form

$$f\left(X_{r+1:n}\left|X_{r:n}=x\right.\right) = \frac{(n-r)\left[1-F(y)\right]^{n-r-1} f(y)}{\left[1-F(x)\right]^{n-r}}, x \leq y \tag{4.2}$$

Similarily, if the parent distribution truncated from the right at y ($x \leq y$ and $r < s$), then

$$f\left(X_{r:n}\left|X_{s:n}=y\right.\right) = \frac{(s-1)!\left[F(x)\right]^{r-1}\left[F(y)-F(x)\right]^{s-r-1} f(x)}{(r-1)!\,(s-r-1)!\left[F(y)\right]^{s-1}} \tag{4.3}$$

In the case of the right truncation at y, we have

$$Q_1 = 0, \quad P_1 = x, \quad P = F(x), \quad Q = 0,$$
$$P_2 = \frac{1-F(x)}{F(x)}, \quad Q_2 = \frac{1}{F(x)}$$

and by putting $r = 1,\ s = 2$ then (4.3) takes the form

$$f\left(X_{1:n}\left|X_{2:n}=y\right.\right) = \frac{f(y)}{F(x)}, \ \ x \leq y\,.$$

Theorem 3. *If $F(x) < 1$, $(0 < x < \infty)$ is the cummulative distribution function of a random variable X and $F(0) = 0$, then*

$$\begin{aligned} 1-F(x) &= e^{-x-\theta\left(x+e^{-x}-1\right)} \\ \Longleftrightarrow\ & E\left(X_{r+1,n}+\theta\left(X_{r+1,n}+e^{-X_{r+1,n}}-1\right)\left|X_{r:n}=x\right.\right) \\ &= x+\theta\left(x+e^{-x}-1\right)+\frac{1}{n-r} \end{aligned}$$

Proof. The proof of the necessity condition starts by subsituting $h(x) = x + \theta\left(x + e^{-x} - 1\right)$, $r = 1$ in (2.1)

$$\begin{aligned} \alpha_{1;n} &= -P_2\alpha_{1;n-1} + Q_2\alpha_{0;n-1} + \frac{1}{n}E(1) \\ &= -P_2\alpha_{1:n-1} + Q_2\alpha_{0;n-1} + \frac{1}{n} \end{aligned}$$

which means that

$$E\left(X_{1:n} + \theta\left(X_{1:n} + e^{-X_{1:n}} - 1\right)\right) =$$

$$E\left(X_{1:n-1} + \theta\left(X_{1:n-1} + e^{-X_{1:n-1}} - 1\right)\right) E\left(X_{0:n-1} + \theta\left(X_{0:n-1} + e^{-X_{0:n-1}} - 1\right)\right) + \frac{1}{n}$$

In the case of the left truncation at x, we get

$$E\left(X_{1:n-r} + \theta\left(X_{1:n-r} + e^{-X_{1:n-r}} - 1\right)\right) =$$

$$E\left(X_{r+1:n} + \theta\left(X_{r+1:n} + e^{-X_{r+1:n}} - 1\right) | X_{r:n} = x\right) = x + \theta\left(x + e^{-x} - 1\right) + \frac{1}{n-r}$$

To prove the sufficient condition, we use (4.2) and (1.2)

$$(n-r)\int_x^\infty \left[x + \theta\left(x + e^{-x} - 1\right)\right]\left[1 - F(y)\right]^{n-r-1} f(y)\,dy =$$

$$\left[x + \theta\left(x + e^{-x} - 1\right) + \frac{1}{n-r}\right]\left[1 - F(x)\right]^{n-r}$$

By differentiating both side w.r.t. x, we get

$$\frac{f(x)}{1 - F(x)} = 1 + \theta\left(1 - e^{-x}\right).$$

$\square$

Theorem 4. *If* $F(x) < 1$, $(0 < x < \infty)$ *is the cummulative distribution function of a random variable* X *and* $F(0) = 0$*, then*

$$1 - F(x) = e^{-x-\theta\left(x+e^{-x}-1\right)} \Longleftrightarrow E\left(X_{1,n} + \theta\left(X_{1,n} + e^{-X_{1,n}} - 1\right) | X_{2:n} = x\right)$$
$$= -\frac{[1 - F(x)]}{F(x)}\left[x + \theta\left(x + e^{-x} - 1\right)\right] + \frac{1}{F(x)} + 1$$

Proof. To prove the necessity condition, let $n = 1$, $r = 1$ in (2.1)

$$\begin{aligned}\alpha_{1;1} &= -P_2\alpha_{1;0} + Q_2\alpha_{0;0} + E(1)\\ &= -P_2\alpha_{1:0} + Q_2\alpha_{0;0} + 1\end{aligned}$$

which means that

$$\begin{aligned}&E\left(X_{1:1} + \theta\left(X_{1:1} + e^{-X_{1:1}} - 1\right)\right)\\ &= -P_2 E\left(X_{1:0} + \theta\left(X_{1:0} + e^{-X_{1:0}} - 1\right)\right)\\ &\quad + Q_2 E\left(X_{0:0} + \theta\left(X_{0:0} + e^{-X_{0:0}} - 1\right)\right) + 1\\ &= -P_2\left[\mu_{1:0} + \theta\left(\mu_{1:0} + e^{-P_1} - 1\right)\right]\\ &\quad + Q_2\left[\mu_{0:0} + \theta\left(\mu_{0:0} + e^{-Q_1} - 1\right)\right]\end{aligned}$$

To prove the sufficient condition, by using (4.2) and (1.2)

$$\int_0^x \left[y + \theta\left(y + e^{-y} - 1\right)\right] f(y)\,dy = \\ -\left[1 - F(x)\right]\left[x + \theta\left(x + e^{-x} - 1\right)\right] + 1 + F(x)$$

Differentiating both sides w.r.t. x

$$\begin{aligned}\left[x + \theta\left(x + e^{-x} - 1\right)\right] f(x) &= -\left[1 - F(x)\right]\left[1 + \theta\left(1 - e^{-x}\right)\right]\\ &\quad + f(x)\left[x + \theta\left(x + e^{-x} - 1\right)\right] + f(x)\end{aligned}$$

Then

$$\left[1 - F(x)\right] = \frac{f(x)}{\left[1+\theta\left(1-e^{-x}\right)\right]} = e^{-x-\theta\left(x+e^{-x}-1\right)}$$

□

5 Some numerical results

According to Khan et al. [7, 8], we have the following special cases of the moments of order statistics for any distribution

$$\mu_{0:n}^{(k)} = Q_1^k$$

$$\mu_{n:n-1}^{(k)} = P_1^k,\ n \geq 1$$

$$\mu_{r,r:n}^{(j,k)} = \mu_{r:n}^{(j+k)},\ 1 \leq r \leq n$$

$$\mu_{n-1,n:n-1}^{(j,k)} = P_1^k \mu_{n-1:n-1}^{(j)}$$

n	r	$\mu_{r:n}^{(1)}$	n	r	$\mu_{r:n}^{(1)}$	n	r	$\mu_{r:n}^{(1)}$	n	r	$\mu_{r:n}^{(1)}$
1	1	2.6851		5	3.5551	8	1	2.0869		7	2.9118
2	1	2.3449	6	1	2.1158		2	2.1859		8	3.2500
	2	3.0252		2	2.2540		3	2.3009		9	3.9225
3	1	2.2307		3	2.4259		4	2.4384	10	1	2.0696
	2	2.5734		4	2.6538		5	2.6094		2	2.1467
	3	3.2511		5	2.9934		6	2.8363		3	2.2332
4	1	2.1733		6	3.6675		7	3.1749		4	2.3316
	2	2.4027	7	1	2.0993		8	3.8478		5	2.4461
	3	2.7440		2	2.2147	9	1	2.0773		6	2.5830
	4	3.4201		3	2.3525		2	2.1640		7	2.7534
5	1	2.1388		4	2.5239		3	2.2627		8	2.9797
	2	2.3113		5	2.7512		4	2.3774		9	3.3176
	3	2.5399		6	3.0902		5	2.5145		10	3.9897
	4	2.8802		7	3.7637		6	2.6852			

Table 1 – Some numerical values generated by the recurrence relations of order statistics from doubly truncated Makeham distribution.

These special cases are used as initial conditions for generating numerical values for the moments.

We implemented the two recurrence relations (2.1) and (3.1) using Matlab. The first table gives the numerical results for the single moments of order statistics for a random sample of size $n = 10$ from the doubly truncated Makeham distribution.

Acknowledgement. The authors would like to thank the anonymous referee and the associate editor for their comments and suggestions, which were helpful in improving the manuscript.

REFERENCES

[1] M.I. Hendi, S.E. Abu-Youssef and A.A. Alraddadi, *Order Statistics from Doubly Truncated Gompertz Distribution and its Characterizations*. The Egyptian Statistical Journal, **50**(1) (2006), 21–31.

[2] N. Blakrishnan and H.J. Malik, *Some general identities involving order statistics*. Commun. Statist. Theor. Meth., **14**(2) (1985), 333–339.

n	s	r	$\alpha_{r;s;n}$
1	1	1	84.6704
2	1	1	35.6337
	2	1	33.1467
		2	100.5605
3	1	1	16.8010
	2	1	24.6928
		2	48.6062
	3	2	44.8326
		3	104.1213
4	1	1	7.4132
	2	1	15.8482
		2	29.1163
	3	2	42.7296
		3	46.7313
	4	3	42.4265
		4	109.1091
5	1	1	3.0860
	2	1	8.7632
		2	15.9588
	3	2	34.1103
		3	31.7974
	4	3	46.5175
		4	41.1813
	5	4	36.4412
		5	116.9808

n	s	r	$\alpha_{r;s;n}$
6	1	1	1.2342
	2	1	4.3790
		2	7.9660
	3	2	22.6199
		3	20.6344
	4	3	44.0847
		4	28.2656
	5	4	41.0797
		5	37.3693
	6	5	31.9840
		6	126.5063
7	1	1	0.4800
	2	1	2.0433
		2	3.7166
	3	2	13.1861
		3	11.9965
	4	3	34.0627
		4	20.7973
	5	4	44.3909
		5	22.7690
	6	5	32.6611
		6	36.6772
	7	6	30.3789
		7	136.4146

n	s	r	$\alpha_{r;s;n}$
8	1	1	0.1828
	2	1	0.9081
		2	1.6517
	3	2	7.0318
		3	6.3953
	4	3	22.6896
		4	13.7684
	5	4	39.0893
		5	18.0539
	6	5	38.4569
		6	17.9067
	7	6	25.0835
		7	38.7534
	8	7	31.3204
		8	145.8919
9	1	1	0.0686
	2	1	0.3892
		2	0.7079
	3	2	3.5158
		3	3.1974
	4	3	13.6121
		4	8.2538
	5	4	29.2827
		5	13.3410
	6	5	37.8661
		6	14.2510

n	s	r	$\alpha_{r;s;n}$
	7	6	30.2252
		7	14.6970
	8	7	19.8457
		8	42.7916
	9	8	34.0730
		9	154.5203
10	1	1	0.0254
	2	1	0.1622
		2	0.2949
	3	2	1.6741
		3	1.5225
	4	3	7.5620
		4	4.5848
	5	4	19.5186
		5	8.8776
	6	5	31.4948
		6	11.5055
	7	6	32.6379
		7	10.6417
	8	7	22.3694
		8	13.2393
	9	8	17.0967
		9	48.0426
	10	9	37.9518
		10	162.1343

Table 2 – Some numerical values generated by the recurrence relations of order statistics from doubly truncated Makeham distribution.

[3] N. Balakrishnan and H.J. Malik, *Order statistics from the linear-exponential distribution, part I: Increasing hazard rate case.* Commun. Statist. Theor. Meth., **15**(1) (1986), 179–203.

[4] N. Blakrishnan, H.J. Malik and S.E. Ahmed, *Recurrence relations and identities for moments of order statistics, II: Specific continuous distributions.* Commun. Statist. Theor. Meth., **17**(8) (1988), 2657–2694.

[5] A.H. Khan and I.A. Khan, *Moments of order statistics from Burr distribution and its characterizations.* Metron, **45** (1987), 21–29.

[6] A.H. Khan and I.A. Khan, *Recurrence relations between moments of order statistics from generalized gamma distribution.* Journal of Statistical Research, **17**(1-2) (1983), 75–82.

[7] A.H. Khan, S. Parvez and M. Yaqub, *Recurrence relations between product moments of order statistics*. J. Statist. Plan. Inf., **8** (1983a), 175–183.

[8] A.H. Khan, M. Yaqub and S. Parvez *Recurrence relations between moments of order statistics*. Naval Res. Logist. Quart., **30** (1983b), 419–441.

[9] M.M. Mohie El-Din, M.A.W. Mahmoud, S.E. Abu-Youssef and K.S. Sultan, *Order Statistics from the doubly truncated linear exponential distribution and its characterizations.* Commun. Statist.-Simula., **26** (1997), 281–290.

[10] M.M. Mohie El-Din, M.A.W. Mahmoud and S.E. Abu-Youssef, *Moments of order statistics from parabolic and skewed distributions and characterization of Weilbull distribution.* Commun. Statist. Simul. Comput., **20**(2,3) (1991), 639–645.