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Simultaneous exact control of piezoelectric systems in multilayered media

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Abstract. This paper considers a pair of transmission problems for the system of piezoelectricity having piecewise constant coefficients. Under suitable monotonicity conditions on the coefficients and certain geometric conditions on the domain and the interfaces where the coefficients have a jump discontinuity, results on simultaneous boundary observation and simultaneous exact control are established.

Mathematical subject classification: 35L50; 35Q60; 35B40.

Key words: Piezoelectricity; transmission problem; simultaneous exact controllability.

1 Introduction

Throughout this paper Ω will be a bounded domain in $\mathbb{R}^3$ with sufficiently smooth boundary S. For $k = 1, 2, \cdots, n$, let $\mathcal{B}_k$ be open, bounded and connected subsets of Ω with smooth boundary S_k, and such that $\overline{\mathcal{B}}_k \subset \mathcal{B}_{k+1}$. We set

$$\Omega_0 = \mathcal{B}_1, \quad \Omega_k = \mathcal{B}_{k+1} \backslash \overline{\mathcal{B}}_k \ \text{ for } \ k = 1, 2, \cdots, n-1, \quad \Omega_n = \Omega \backslash \overline{\mathcal{B}}_n.$$

Assume that Ω is occupied by a linear multilayered piezoelectric body whose motion is governed by the following system ([4], [6])

$$\left\{ \begin{array}{lll} \rho\, \ddot{u} = \nabla \cdot T, & \mu \dot{H} = -\text{curl } E, & \dot{\mathcal{D}} = \text{curl } H, \\ \nabla \cdot \mathcal{D} = 0, & \nabla \cdot H = 0, & \\ T = c \cdot \widehat{\nabla} u - E \cdot e, & \mathcal{D} = e \cdot \widehat{\nabla} u + b \cdot E, & \end{array} \right. \tag{1.1}$$

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where ρ is the mass density, u is the displacement vector, $\widehat{\nabla} u$ is the symmetric part of ∇u, T is the stress tensor, H is the magnetic field, E is the electric field, $\mathcal{D}$ is the electric displacement vector, μ is the magnetic permeability, c, e, b are the elastic, piezoelectric and electric permittivity tensors respectively whose Cartesian components satisfy the following properties:

$$\begin{aligned} &c_{ijkl} = c_{jikl} = c_{klij}, \qquad b_{ij} = b_{ji}, \qquad e_{ijk} = e_{ikj}, \\ &b_{ij}\xi^j\xi^i \geq b_0|\xi|^2, \qquad b_0 > 0 \end{aligned}$$

for any real vector $\xi = (\xi^1, \xi^2, \xi^3) \in \mathbb{R}^3$,

$$c_{ijkl}\varepsilon_{kl}\varepsilon_{ij} \geq c_0\varepsilon_{ij}\varepsilon_{ij}, \qquad c_0 > 0$$

for any real symmetric matrices $\{\varepsilon_{ij}\}$ of order 3.

We introduce the following matrices of order 3

$$A_i = \{e_{khi}\}, \qquad B = \{b_{kh}\}, \qquad A_{ij} = \{a_{kh}^{ij}\},$$

where

$$a_{kh}^{ij} = (1 - \delta_{ih}\delta_{jk})c_{ikjh} + \delta_{ik}\delta_{jh}c_{ihjk}.$$

It follows from the symmetry of c_{ijkl} that

$$A_{ij}^* = A_{ji}.$$

Using these notations, we write the system (1.1) in the matrix form

$$\begin{cases} \rho \dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial}{\partial x_i}\left\{A_{ij}\dfrac{\partial u}{\partial x_j} - A_i^* E\right\} = 0, \\[2ex] \dfrac{\partial}{\partial t}\left\{BE + A_i\dfrac{\partial u}{\partial x_i}\right\} - \operatorname{curl} H = 0, \\[2ex] \mu \dfrac{\partial H}{\partial t} + \operatorname{curl}\ E = 0, \\[2ex] \operatorname{div}\ \left\{BE + A_i\dfrac{\partial u}{\partial x_i}\right\} = 0, \qquad \operatorname{div}\ H = 0. \end{cases}$$

We note that

$$\frac{\partial}{\partial x_i}\left\{A_{ij}\frac{\partial u}{\partial x_j}\right\} = \frac{\partial}{\partial x_j}\left\{c_{1jkh}\varepsilon_{kh}(u), c_{2jkh}\varepsilon_{kh}(u), c_{3jkh}\varepsilon_{kh}(u)\right\},$$

where

$$\varepsilon_{kh}(u) = \frac{1}{2}\left(\frac{\partial u^k}{\partial x_h} + \frac{\partial u^h}{\partial x_k}\right).$$

It is assumed that

$$\sum_{i,j=1}^{3}(A_{ij}\eta_j, \eta_i) \geq C_0 \sum_{i=1}^{3}|\eta_i|^2, \qquad C_0 > 0$$

for any real vector $\eta_i \in \mathbb{R}^3$. Here $(\cdot,\cdot)$ denotes the inner product in $\mathbb{R}^3$. We remark that this assumption holds for an isotropic medium ($c_{ijkh} = \widetilde{\lambda}\delta_{ij}\delta_{kh} + \widetilde{\mu}\delta_{ik}\delta_{jh} + \widetilde{\mu}\delta_{ih}\delta_{jk}$) with a constant $C_0 = \widetilde{\mu}$:

$$\sum_{i,j=1}^{3}(A_{ij}\eta_j, \eta_i) = (\widetilde{\lambda} + \widetilde{\mu})\left(\sum_{i=1}^{3}\eta_i^i\right)^2 + \widetilde{\mu}\sum_{i,j=1}^{3}(\eta_i^j)^2 \geq \widetilde{\mu}\sum_{i=1}^{3}|\eta_i|^2.$$

It is assumed that $c_{ijkh}(x), b_{ij}(x), \mu(x)$ are piecewise constant functions which lose the continuity on $S_1, S_2, \cdots, S_n$, ρ and e_{khi} are constants, $\rho > 0$.

We consider the following transmission problems

$$\left\{\begin{array}{l} \rho\,\dfrac{\partial^2 u}{\partial t^2} - \dfrac{\partial}{\partial x_i}\left\{A_{ij}\,\dfrac{\partial u}{\partial x_j} - A_i^* E\right\} = 0, \\[2ex] \dfrac{\partial}{\partial t}\left\{BE + A_i\dfrac{\partial u}{\partial x_i}\right\} - \operatorname{curl} H = 0, \\[2ex] \qquad (x,t) \in \Omega_m \times (0,T) \;\; m = 0, 1, \cdots, n \\[2ex] \mu\,\dfrac{\partial H}{\partial t} + \operatorname{curl}\; E = 0, \\[2ex] \operatorname{div}\;\left\{BE + A_i\dfrac{\partial u}{\partial x_i}\right\} = 0, \qquad \operatorname{div}\; H = 0. \end{array}\right. \tag{1.2}$$

$$u|_{t=0} = f_1(x), \quad \frac{\partial u}{\partial t}|_{t=0} = f_2(x), \quad E|_{t=0} = f_3(x), \quad H|_{t=0} = f_4(x), \tag{1.3}$$

$$\begin{cases} u^{(m-1)} = u^{(m)}, \left(A_{ij}^{(m-1)} \dfrac{\partial u^{(m-1)}}{\partial x_j} - A_i^* E^{(m-1)} \right) \nu_i \\ \quad = \left(A_{ij}^{(m)} \dfrac{\partial u^{(m)}}{\partial x_j} - A_i^* E^{(m)} \right) \nu_i, \\ \quad (x,t) \in S_m \times (0,T), \quad m = 1, \cdots, n \\ [\nu, E^{(m-1)}] = [\nu, E^{(m)}], \ [\nu, H^{(m-1)}] = [\nu, H^{(m)}], \end{cases} \tag{1.4}$$

$$\begin{aligned} &\left\{ \left(A_{ij} \frac{\partial u}{\partial x_j} - A_i^* E \right) \nu_i + \beta u \right\} |_{S\times(0,T)} = Q(x,t), \\ &\quad [E, \nu]|_{S\times(0,T)} = G(x,t). \end{aligned} \tag{1.5}$$

and

$$\begin{cases} \rho \dfrac{\partial^2 v}{\partial t^2} - \dfrac{\partial}{\partial x_i} \left\{ A_{ij} \dfrac{\partial v}{\partial x_j} - A_i^* \Phi \right\} = 0, \\ \dfrac{\partial}{\partial t} \left\{ B\Phi + A_i \dfrac{\partial v}{\partial x_i} \right\} - \text{curl } \Psi = 0, \\ \quad (x,t) \in \Omega_m \times (0,T) \quad m = 0, 1, \cdots, n \\ \mu \dfrac{\partial \Psi}{\partial t} + \text{curl } \Phi = 0, \\ \text{div } \left\{ B\Phi + A_i \dfrac{\partial v}{\partial x_i} \right\} = 0, \qquad \text{div } \Psi = 0, \end{cases} \tag{1.6}$$

$$v|_{t=0} = \varphi_1(x), \quad \frac{\partial v}{\partial t}|_{t=0} = \varphi_2(x), \quad \Phi|_{t=0} = \varphi_3(x), \quad \Psi|_{t=0} = \varphi_4(x), \tag{1.7}$$

$$\begin{cases} v^{(m-1)} = v^{(m)}, \left(A_{ij}^{(m-1)} \dfrac{\partial v^{(m-1)}}{\partial x_j} - A_i^* \Phi^{(m-1)} \right) \nu_i \\ \quad = \left(A_{ij}^{(m)} \dfrac{\partial v^{(m)}}{\partial x_j} - A_i^* \Phi^{(m)} \right) \nu_i, \\ \quad (x,t) \in S_m \times (0,T), \quad m = 1, \cdots, n \\ [\nu, \Phi^{(m-1)}] = [\nu, \Phi^{(m)}], \ [\nu, \Psi^{(m-1)}] = [\nu, \Psi^{(m)}], \end{cases} \tag{1.8}$$

$$v\,|_{S\times(0,t)}=\mathcal{D}(x,t),\qquad [\Psi,\nu]\,|_{S\times(0,T)}=\mathcal{P}(x,t),\tag{1.9}$$

where $[\cdot,\cdot]$ is the vector product, $\nu=\nu(x)=(\nu_1,\nu_2,\nu_3)$ (for $x\in S_m,\ x\in S$) is the unit normal vector pointing into the exterior of $\mathcal{B}_m$ or Ω; $A_{ij}^{(m)}$, $u^{(m)}$, $v^{(m)}$, $E^{(m)}$, $\Phi^{(m)}$, $H^{(m)}$, $\Psi^{(m)}$ are the restrictions of the corresponding matrices and vector-functions on Ω_m. In (1.5) $\beta=\beta(x)$ is a continuously differentiable positive function on S.

The problem of exact boundary control for the system (1.2)–(1.5) ((1.6)–(1.9)) is formulated as follows:

Given the initial distribution $f=\{f_1,f_2,f_3,f_4\}(\varphi=\{\varphi_1,\varphi_2,\varphi_3,\varphi_4\})$, time $T>0$, and a desired terminal state $g=\{g_1,g_2,g_3,g_4\}(\psi=\{\psi_1,\psi_2,\psi_3,\psi_4\})$ with f, $g(\varphi,\psi)$ in appropriate function spaces, find a vector-valued functions $Q(x,t)$, $G(x,t)(\mathcal{D}(x,t),\mathcal{P}(x,t))$ in a suitable function spaces such that the solution of (1.2)–(1.5) ((1.6)–(1.9)) satisfies the conditions

$$\left\{u,\frac{\partial u}{\partial t},E,H\right\}|_{t=T}=g(x)\left(\left\{v,\frac{\partial v}{\partial t},\Phi,\Psi\right\}|_{t=T}=\psi(x)\right).$$

Our purpose is to obtain simultaneous exact boundary control of these problems, $\{\mathcal{D}(x,t),\mathcal{P}(x,t)\}$ serving as a control in problem (1.6)–(1.9), while the vector-valued functions

$$Q(x,t)=\frac{\partial}{\partial t}\,\mathcal{D}(x,t),\,G(x,t)=[\nu,\mathcal{P}(x,t)]$$

is a control in (1.2)–(1.5).

Spatial energy estimates for a semi-infinite piezoelectric beam have been studied by A. Borrelli and M.C. Patria [2].

Boundary controllability for some partial case of the system (1.1) with another boundary and interface conditions was investigated in [14].

For $A_i\equiv 0$, the piezoelectric system (1.2) decouples into a pair of hyperbolic systems: the Maxwell system and the hyperbolic system of second order.

The exact controllability problem for the Maxwell system has been studied by D. Russell [27] for a circular cylindrical region, by K. Kime [16] for a spherical region, and by J. Lagnese [20] for a general region. Stabilization for the Maxwell system with the Silver-Müller absorbing boundary conditions and exact controllability for corresponding initial boundary value problem have been studied by

V. Komornik [17], P. Martinez [24] and N. Weck [28]. The uniform exponential decay of solutions of Maxwell's equations with boundary dissipation and exact boundary controllability was proved in [7], [8].

Stabilization and exact boundary controllability for the system of elasticity have been studied by J. Lagnese [18], [19], F. Alabau and V. Komornik [1] and M. Horn [5] among others. In [7], [9] boundary observation, stabilization and exact controllability were studied for a class of hyperbolic systems which includes the system of elasticity.

Boundary controllability in transmission problems for a class of second order hyperbolic systems has been studied by J. Lagnese [19]. Uniform stabilization and exact control for the Maxwell system in multilayered media were investigated in [8]. The question of boundary controllability in transmission problems for the wave equation has been considered by J.-L. Lions [23], and S. Nicaise [25], [26].

The main novelty of this note is that we study the simultaneous exact control. Simultaneous exact control for the wave equation has been established by D. Russell [27] for a circular cylindrical region and by J.-L. Lions [22], F. Khodja and A. Bader [15].

In [9]–[13] simultaneous controllability were studied for a class of hyperbolic systems of second order, for a pair of Maxwell's equations and for a class of evolution systems which includes the Schrödinger equation.

This article is organized as follows: simultaneous boundary observation for the problems (1.2)–(1.5) and (1.6)–(1.9) with zero boundary conditions ($\mathcal{Q} \equiv G \equiv \mathcal{D} \equiv \mathcal{P} \equiv 0$) is established in Section 2. In Section 3 the simultaneous exact controllability is studied by means of the Hilbert Uniqueness Method, introduced by J.-L. Lions [21], [22].

2 Boundary observability

Throughout this paper $H_k(\Omega)$ and $H_q(S)$ denote the usual Sobolev spaces.

We denote by $\mathcal{H}_0$ the Hilbert space of pairs $u = \{u_1(x), u_2(x)\}$ of three-component vector-valued functions

$$u_i \in L_2(\Omega),\ \ \operatorname{curl} u_i \in L_2(\Omega_m),\ \ m = 0, 1, \cdots, n,$$

with the inner product

$$\langle u, v\rangle_0 = \sum_{m=0}^{n} \int_{\Omega_m} \big\{(\text{curl}\, u_1, \text{curl}\, v_1) + (\text{curl}\, u_2, \text{curl}\, v_2) + (B^{(m)} u_1, v_1) + \mu^{(m)}(u_2, v_2)\big\} dx.$$

From the results of [3] it follows that the expressions $[\nu, u_1]$, $[\nu, u_2]$, $(\nu = \nu(x), x \in S, x \in S_m, m = 1, 2, \cdots, n)$ are well defined on S, S_m and belong to $H_{-\frac{1}{2}}(S)$, $H_{-\frac{1}{2}}(S_m)$.

This enables us to introduce in $\mathcal{H}_0$ the closed subspaces $\mathcal{H}_1$, $\widetilde{\mathcal{H}}_1$:

$$\begin{aligned}\mathcal{H}_1 &= \big\{u = \{u_1, u_2\} \in \mathcal{H}_0 : [\nu, u_1^{(m-1)}] = [\nu, u_1^{(m)}], [\nu, u_2^{(m-1)}] \\ &= [\nu, u_2^{(m)}] \text{ on } S_m, \quad m = 1, 2, \cdots, n; \ [\nu, u_1] = 0 \text{ on } S\big\},\end{aligned}$$

the space $\widetilde{\mathcal{H}}_1$ is defined just as $\mathcal{H}_1$ with the only difference that $[\nu, u_2]$ vanishes on S.

We denote by $\mathcal{H}$ the real Hilbert space of quadruples $w = \{w_1, w_2, w_3, w_4\}$ of three-component vector-valued functions $w_i(x)$ such that

$$w_1^{(m)} \in H_1(\Omega_m), w_2^{(m)}, w_3^{(m)}, w_4^{(m)} \in L_2(\Omega_m), \qquad m = 0, 1, \cdots, n,$$

where $w_i^{(m)}$ is the restriction of w_i on Ω_m. The inner product in $\mathcal{H}$ is given by

$$\begin{aligned}\langle v, w\rangle_{\mathcal{H}} = \sum_{m=0}^{n} \int_{\Omega_m} \bigg\{\bigg(A_{ij}^{(m)} \frac{\partial v_1^{(m)}}{\partial x_j}, \frac{\partial w_1^{(m)}}{\partial x_i}\bigg) + \rho(v_2^{(m)}, w_2^{(m)}) \\ + (B^{(m)} v_3^{(m)}, w_3^{(m)}) + \mu^{(m)}(v_4^{(m)}, w_4^{(m)})\bigg\} dx + \int_S \beta(v_1, w_1) dS.\end{aligned}$$

The space $\widetilde{\mathcal{H}}$ is defined just as $\mathcal{H}$ with the only difference that the first vector-valued function w_1 vanishes on S.

In $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ we define unbounded operators $\mathcal{A}$ and $\widetilde{\mathcal{A}}$:

$\mathcal{D}(\mathcal{A})$ consists of the elements $u = \{u_1, u_2, u_3, u_4\} \in \mathcal{H}$ such that

$$A_{ij}^{(m)} \frac{\partial u_1^{(m)}}{\partial x_j} - A_i^* u_3^{(m)} \in H_1(\Omega_m), \ u_2^{(m)} \in H_1(\Omega_m), \ \{u_3, u_4\} \in \mathcal{H}_1,$$

$$\bigg(A_{ij} \frac{\partial u_1}{\partial x_j} - A_i^* u_3\bigg) \nu_i + \beta u_1 = 0, [u_3, \nu] = 0 \text{ on } S,$$

$$\begin{cases} u_1^{(m-1)} = u_1^{(m)}, \qquad x \in S_m, \quad m = 1, 2, \cdots, n \\ \left(A_{ij}^{(m-1)} \dfrac{\partial u_1^{(m-1)}}{\partial x_j} - A_i^* u_3^{(m-1)} \right) \nu_i = \left(A_{ij}^{(m)} \dfrac{\partial u_1^{(m)}}{\partial x_j} - A_i^* u_3^{(m)} \right) \nu_i . \end{cases}$$

$$\begin{aligned} \mathcal{A}u \;=\; & \left\{ u_2, \rho^{-1} \frac{\partial}{\partial x_i} \left(A_{ij} \frac{\partial u_1}{\partial x_j} - A_i^* u_3 \right), \right. \\ & \left. B^{-1} \left(\text{curl}\, u_4 - A_i \frac{\partial u_2}{\partial x_i} \right), -\mu^{-1} \text{curl}\, u_3 \right\} \end{aligned}$$

for $u = \{u_1, u_2, u_3, u_4\} \in \mathcal{D}(\mathcal{A})$.

The operator $\widetilde{\mathcal{A}}$ is defined just as $\mathcal{A}$ with the only difference that elements

$$v = \{v_1, v_2, v_3, v_4\} \in \mathcal{D}(\widetilde{\mathcal{A}}) \ (v \in \widetilde{\mathcal{H}}, \{v_3, v_4\} \in \widetilde{\mathcal{H}}_1)$$

satisfy another boundary conditions

$$v_1 = v_2 = 0, \quad [\nu, v_4] = 0 \quad \text{on} \quad S.$$

The skew-selfadjointness of $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ can be verified in the standard way.

Let $\mathcal{U}(t)$ and $\widetilde{\mathcal{U}}(t)$ be the strongly continuous groups of unitary operators generated by $\mathcal{A}$ and $\widetilde{\mathcal{A}}$.

We set

$$\begin{aligned} M &= \{w \in \mathcal{D}(\mathcal{A}^*) : \mathcal{A}^* w = 0\}, \\ \widetilde{M} &= \{w \in \mathcal{D}(\widetilde{\mathcal{A}}^*) : \widetilde{\mathcal{A}}^* w = 0\}. \end{aligned}$$

Denote by M_1 and $\widetilde{M}_1$ the orthogonal complements of M and $\widetilde{M}$ in $\mathcal{H}$ and $\widetilde{\mathcal{H}}$ respectively.

Let us consider the problem (1.2)–(1.5) with homogeneous boundary conditions ($Q \equiv G \equiv 0$). The kernel M of $\mathcal{A}^*$ is nonempty, since it contains the quadruples $w = \{w_1, 0, \nabla g_1, \nabla g_2\}$, where $g_1 \in H_2(\Omega)$, $g_2 \in H_2(\Omega) \cap \overset{\circ}{H}_1(\Omega)$, w_1 is a solution of the following problem

$$\frac{\partial}{\partial x_i} \left(A_{ij} \frac{\partial w_1}{\partial x_j} \right) = \frac{\partial}{\partial x_i} (A_i^* \nabla g_1), \qquad x \in \Omega_m, \qquad m = 0, 1, \cdots, n$$

$$\begin{cases} w_1^{(m-1)} = w_1^{(m)}, \qquad x \in S_m, \quad m = 1, 2, \cdots, n \\ \left(A_{ij}^{(m-1)} \dfrac{\partial w_1^{(m-1)}}{\partial x_j} \right) \nu_i = A_{ij}^{(m)} \dfrac{\partial w_1^{(m)}}{\partial x_j} \nu_i, \end{cases} \tag{2.1}$$

$$\left(A_{ij} \frac{\partial w_1}{\partial x_j} \nu_i + \beta w_1 \right) |_S = A_k^* \nabla g_1 \nu_k \ |_S \ .$$

It is obvious that $\mathcal{U}(t)$ takes $M_1 \cap \mathcal{D}(\mathcal{A})$ into itself. Indeed, if $w \in M$ and $v \in M_2 \cap \mathcal{D}(\mathcal{A})$, then

$$\frac{d}{dt} \langle \mathcal{U}(t)v, w \rangle_{\mathcal{H}} = \langle \mathcal{A}\mathcal{U}(t)v, w \rangle_{\mathcal{H}} = \left\langle \mathcal{U}(t)v, \mathcal{A}^* w \right\rangle_{\mathcal{H}} = 0.$$

We remark that element $v = \{v_1, v_2, v_3, v_4\} \in M_1 \cap \mathcal{D}(\mathcal{A})$ possess the following property:

$$\text{div} \left\{ B^{(m)} v_3^{(m)} + A_k \frac{\partial v_1^{(m)}}{\partial x_k} \right\} = 0, \quad \text{div } v_4^{(m)} = 0, \quad m = 0, 1, \cdots, n \tag{2.2}$$

in the sense of distributions.

Indeed, element $w = \{w_1, 0, \nabla g_1, 0\}$ where $g_1 \in H_2(\Omega)$, supp $g_1 \subset \Omega_m$, w_1 is a solution of the problem (2.1), belongs to M. Let $v \in M_1 \cap \mathcal{D}(\mathcal{A})$. Then

$$0 = \langle v, w \rangle_{\mathcal{H}} = \int_{\Omega_m} \left(A_k \ \frac{\partial v_1^{(m)}}{\partial x_k} + B^{(m)} v_3^{(m)}, \nabla g_1 \right) dx$$

for an arbitrary $g_1 \in H_2(\Omega)$, supp $g_1 \subset \Omega_m$, which implies

$$\text{div} \left\{ B^{(m)} v_3^{(m)} + A_k \frac{\partial v_1^{(m)}}{\partial x_k} \right\} = 0$$

in the sense of distributions.

It can be shown in a similar way (element $\{0, 0, 0, \nabla g_2\}$ belongs to M for an arbitrary $g_2 \in H_2(\Omega)$, supp $g_2 \subset \Omega_m$) that div $v_4^{(m)} = 0$ in the sense of distributions.

Let us show that elements $v = \{v_1, v_2, v_3, v_4\} \in M_1 \cap \mathcal{D}(\mathcal{A})$ satisfy the boundary condition

$$(v_4, \nu) \ |_S = 0. \tag{2.3}$$

We note that element $w = \{0, 0, 0, \nabla g_2\}$ belongs to the kernel of $\mathcal{A}^*$ for an arbitrary $g_2 \in H_2(\Omega)$, $g_2 = 0$ in $\overline{\Omega}_0 \cup \overline{\Omega}_1 \cup \cdots \cup \overline{\Omega}_{n-1}$.

Thus, for $v = \{v_1, v_2, v_3, v_4\} \in M_1 \cap \mathcal{D}(\mathcal{A})$ we have

$$0 = \langle v, w\rangle_{\mathcal{H}} = \int_{\Omega_n} \mu\,(v_4, \nabla g_2)\,dx = \int_S \mu g_2\,(v_4, \nu)\,dS,$$

which implies (2.3).

Our next goal is to show that elements $v = \{v_1, v_2, v_3, v_4\} \in M_1 \cap \mathcal{D}(\mathcal{A})$ satisfy the following interface conditions

$$\begin{cases} \mu^{(m-1)}(v_4^{(m-1)}, \nu) = \mu^{(m)}(v_4^{(m)}, \nu) \qquad x \in S_m, \quad m = 1, 2, \cdots, n \\ \left(B^{(m-1)}v_3^{(m-1)} + A_k \dfrac{\partial v_1^{(m-1)}}{\partial x_k}, \nu\right) = \left(B^{(m)}v_3^{(m)} + A_k \dfrac{\partial v_1^{(m)}}{\partial x_k}, \nu\right). \end{cases} \tag{2.4}$$

Since $w = \{0, 0, 0, \nabla g_2\}$ belongs to the kernel of $\widetilde{\mathcal{A}}^*$ for an arbitrary $g_2 \in H_2(\Omega) \cap \overset{\circ}{H}_1(\Omega)$, it follows that

$$\begin{aligned} 0 \;=\; \langle v, w\rangle_{\mathcal{H}} &= \sum_{m=0}^{n} \int_{\Omega_m} \mu^{(m)}(v_4^{(m)}, \nabla g_2)dx \\ &= \int_{S_1} \mu^{(0)}(v_4^{(0)}, \nu)g_2\,dS - \int_{S_1} \mu^{(1)}(v_4^{(1)}, \nu)g_2\,dS \\ &\quad + \cdots + \int_{S_n} \mu^{(n-1)}(v_4^{(n-1)}, \nu)g_2\,dS - \int_{S_n} \mu^{(n)}(v_4^{(n)}, \nu)g_2\,dS. \end{aligned}$$

Now we choose g_2 such that $g_2 = 0$ on $S_1, \cdots, S_{m-1}, S_{m+1}, \cdots, S_n$. Then

$$\int_{S_m} \left\{\mu^{(m-1)}(v_4^{(m-1)}, \nu) - \mu^{(m)}(v_4^{(m)}, \nu)\right\} g_2\,dS = 0$$

and we have

$$\mu^{(m-1)}(v_4^{(m-1)}, \nu) = \mu^{(m)}(v_4^{(m)}, \nu) \;\text{ on }\; S_m,\; m = 1, 2, \cdots, n.$$

Moreover, element $w = \{w_1, 0, \nabla g_1, 0\}$ belongs to the kernel of $\mathcal{A}^*$ for an arbitrary $g_1 \in \overset{\circ}{H}_2(\Omega)$ (w_1 is a solution of (2.1)). We have

$$
\begin{aligned}
0 = \langle v, w\rangle_{\mathcal{H}} &= \sum_{m=0}^{n} \int_{\Omega_m} \left(A_k \frac{\partial v_1^{(m)}}{\partial x_k} + B^{(m)} v_3^{(m)}, \nabla g_1 \right) dx \\
&= \sum_{m=1}^{n} \int_{S_m} \left\{ \left(B^{(m-1)} v_3^{(m-1)} + A_k \frac{\partial v_1^{(m-1)}}{\partial x_k}, \nu \right) \right. \\
&\quad \left. - \left(B^{(m)} v_3^{(m)} + A_k \frac{\partial v_1^{(m)}}{\partial x_k}, \nu \right) \right\} g_1 \, dS.
\end{aligned}
$$

We choose g_1 such that $g_1 = 0$ on $S_1, \cdots, S_{m-1}, S_{m+1}, \cdots, S_n$. This gives us that

$$
\left(B^{(m-1)} v_3^{(m-1)} + A_k \frac{\partial v_1^{(m-1)}}{\partial x_k}, \nu \right) = \left(B^{(m)} v_3^{(m)} + A_k \frac{\partial v_1^{(m)}}{\partial x_k}, \nu \right) \quad \text{on} \quad S_m .
$$

Let us consider now the problem (1.6)–(1.9) with homogeneous boundary conditions ($\mathcal{D} \equiv \mathcal{P} \equiv 0$). We remark that the kernel $\widetilde{M}$ of $\widetilde{\mathcal{A}}^*$ contains the quadruples $w = \{w_1, 0, \nabla g_1, \nabla g_2\}$, where $g_1 \in H_2(\Omega)$, $g_2 \in H_2(\Omega) \cap \overset{\circ}{H}_1(\Omega)$, w_1 is a solution of (2.1) with the only difference that the functions w_1 satisfy the boundary condition:

$$
w_1 \mid_S = 0. \tag{2.5}
$$

It can be shown in the same way that elements $v = \{v_1, v_2, v_3, v_4\} \in \widetilde{M}_1 \cap \mathcal{D}(\widetilde{\mathcal{A}})$ satisfy (2.2), (2.4).

Let us show that elements $v = \{v_1, v_2, v_3, v_4\} \in \widetilde{M}_1 \cap \mathcal{D}(\widetilde{\mathcal{A}})$ satisfy the additional boundary condition

$$
\left(B v_3 + A_k \frac{\partial v_1}{\partial x_k}, \nu \right) \mid_S = 0. \tag{2.6}
$$

We remark that element $w = \{w_1, 0, \nabla g_1, 0\}$ belongs to the kernel of $\widetilde{\mathcal{A}}^*$ for an arbitrary $g_1 \in H_2(\Omega)$, $g_1 \equiv 0$ in $\overline{\Omega}_0 \cup \overline{\Omega}_1 \cup \cdots \cup \overline{\Omega}_{n-1}$, w_1 is a solution of (2.1) with boundary condition (2.5).

Thus, for $\{v_1, v_2, v_3, v_4\} \in \widetilde{M}_1 \cap \mathcal{D}(\widetilde{\mathcal{A}})$ we have

$$0 = \langle v, w\rangle_{\widetilde{\mathcal{H}}} = \int_{\Omega_n} \left(A_k \frac{\partial v_1}{\partial x_k} + B v_3^{(m)}, \nabla g_1 \right) dx$$
$$= \int_S g_1 \left(B v_3 + A_k \frac{\partial v_1}{\partial x_k}, \nu \right) dS,$$

which implies (2.6).

We arrive at the following assertion.

Theorem 2.1. *Suppose that* $f = \{f_1, f_2, f_3, f_4\} \in M_1 \cap \mathcal{D}(\mathcal{A})$ $(\varphi = \{\varphi_1, \varphi_2, \varphi_3, \varphi_4\} \in \widetilde{M}_1 \cap \mathcal{D}(\widetilde{\mathcal{A}}))$. *Then there exists a unique solution* $\{u(x,t), E(x,t), H(x,t)\}$ $(v(x,t), \Phi(x,t), \Psi(x,t))$ *of* (1.2)–(1.5) ((1.6)–(1.9)) *with zero boundary conditions such that for all* $t \geq 0$

$$u(x,t) \in H_2(\Omega_m), \frac{\partial u}{\partial t}(x,t), E(x,t), H(x,t) \in H_1(\Omega_m), \frac{\partial^2 u}{\partial t^2}(x,t) \in L_2(\Omega_m)$$

$$\left(v(x,t) \in H_2(\Omega_m), \frac{\partial v}{\partial t}(x,t), \Phi(x,t), \Psi(x,t) \in H_1(\Omega_m), \frac{\partial^2 v}{\partial t^2}(x,t) \in L_2(\Omega_m) \right),$$

$m = 0, 1, \cdots, n;$

$$(H, \nu) = 0 \left((B\Phi + A_k \frac{\partial v}{\partial x_k}, \nu) = 0 \right), \qquad x \in S, \quad t \geq 0.$$

Moreover, $\{u, E, H\}(\{v, \Phi, \Psi\})$ satisfies the additional interface conditions (2.4), where

$$v_1 = u, \quad v_3 = E, \quad v_4 = H \quad (v_1 = v, v_3 = \Phi, v_4 = \Psi)$$

and

$$||\{u, \frac{\partial u}{\partial t}, E, H\}||_{\mathcal{H}} = ||f||_{\mathcal{H}} \left(||\{v, \frac{\partial v}{\partial t}, \Phi, \Psi\}||_{\widetilde{\mathcal{H}}} = ||\varphi||_{\widetilde{\mathcal{H}}} \right).$$

Let $f = \{f_1, f_2, f_3, f_4\} \in \mathcal{H}$ and $f^n = \{f_1^n, f_2^n, f_3^n, f_4^n\} \in \mathcal{D}(\mathcal{A})$, such that $||f - f^n||_{\mathcal{H}} \to 0$ as $n \to \infty$. Then, $\mathcal{U}(t)f^n$ satisfies the following identity

$$\int_0^T \left\{ \left\langle \mathcal{U}(t)f^n, \frac{dV}{dt} \right\rangle_{\mathcal{H}} + \left\langle \mathcal{U}(t)f^n, \mathcal{A}^* V \right\rangle_{\mathcal{H}} \right\} dt = -\left\langle f^n, V(0) \right\rangle_{\mathcal{H}},$$

where $V(t) \in L_2(0, T; \mathcal{D}(\mathcal{A}^*))$, $V_t(t) \in L_2(0, T; \mathcal{H})$, $V(T) = 0$.

From this we easily obtain that

$$\int_0^T \left\{ \left\langle \mathcal{U}(t) f, \frac{dV}{dt} \right\rangle_{\mathcal{H}} + \left\langle \mathcal{U}(t) f, \mathcal{A}^* V \right\rangle_{\mathcal{H}} \right\} dt = - \langle f, V(0) \rangle_{\mathcal{H}} , \qquad (2.7)$$

i.e., $\mathcal{U}(t) f$ is the weak solution of the abstract Cauchy problem

$$\frac{du}{dt} = \mathcal{A} u, \qquad u|_{t=0} = f.$$

We note that $\mathcal{U}(t)$ takes M_1 into itself. Indeed, if $g \in M$ and $V(t) = (T - t) g$, then from (2.7) it follows that

$$\int_0^T \langle \mathcal{U}(t) f, g \rangle_{\mathcal{H}} \, dt = T \langle f, g \rangle_{\mathcal{H}} .$$

Thus,

$$\langle \mathcal{U}(t) f, g \rangle_{\mathcal{H}} = \langle f, g \rangle_{\mathcal{H}} \quad \text{for} \quad t \geq 0.$$

In the same way we get the corresponding properties for $\widetilde{\mathcal{U}}(t)$.

Let us now concern ourselves with the simultaneous boundary observability for a pair of piezoelectric systems. The proof is based on the invariance of the piezoelectric system relative to the one-parameter group of dilations in all variables. This property of the system leads to the following identity:

$$\begin{aligned}
&2 \left(t \, \frac{\partial u}{\partial t} + (\nabla g, \nabla) u + u, \rho \, \frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x_i} \left\{ A_{ij} \, \frac{\partial u}{\partial x_j} - A_i^* E \right\} \right) \\
&+ 2 \left(t E + \mu [\nabla g, H], \frac{\partial}{\partial t} \left\{ B E + A_i \frac{\partial u}{\partial x_i} \right\} - \text{curl } H \right) \\
&+ 2 \left(t H - \left[\nabla g, B E + A_k \frac{\partial u}{\partial x_k} \right], \mu \frac{\partial H}{\partial t} + \text{curl } E \right) \\
&+ 2 (E, \nabla g) \text{ div } \left\{ B E + A_k \frac{\partial u}{\partial x_k} \right\} + 2 \mu (H, \nabla g) \text{ div } H \\
&= \frac{\partial}{\partial t} \left\{ t \left[\rho \left| \frac{\partial u}{\partial t} \right|^2 + \left(A_{ij} \frac{\partial u}{\partial x_j} \, \frac{\partial u}{\partial x_i} \right) + (B E, E) + \mu |H|^2 \right] \right. \\
&\left. + 2 \rho \left(\frac{\partial u}{\partial t}, (\nabla g, \nabla) u + u \right) + 2 \mu \left([\nabla g, H], B E + A_k \frac{\partial u}{\partial x_k} \right) \right\}
\end{aligned}$$

$$
\begin{aligned}
&-\frac{\partial}{\partial x_i}\left\{2\left(t\,\frac{\partial u}{\partial t}+(\nabla g,\nabla)u+u,\,A_{ij}\frac{\partial u}{\partial x_j}-A_i^*E\right)\right.\\
&\left.+\frac{\partial g}{\partial x_i}\left[\rho\,\left|\frac{\partial u}{\partial t}\right|^2-\left(A_{pq}\,\frac{\partial u}{\partial x_q},\,\frac{\partial u}{\partial x_p}\right)\right]\right\}\\
&-\operatorname{div}\left\{2t[H,E]+\nabla g(BE,E)+\nabla g\mu|H|^2-2BE(E,\nabla g)\right.\\
&\left.-2\mu H(H,\nabla g)+2\left[E\left[\nabla g,\,A_k\frac{\partial u}{\partial x_k}\right]\right]\right\}\\
&-\left\{(\Delta g-1)\left(A_{ij}\frac{\partial u}{\partial x_j},\,\frac{\partial u}{\partial x_i}\right)-2\frac{\partial^2 g}{\partial x_p\partial x_i}\left(A_{ij}\frac{\partial u}{\partial x_j},\,\frac{\partial u}{\partial x_p}\right)+(3-\Delta g)\rho\left|\frac{\partial u}{\partial t}\right|^2\right\}\\
&-\left\{2\frac{\partial^2 g}{\partial x_i\partial x_k}b_{ij}E^jE^k-(\Delta g-1)(BE,E)+2\frac{\partial^2 g}{\partial x_i\partial x_j}\mu H^iH^j-(\Delta g-1)\mu|H|^2\right\}\\
&-2\left(E,\,\frac{\partial^2 g}{\partial x_i\partial x_k}A_k\frac{\partial u}{\partial x_i}+\left(A_k\frac{\partial u}{\partial x_k},\nabla\right)\nabla g-(\Delta g-1)A_k\frac{\partial u}{\partial x_k}\right),
\end{aligned}
\tag{2.8}
$$

where $g(x)$ is an arbitrary smooth function, $\nabla=\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial x_2},\frac{\partial}{\partial x_3}\right)$. For $g(x)=2^{-1}|x-x^0|^2$, (2.8) represents a conservation law.

Let $f=\{f_1,f_2,f_3,f_4\}\in M_1\cap\mathcal{D}(\mathcal{A})$ and $\{u(x,t),E(x,t),H(x,t)\}$ is the corresponding solution of (1.2)–(1.5) with zero boundary conditions.

From (2.8) after integration over $\Omega_m\times(0,T)$ and summation over m we get

$$
\begin{aligned}
&T\sum_{m=0}^{n}\int_{\Omega_m}\left\{\rho\left|\frac{\partial u}{\partial t}\right|^2+\left(A_{ij}\frac{\partial u}{\partial x_j},\,\frac{\partial u}{\partial x_i}\right)+(BE,E)+\mu|H|^2\right\}dx\\
&\quad+\;2\sum_{m=0}^{n}\int_{\Omega_m}\left\{\rho\left(\frac{\partial u}{\partial t},(\nabla g,\nabla)u+u\right)\right.\\
&\quad\left.+\mu\left([\nabla g,H],\,BE+A_k\frac{\partial u}{\partial x_k}\right)\right\}dx\Big|_{t=0}^{t=T}\\
&\quad=\sum_{m=1}^{n}\int_0^T\int_{S_m}\{\mathcal{B}_{m-1}(u,E,H)-\mathcal{B}_m(u,E,H)\}\,dS\,dt\\
&\quad+\int_0^T\int_S\mathcal{B}_n(u,E,H)dS\,dt+\sum_{m=1}^{n}\int_0^T\int_{\Omega_m}\mathcal{F}_m(u,E,H;g)dx\,dt,
\end{aligned}
\tag{2.9}
$$

where $\mathcal{F}_m(u,E,H;g)$ is the restriction of the last three terms on the right-hand side of (2.8) on Ω_m and

$$
\begin{aligned}
\mathcal{B}_m(u,E,H) &= 2\left(t\,\frac{\partial u^{(m)}}{\partial t} + (\nabla g,\nabla)u^{(m)}\right.\\
&+ \left. u^{(m)}, A_{ij}^{(m)}\frac{\partial u^{(m)}}{\partial x_j}\nu_i - A_i^* E^{(m)}\nu_i\right)\\
&+ \frac{\partial g}{\partial \nu}\left[\rho\left|\frac{\partial u^{(m)}}{\partial t}\right|^2 - \left(A_{pq}^{(m)}\frac{\partial u^{(m)}}{\partial x_q}, \frac{\partial u^{(m)}}{\partial x_p}\right)\right]\\
&+ 2t(\nu,[H^{(m)},E^{(m)}]) + \frac{\partial g}{\partial \nu}(B^{(m)}E^{(m)},E^{(m)})\\
&+ \frac{\partial g}{\partial \nu}\mu^{(m)}|H^{(m)}|^2 - 2(B^{(m)}E^{(m)},\nu)(E^{(m)},\nabla g)\\
&- 2\mu^{(m)}(H^{(m)},\nu)(H^{(m)},\nabla g) + 2\left(\left[\nabla g, A_k\frac{\partial u^{(m)}}{\partial x_k}\right],[\nu,E^{(m)}]\right).
\end{aligned}
\tag{2.10}
$$

The next assertion is of a technical nature and can be proved by direct computations.

Lemma 2.2. *The following representation holds:*

$$
\begin{aligned}
&\mathcal{B}_{m-1}(u,E,H) - \mathcal{B}_m(u,E,H)\\
&\quad = -\frac{\partial g}{\partial \nu}\left\{\left((A_{ij}^{(m-1)} - A_{ij}^{(m)})\,\frac{\partial u^{(m-1)}}{\partial x_j}, \frac{\partial u^{(m-1)}}{\partial x_i}\right)\right.\\
&\quad + \left(A_{ij}^{(m)}\left(\frac{\partial u^{(m)}}{\partial x_j} - \frac{\partial u^{(m-1)}}{\partial x_j}\right), \left(\frac{\partial u^{(m)}}{\partial x_i} - \frac{\partial u^{(m-1)}}{\partial x_i}\right)\right)\\
&\quad + (\mu^{(m)} - \mu^{(m-1)})\left(|[H^{(m)},\nu]|^2 + \frac{\mu^{(m)}}{\mu^{(m-1)}}|(H^{(m)},\nu)|^2\right)\\
&\quad + \left(\left(B^{(m)} - B^{(m-1)}\right)E^{(m)},E^{(m)}\right)\\
&\quad + \left.\left(B^{(m-1)}(E^{(m)} - E^{(m-1)}), E^{(m)} - E^{(m-1)}\right)\right\}.
\end{aligned}
\tag{2.11}
$$

Let us now concern ourselves with an estimate of the integral of $\mathcal{F}_m(u,E,H;g)$ over $\Omega_m \times (0,T)$.

We consider the elliptic problem

$$
\Delta W = 1 \quad \text{on} \quad \Omega, \qquad \frac{\partial W}{\partial \nu}\,|_S = \frac{\text{mes }\Omega}{\text{mes }S},
$$

which admits a solution $W(x) \in C^2(\Omega) \cap C^1(\overline{\Omega})$.

We set

$$g(x) = \delta W(x) + \frac{1}{2}|x - x^0|^2, \qquad \delta > 0.$$

Direct computations give us that (the index m is omitted for simplicity of notations)

$$\begin{aligned}\mathcal{F}_m(u, E, H; g) &= \delta\left(A_{ij}\frac{\partial u}{\partial x_j}, \frac{\partial u}{\partial x_i}\right) - 2\delta\frac{\partial^2 W}{\partial x_p \partial x_i}\left(A_{ij}\frac{\partial u}{\partial x_j}, \frac{\partial u}{\partial x_p}\right) - \delta\rho\left|\frac{\partial u}{\partial t}\right|^2 \\ &+ 2\delta\frac{\partial^2 W}{\partial x_i \partial x_k} b_{ij} E^j E^k + 2\delta\frac{\partial^2 W}{\partial x_i \partial x_j}\ \mu H^i H^j - \delta((BE, E) + \mu|H|^2) \\ &+ 2\delta\left(E, \frac{\partial^2 W}{\partial x_i \partial x_k} A_k \frac{\partial u}{\partial x_i} + \left(A_k\frac{\partial u}{\partial x_k}, \nabla\right)\nabla W - A_k\frac{\partial u}{\partial x_k}\right).\end{aligned}$$

We have

$$\begin{aligned}&- 2\delta\frac{\partial^2 W}{\partial x_p \partial x_i}\left(A_{ij}\frac{\partial u}{\partial x_j}, \frac{\partial u}{\partial x_p}\right) \\ &\le \delta\left[\sigma_1 + \frac{27}{C_0\sigma_1}\ C(A)C^2(W)\right]\left(A_{ij}\ \frac{\partial u}{\partial x_j}, \frac{\partial u}{\partial x_i}\right),\end{aligned} \tag{2.12}$$

where $\sigma_1 > 0$, C_0 is such that

$$\sum_{i,j=1}^{3}(A_{ij}\eta_j, \eta_i) \ge C_0\sum_{i=1}^{3}|\eta_i|^2, \qquad \eta_i \in \mathbb{R}^3,$$

and

$$C(A) = \max_{x\in\overline{\Omega} i,j=1,2,3}||A_{ij}(x)||, \qquad C(W) = \max_{x\in\overline{\Omega} i,j=1,2,3}\left|\frac{\partial^2 W(x)}{\partial x_i \partial x_j}\right|.$$

We note that $C(W) \ge 1/3$.

Next, we get the estimate

$$\begin{aligned}&2\delta\left(E, \frac{\partial^2 W}{\partial x_i \partial x_k} A_k \frac{\partial u}{\partial x_i} + \left(A_k\frac{\partial u}{\partial x_k}, \nabla\right)\nabla W - A_k\frac{\partial u}{\partial x_k}\right) \\ &\le \delta\sigma_2(1 + 2C(W))C_1(A)|E|^2 \\ &+ \delta\frac{1}{\sigma_2}(1 + 2C(W))C_1(A)\frac{1}{C_0}\left(A_{ij}\frac{\partial u}{\partial x_j}, \frac{\partial u}{\partial x_i}\right),\end{aligned} \tag{2.13}$$

where σ_2 is an arbitrary positive number,

$$C_1(A) = \max_{k=1,2,3} ||A_k||.$$

We have

$$\begin{aligned} &2\delta \frac{\partial^2 W}{\partial x_i \partial x_k} b_{ij} E^j E^k + 2\delta \frac{\partial^2 W}{\partial x_i \partial x_j} \mu H^i H^j \\ &\leq \delta 6 C(W) \mu |H|^2 + \delta 6 C(W) ||B|| \frac{1}{b_0} (BE, E). \end{aligned} \tag{2.14}$$

Thus, from (2.12)–(2.14) we get the estimate

$$\begin{aligned} \mathcal{F}_m(u, E, H; g) \leq \delta \Bigg[1 + \sigma_1 + \frac{27}{C_0 \sigma_1} C(A) C^2(W) \\ + \frac{1}{C_0 \sigma_2} C_1(A)(1 + 2C(W)) \Bigg] \left(A_{ij} \frac{\partial u}{\partial x_j}, \frac{\partial u}{\partial x_i} \right) + \delta [6C(W) - 1]\, \mu |H|^2 \\ + \delta \left[6C(W) \frac{1}{b_0} ||B|| + \sigma_2 (1 + 2C(W)) C_1(A) \frac{1}{b_0} \right] (BE, E). \end{aligned} \tag{2.15}$$

We now choose σ_1 and σ_2. We set

$$\sigma_1 = C(W) \sqrt{\frac{C(A)}{C_0}}, \qquad \sigma_2 = \sqrt{\frac{b_0}{C_0}}.$$

From the inequality (2.15) it follows that

$$\begin{aligned} &\sum_{m=0}^{n} \int_0^T \int_{\Omega_m} \mathcal{F}_m(u, E, H; g) dx\, dt \leq \delta C_1 \sum_{m=0}^{n} \int_0^T \int_{\Omega_m} \\ &\left\{ \left(A_{ij}^{(m)} \frac{\partial u^{(m)}}{\partial x_j}, \frac{\partial u^{(m)}}{\partial x_i} \right) + (B^{(m)} E^{(m)}, E^{(m)}) + \mu^{(m)} |H^{(m)}|^2 \right\} dx\, dt \\ &\leq \delta C_1 T \left|\left| \left\{ u, \frac{\partial u}{\partial t}, E, H \right\} \right|\right|_{\mathcal{H}}^2, \end{aligned} \tag{2.16}$$

where δ is an arbitrary positive number and

$$C_1 = \frac{C_1(A)(1 + 2C(W))}{\sqrt{C_0 b_0}} + \max \left\{ 1 + 28 C(W) \sqrt{\frac{C(A)}{C_0}}, 6C(W) ||B|| \frac{1}{b_0} \right\}.$$

From here on we will assume that Ω and S_m satisfy the following conditions: there exists $\delta_1 \geq 0$ such that

(i) $\delta_1 C_1 < 1$, (2.17)

(ii) $\delta_1 \dfrac{\partial W}{\partial \nu} + (x - x^0, \nu) \geq 0$ for some point $x^0 \in \Omega, x \in S_m, m = 1, 2, \cdots, n,$

(iii) $\delta_1 \dfrac{\text{mes } \Omega}{\text{mes } S} + (x - x^0, \nu) > 0, x \in S, x^0$ is defined in (ii).

We note that the above conditions are valid when $\delta_1 = 0$ for star-shaped surfaces $S_1, S_2, \cdots, S_n$, and strongly star-shaped surface S, i.e.,

$$(x - x^0, \nu) > 0, \qquad x \in S.$$

Moreover, if $S_1, S_2, \cdots, S_n$ are strongly star-shaped with respect to a point $x^0 \in \Omega$, then, the above conditions hold with $\delta_1 > 0$ for a class of domains which includes star-shaped domains.

Henceforth we set

$$g(x) = \delta_1 W(x) + \frac{1}{2}\, |x - x^0|^2,$$

where δ_1 is defined in (2.17).

Our next goal is to estimate the second integral on the left-hand side of (2.9).

The following inequality is proved by standard arguments

$$\begin{aligned} 2\sum_{m=0}^{n} \int_{\Omega_m} \Bigg\{\rho \left(\frac{\partial u}{\partial t}, (\nabla\varphi, \nabla)u + u\right) \\ + \mu \left([\nabla\varphi, H], BE + A_k \frac{\partial u}{\partial x_k}\right)\Bigg\} dx \leq C_2 \left\| \left\{u, \frac{\partial u}{\partial t}, E, H\right\} \right\|^2_{\mathcal{H}}, \end{aligned} \tag{2.18}$$

where

$$C_2 = \max\left\{2, \frac{1 + C(\Omega)}{c_2}(\rho + \mu_1 C_1(A)), 2C(\Omega), \frac{\mu_1 b_1}{b_0} C(\Omega)\right\},$$
$$C(\Omega) = \max_{x \in \overline{\Omega}} \{|x - x^0| + \delta_1 |\nabla W(x)|\}, \quad \mu_1 = \max_{\overline{\Omega}} \mu, \quad b_1 = \max_{x \in \overline{\Omega}} ||B(x)||,$$

$c_2 > 0$ is such that

$$c_2 \sum_{m=0}^{n} ||u^{(m)}||^2_{H_1(\Omega_m)} \leq \sum_{m=0}^{n} \int_{\Omega_m} (A_{ij}^{(m)} \frac{\partial u^{(m)}}{\partial x_j}, \frac{\partial u^{(m)}}{\partial x_i}) dx + \int_S \beta |u^{(n)}|^2 dS,$$

$u^{(m)} \in H_2(\Omega_m)$, $u^{(m-1)} = u^{(m)}$ on S_m, $m = 1, 2, \cdots, n$.

Now, we are concern with an estimate of the surface integral (over $S \times (0, T)$) in (2.9).

Using the boundary conditions (1.5) ($Q \equiv G \equiv 0$) and additional boundary condition

$$(H, \nu) \mid_{S\times(0,T)} = 0,$$

we get

$$\begin{aligned}\int_0^T \int_S \mathcal{B}_n(u, E, H) dS\, dt = -T \int_S \beta |u(x, T)|^2 dS \\ + \int_0^T \int_S \Bigg\{ \frac{\partial g}{\partial \nu} \left(\rho \left| \frac{\partial u}{\partial t} \right|^2 + \mu |[H, \nu]|^2 \right) \\ + \beta |(\nabla g, \nabla) u|^2 - \frac{\partial g}{\partial \nu} \left(A_{ij} \frac{\partial u}{\partial x_j}, \frac{\partial u}{\partial x_i} \right) \\ - \frac{\partial g}{\partial \nu} (B\nu, \nu) |(E, \nu)|^2 - \beta |u + (\nabla g, \nabla) u|^2 \Bigg\} dS\, dt.\end{aligned} \tag{2.19}$$

Let $\chi > 0$ be such that

$$\frac{\partial g}{\partial \nu} \geq \chi |\nabla g|, \qquad x \in S.$$

We have

$$\beta |(\nabla g, \nabla) u|^2 \leq \beta \frac{1}{C_0} |\nabla g|^2 \left(A_{ij} \frac{\partial u}{\partial x_j}, \frac{\partial u}{\partial x_i} \right). \tag{2.20}$$

Assume that $\beta = \beta(x)$ satisfies the following condition

$$\beta(x) \leq \frac{C_0 \chi}{C(\Omega)}.$$

Thus, from (2.20) we obtain the inequality

$$\beta |(\nabla g, \nabla) u|^2 \leq \chi |\nabla g| \left(A_{ij} \frac{\partial u}{\partial x_j}, \frac{\partial u}{\partial x_i} \right) \leq \frac{\partial g}{\partial \nu} \left(A_{ij} \frac{\partial u}{\partial x_j}, \frac{\partial u}{\partial x_i} \right),$$

from which it follows that

$$\begin{aligned}&\int_0^T \int_S \mathcal{B}_n(u, E, H) dS\, dt \\ &\le \int_0^T \int_S \frac{\partial g}{\partial \nu} \left\{ \rho \left| \frac{\partial u}{\partial t} \right|^2 + \mu |[H, \nu]|^2 \right\} dS\, dt - T \int_S \beta |u(x, T)|^2 dS.\end{aligned} \tag{2.21}$$

Suppose that the coefficients of the systems (1.2) satisfy the following monotonicity conditions

$$\begin{cases} \left((A_{ij}^{(m-1)} - A_{ij}^{(m)}) \eta_j, \eta_i \right) \ge 0, \quad \eta_i \in \mathbb{R}^3, \\ ((B^{(m)} - B^{(m-1)}) \eta, \eta) \ge 0, \quad \eta \in \mathbb{R}^3, \quad m = 1, 2, \cdots, n \\ \mu^{(m)} \ge \mu^{(m-1)}. \end{cases} \tag{2.22}$$

Using these conditions and Lemma 2.2, we obtain

$$\mathcal{B}_{m-1}(u, E, H) - \mathcal{B}_m(u, E, H) \le 0, \qquad m = 1, 2, \cdots, n. \tag{2.23}$$

Thus, from the identity (2.9) and the inequalities (2.16), (2.18), (2.21), (2.23) we get

$$\begin{aligned}&(1 - \delta_1 C_1)(T - T_1) \\ &\quad \left\{ \sum_{m=0}^n \int_{\Omega_m} \left[\rho \left| \frac{\partial u}{\partial t} \right|^2 + \left(A_{ij} \frac{\partial u}{\partial x_j}, \frac{\partial u}{\partial x_i} \right) + (BE, E) + \mu |H|^2 \right] \right. \\ &\quad \left. + \int_S \beta |u(x, T)|^2 dS \right\} \le \int_0^T \int_S \frac{\partial g}{\partial \nu} \left\{ \rho \left| \frac{\partial u}{\partial t} \right|^2 + \mu |[H, \nu]|^2 \right\} dS\, dt.\end{aligned} \tag{2.24}$$

where

$$T_1 = \frac{2C_2}{1 - \delta_1 C_1}.$$

We now consider the problem (1.6)–(1.9) with zero boundary conditions. Let $\varphi = \{\varphi_1, \varphi_2, \varphi_3, \varphi_4\} \in \widetilde{M}_1 \cap \mathcal{D}(\widetilde{\mathcal{A}})$ and $\{v, \frac{\partial v}{\partial t}, \Phi, \Psi\} = \widetilde{\mathcal{U}}(t)\varphi$.

In this case we have

$$\begin{aligned}&\int_0^T \int_S \mathcal{B}_n(v, \Phi, \Psi) dS\, dt \\ &\quad = \int_0^T \int_S \frac{\partial g}{\partial \nu} \left\{ \left(A_{ij} \frac{\partial v}{\partial x_j}, \frac{\partial v}{\partial x_i} \right) + (B\Phi, \Phi) - \mu |(\Psi, \nu)|^2 \right\} dS\, dt.\end{aligned}$$

In the same way we get the estimate

$$
\begin{aligned}
&(1-\delta_1 C_1)(T-T_1) \\
&\quad \sum_{m=0}^{n} \int_{\Omega_m} \left\{ \rho \left| \frac{\partial v}{\partial t} \right|^2 + \left(A_{ij} \frac{\partial v}{\partial x_j}, \frac{\partial v}{\partial x_i} \right) + (B\Phi, \Phi) + \mu |\Psi|^2 \right\} dx \qquad (2.25) \\
&\quad \leq \int_0^T \int_S \frac{\partial g}{\partial \nu} \left\{ \left(A_{ij} \frac{\partial v}{\partial x_j}, \frac{\partial v}{\partial x_i} \right) + (B\Phi, \Phi) \right\} dS\, dt.
\end{aligned}
$$

Our next goal is to obtain the simultaneous boundary observation for a pair of systems (1.2), (1.6).

Let $f \in M_1 \cap \mathcal{D}(\mathcal{A})$, $\varphi \in \widetilde{M}_1 \cap \mathcal{D}(\widetilde{\mathcal{A}})$ and

$$
\left\{ u, \frac{\partial u}{\partial t}, E, H \right\} = \mathcal{U}(t) f, \quad \left\{ v, \frac{\partial v}{\partial t}, \Phi, \Psi \right\} = \widetilde{\mathcal{U}}(t) \varphi.
$$

We can immediately verify the identity

$$
\begin{aligned}
&\sum_{m=0}^{n} \int_{\Omega_m} \left\{ \rho \left(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t} \right) + \left(A_{ij} \, \frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_i} \right) + (BE, \Phi) + \mu (H, \Psi) \right\} dx \; \Big|_{t=0}^{t=T} \\
&\quad = \int_0^T \int_S \left\{ \left(\frac{\partial u}{\partial t}, \left(A_{ij} \, \frac{\partial v}{\partial x_j} - A_i^* \Phi \right) \nu_i \right) + (\Phi, [\nu, H]) \right\} dS\, dt. \qquad (2.26)
\end{aligned}
$$

The following formula can be proved by direct computations:

$$
\begin{aligned}
&\left(\frac{\partial u}{\partial t}, \left(A_{ij} \frac{\partial v}{\partial x_j} - A_i^* \Phi \right) \nu_i \right) + (\Phi, [\nu, H]) \\
&\quad = \frac{1}{2} \left| \frac{\partial u}{\partial t} + \left(A_{ij} \frac{\partial v}{\partial x_j} - A_i^* \Phi \right) \nu_i \right|^2 + \frac{1}{2} |\Phi - [H, \nu]|^2 \\
&\quad - \frac{1}{2} \left| \frac{\partial u}{\partial t} \right|^2 - \frac{1}{2} \left| A_{ij} \frac{\partial v}{\partial x_j} \nu_i \right|^2 - \frac{1}{2} |A_i^* \Phi \nu_i|^2 - \frac{1}{2} |\Phi|^2 \qquad (2.27) \\
&\quad - \frac{1}{2} |[H, \nu]|^2 + \left(A_{ij} \frac{\partial v}{\partial x_j} \nu_i, A_k^* \Phi \nu_k \right).
\end{aligned}
$$

We have

$$
\left(A_{ij} \frac{\partial v}{\partial x_j} \nu_i, A_k^* \Phi \nu_k \right) \leq \frac{1}{2} \, \frac{1}{1+\varepsilon} \left| A_{ij} \frac{\partial v}{\partial x_j} \nu_i \right|^2 + \frac{1}{2} (1+\varepsilon) \left| A_k^* \Phi \nu_k \right|^2,
$$

$$
|A_k^* \Phi \nu_k|^2 \leq 9 C_1^2(A) |\Phi|^2.
$$

We set

$$\varepsilon = \frac{1-\delta_0}{9C_1^2(A)}, \qquad 0 < \delta_0 < 1.$$

Then (2.27) implies the inequality

$$\begin{aligned}
&\left(\frac{\partial u}{\partial t}, \left(A_{ij}\frac{\partial v}{\partial x_j} - A_i^*\Phi\right)\nu_i\right) + (\Phi, [\nu, H]) \\
&\leq \frac{1}{2}\left|\frac{\partial u}{\partial t} + \left(A_{ij}\frac{\partial v}{\partial x_j} - A_i^*\Phi\right)\nu_i\right|^2 + \frac{1}{2}|\Phi - [H,\nu]|^2 - \frac{1}{2}\left|\frac{\partial u}{\partial t}\right|^2 \qquad (2.28)\\
&- \frac{1}{2}|[H,\nu]|^2 - \frac{1-\delta_0}{2(9C_1^2(A)+1-\delta_0)}\left|A_{ij}\frac{\partial v}{\partial x_j}\nu_i\right|^2 - \frac{\delta_0}{2}|\Phi|^2.
\end{aligned}$$

Henceforth we assume that matrices A_{ij} satisfy the following condition

$$|A_{ij}^{(n)}\nu_j\nu_i\xi| \geq a_0|\xi|, \qquad \xi \in \mathbb{R}^3. \qquad (2.29)$$

Taking (2.29) into account, from (2.28) we find that

$$\begin{aligned}
&\left(\frac{\partial u}{\partial t}, \left(A_{ij}\frac{\partial v}{\partial x_j} - A_i^*\Phi\right)\nu_i\right) + (\Phi, [\nu, H]) \\
&\leq \frac{1}{2}\left|\frac{\partial u}{\partial t} + \left(A_{ij}\frac{\partial v}{\partial x_j} - A_i^*\Phi\right)\nu_i\right|^2 + \frac{1}{2}|\Phi - [H,\nu]|^2 \\
&- \frac{1}{2}\left|\frac{\partial u}{\partial t}\right|^2 - \frac{1}{2}|[H,\nu]|^2 - (1-\delta_0)C_3\left(A_{ij}\frac{\partial v}{\partial x_j}, \frac{\partial v}{\partial x_i}\right) - \frac{\delta_0}{2}|\Phi|^2,
\end{aligned}$$

where

$$C_3 = \frac{a_0}{2(9C_1^2(A)+1-\delta_0)}.$$

From this and the identity (2.26) we obtain

$$\begin{aligned}
&\sum_{m=0}^{n}\int_{\Omega_m}\left\{\rho\left(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}\right) + \left(A_{ij}\frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_i}\right) + (BE, \Phi) + \mu(H, \Psi)\right\}dx\ \Big|_{t=0}^{t=T} \\
&\leq \frac{1}{2}\int_0^T\int_S\left\{\left|\frac{\partial u}{\partial t} + \left(A_{ij}\frac{\partial v}{\partial x_j} - A_i^*\Phi\right)\nu_i\right|^2 + |\Phi - [\nu, H]|^2\right\}dS\,dt \qquad (2.30)\\
&- \frac{1}{2}\int_0^T\int_S\left\{\left|\frac{\partial u}{\partial t}\right|^2 + |[\nu, H]|^2 + 2(1-\delta_0)C_3\left(A_{ij}\frac{\partial v}{\partial x_j}, \frac{\partial v}{\partial x_i}\right) + \delta_0|\Phi|^2\right\}dS\,dt.
\end{aligned}$$

We multiply (2.24) and (2.25) by

$$d_1 = \frac{1}{2C(\Omega)\max\{\rho, \mu^{(n)}\}}, \quad d_2 = \frac{1}{2C(\Omega)\max\left\{\frac{b_1}{\delta_0}\ \frac{1}{2(1-\delta_0)C_3}\right\}},$$

respectively, and add the inequalities thus obtained to (2.30); using the inequality

$$\begin{aligned}
&\sum_{m=0}^{n}\int_{\Omega_m}\left\{\rho\left(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}\right) + \left(A_{ij}\frac{\partial u}{\partial x_j}, \frac{\partial v}{\partial x_i}\right) + (BE, \Phi) + \mu(H, \Psi)\right\} dx\ |_{t=0}^{t=T}\\
&\quad \le \mathcal{D}_0 \sum_{m=0}^{n}\int_{\Omega_m}\left\{\rho\left|\frac{\partial u}{\partial t}\right|^2 + \left(A_{ij}\frac{\partial u}{\partial x_j}, \frac{\partial u}{\partial x_i}\right) + (BE, E) + \mu|H|^2\right.\\
&\quad + \left.\rho\left|\frac{\partial v}{\partial t}\right|^2 + \left(A_{ij}\frac{\partial v}{\partial x_j}, \frac{\partial v}{\partial x_i}\right) + (B\Phi, \Phi) + \mu|\Psi|^2\right\} dx,
\end{aligned}$$

where

$$\mathcal{D}_0 = \max\{1, b_1 b_0^{-1}, 9C(A)C_0^{-1}\},$$

we arrive at the estimate

$$\begin{aligned}
&[d_1(1-\delta_1 C_1)(T-T_1) - \mathcal{D}_0]||\{u, \frac{\partial u}{\partial t}, E, H\}||_{\mathcal{H}}^2\\
&+\ [d_2(1-\delta_1 C_1)(T-T_1) - \mathcal{D}_0]||\{v, \frac{\partial v}{\partial t}, \Phi, \Psi\}||_{\widetilde{\mathcal{H}}}^2 \qquad (2.31)\\
&\le\ \frac{1}{2}\int_0^T\int_S\left\{\left|\frac{\partial u}{\partial t} + \left(A_{ij}\frac{\partial v}{\partial x_j} - A_i^*\Phi\right)\nu_i\right|^2 + |\Phi - [H, \nu]|^2\right\} dS\, dt.
\end{aligned}$$

From (2.31) we deduce the following uniqueness property.

Theorem 2.3. *Assume that S_m and Ω satisfy conditions* (2.17). *Suppose that the matrices $A_{ij}^{(m)}$, $B^{(m)}$ and the coefficients $\mu^{(m)}$ satisfy the monotonicity conditions* (2.22), *$A_{ij}^{(n)}$ satisfy the condition* (2.29),

$$0 < \beta(x) \le \frac{C_0\chi}{C(\Omega)}.$$

Let $f(x) \in M_1 \cap \mathcal{D}(\mathcal{A})$, $\varphi(x) \in \widetilde{M}_1 \cap \mathcal{D}(\widetilde{\mathcal{A}})$. Suppose that $\{u(x,t), E(x,t), H(x,t)\}$ and $\{v(x,t), \Phi(x,t), \Psi(x,t)\}$ are solutions of problems (1.2)–(1.5) and (1.6)–(1.9) with zero boundary conditions, respectively, and that

$$\frac{\partial u}{\partial t} + \left(A_{ij} \frac{\partial v}{\partial x_j} - A_i^* \Phi \right) \nu_i = 0, \quad \Phi - [H, \nu] = 0 \quad \text{on} \quad S \times (0,T).$$

In this case, if $T > T^* = T_1 + \frac{\mathcal{D}_0}{1-\delta_1 C_1} \max\left\{\frac{1}{d_1}, \frac{1}{d_2}\right\}$, then

$$u(x,t) \equiv E(x,t) \equiv H(x,t) \equiv v(x,t) \equiv \Phi(x,t) \equiv \Psi(x,t) \equiv 0$$
$$\text{in} \quad \Omega \times (0,T).$$

From Theorem 2.3 it follows that for $T > T^*$ the expression

$$\begin{aligned} ||\{f,\varphi\}||_{\mathcal{F}} \equiv \Bigg(\int_0^T \int_S \Bigg\{ \left| \frac{\partial u}{\partial t} + \left(A_{ij} \frac{\partial v}{\partial x_j} - A_i^* \Phi \right) \nu_i \right|^2 \\ + |\Phi - [H,\nu]|^2 \Bigg\} dS\, dt \Bigg)^{\frac{1}{2}} \end{aligned} \tag{2.32}$$

defines a norm on the set of initial data $f = \{f_1, f_2, f_3, f_4\}$ and $\varphi = \{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$ of problems (1.2)–(1.5) and (1.6)–(1.9) with zero boundary conditions. In (2.32)

$$\left\{u, \frac{\partial u}{\partial t}, E, H\right\} = \mathcal{U}(t) f, \qquad \left\{v, \frac{\partial v}{\partial t}, \Phi, \Psi\right\} = \widetilde{\mathcal{U}}(t)\varphi.$$

We denote by $\mathcal{F}$ the Hilbert space obtained by completing $M_1 \cap \mathcal{D}(\mathcal{A}) \times \widetilde{M}_1 \cap \mathcal{D}(\widetilde{\mathcal{A}})$ with respect to the norm (2.32). We have

$$||\{f,\varphi\}||_{\mathcal{E}}^2 \equiv ||f||_{\mathcal{H}}^2 + ||\varphi||_{\widetilde{\mathcal{H}}}^2 \le C ||\{f,\varphi\}||_{\mathcal{F}}^2, \qquad \mathcal{F} \subset \mathcal{E} \equiv \mathcal{H} \times \widetilde{\mathcal{H}}.$$

Our next purpose is to prove simultaneous exact controllability for the problems (1.2)–(1.5), (1.6)–(1.9).

3 Exact controllability

We denote by $\mathcal{F}'$ the dual space of $\mathcal{F}$ with respect to $\mathcal{E}$. Let us consider the pair of problems: (1.2)–(1.4) with boundary conditions

$$\begin{aligned}&\left\{\left(A_{ij}\frac{\partial u}{\partial x_j} - A_i^* E\right)\nu_1 + \beta u\right\} |_{S\times(0,T)} \\ &= Q(x,t), [E,\nu] |_{S\times(0,T)} = [\nu, \mathcal{P}]\end{aligned} \tag{3.1}$$

and (1.6)–(1.8) with boundary conditions

$$\frac{\partial v}{\partial t} |_{S\times(0,T)} = Q(x,t), \qquad [\Psi,\nu] |_{S\times(0,T)} = \mathcal{P}(x,t), \tag{3.2}$$

where $Q(x,t), \mathcal{P}(x,t) \in L_2(S\times(0,T))$, $\{f(x),\varphi(x)\} \in \mathcal{F}'$.

We rewrite systems (1.2) and (1.6) in the form

$$\frac{d}{dt}\{u,u',E,H\} = \mathcal{A}\{u,u',E,H\}, \qquad \frac{d}{dt}\{v,v',\Phi,\Psi\} = \widetilde{\mathcal{A}}\{v,v',\Phi,\Psi\}$$

By definition,

$$\{u(t),u'(t),E(t),H(t),v(t),v'(t),\Phi(t),\Psi(t)\} \in L_\infty(0,T;\mathcal{F}')$$

is a solution of (1.2)–(1.4), (3.1) and (1.6)–(1.8), (3.2) if the identity

$$\begin{aligned}&\left\langle\{u(t),u'(t),E(t),H(t),v(t),v'(t),\Phi(t),\Psi(t)\},\{\mathcal{U}(t)\widetilde{f},\widetilde{\mathcal{U}}(t)\widetilde{\varphi}\}\right\rangle_{\mathcal{E}} \\ &= \int_0^t\int_S\left\{\left(Q,\frac{\partial\widetilde{u}}{\partial\tau} + \left(A_{ij}\frac{\partial\widetilde{v}}{\partial x_j} - A_i^*\widetilde{\Phi}\right)\nu_i\right)\right. \\ &+ \left.(\mathcal{P},[\widetilde{H},\nu] - \widetilde{\Phi})\right\}dS\,d\tau + \left\langle\{f,\varphi\},\{\widetilde{f},\widetilde{\varphi}\}\right\rangle_{\mathcal{E}}\end{aligned} \tag{3.3}$$

holds for all $\{\widetilde{f},\widetilde{\varphi}\} \in \mathcal{F}$, $0 < t < T$. In (3.3)

$$\begin{aligned}&\left\langle\{f,\varphi\},\{\widetilde{f},\widetilde{\varphi}\}\right\rangle_{\mathcal{E}} = \left\langle f,\widetilde{f}\right\rangle_{\mathcal{H}} + \langle\varphi,\widetilde{\varphi}\rangle_{\widetilde{\mathcal{H}}}, \\ &\left\{\widetilde{u},\frac{\partial\widetilde{u}}{\partial t},\widetilde{E},\widetilde{H}\right\} = \mathcal{U}(t)\widetilde{f}, \qquad \left\{\widetilde{v},\frac{\partial\widetilde{v}}{\partial t},\widetilde{\Phi},\widetilde{\Psi}\right\} = \widetilde{\mathcal{U}}(t)\widetilde{\varphi}.\end{aligned}$$

In a similar way we define a solution of (1.2), (1.4), (3.1) and (1.6), (1.8), (3.2) with zero data for $t = T$:

$$\{u(t),u'(t),E(t),H(t),v(t),v'(t),\Phi(t),\Psi(t)\} \in L_\infty(0,T;\mathcal{F}')$$

is a solution of (1.2), (1.4), (3.1), (1.6), (1.8), (3.2) with zero data for $t = T$ if

$$\begin{aligned}
&\left\langle \{u(t), u'(t), E(t), H(t), v(t), v'(t), \Phi(t), \Psi(t)\}, \{\mathcal{U}(t)\widetilde{f}, \widetilde{\mathcal{U}}(t)\widetilde{\varphi}\}\right\rangle_{\mathcal{E}} \\
&\quad = -\int_t^T \int_S \left\{ \left(Q, \frac{\partial \widetilde{u}}{\partial t} + \left(A_{ij} \frac{\partial \widetilde{v}}{\partial x_j} - A_i^* \widetilde{\Phi} \right) \nu_i \right) \right. \\
&\quad \left. + \ (\mathcal{P}, [\widetilde{H}, \nu] - \widetilde{\Phi}) \right\} dS \, d\tau
\end{aligned} \tag{3.4}$$

for all $\{\widetilde{f}, \widetilde{\varphi}\} \in \mathcal{F}$, $0 < t < T$.

Let $\{g, \psi\}$ be an arbitrary element of $\mathcal{F}$, and let $\{u, u', E, H, v, v', \Phi, \Psi\}$ be solution of (1.2), (1.4), (3.1), (1.6), (1.8), (3.2) with zero data for $t = T$, $T > T^*$, and boundary functions

$$Q = -\left(\frac{\partial w}{\partial t} + \left(A_{ij} \frac{\partial m}{\partial x_j} - A_i^* p \right) \nu_i \right), \mathcal{P} = -([h, \nu] - p),$$

where

$$\left\{ w, \frac{\partial w}{\partial t}, e, h \right\} = \mathcal{U}(t) g, \qquad \left\{ m, \frac{\partial m}{\partial t}, p, q \right\} = \widetilde{\mathcal{U}}(t) \psi.$$

We set

$$M\{g, \psi\} = \{u, u', E, H, v, v', \Phi, \Psi\} \left.\right|_{t=0} .$$

From (3.4) it follows that

$$\left\langle M\{g, \psi\}, \{\widetilde{f}, \widetilde{\varphi}\}\right\rangle_{\mathcal{E}} = \left\langle \{g, \psi\}, \{\widetilde{f}, \widetilde{\varphi}\}\right\rangle_{\mathcal{F}} \tag{3.5}$$

for any $\{\widetilde{f}, \widetilde{\varphi}\} \in \mathcal{F}$. This implies that M is an isomorphism of $\mathcal{F}$ onto the whole of $\mathcal{F}'$.

We return to problems (1.2)–(1.4), (3.1) and (1.6)–(1.8), (3.2). Suppose that the initial data $\{f, \varphi\}$ belong to $\mathcal{F}'$. We set

$$\{g, \psi\} = M^{-1}\{f, \varphi\},$$

$$Q = -\left(\frac{\partial w}{\partial t} + \left(A_{ij} \frac{\partial m}{\partial x_j} - A_i^* p \right) \nu_i \right), \qquad \mathcal{P} = -([h, \nu] - p),$$

where

$$\left\{ w, \frac{\partial w}{\partial t}, e, h \right\} = \mathcal{U}(t) g, \qquad \left\{ m, \frac{\partial m}{\partial t}, p, q \right\} = \widetilde{\mathcal{U}}(t) \psi.$$

From (3.3) with $t = T > T^*$ we find that

$$\begin{aligned}&\left\langle \{u(T), u'(T), E(T), H(T), v(T), v'(T), \Phi(T), \Psi(T)\}, \{\mathcal{U}(T)\widetilde{f}, \widetilde{\mathcal{U}}(T)\widetilde{\varphi}\}\right\rangle_{\mathcal{E}} \\ &\quad = \left\langle M\{g, \psi\}, \{\widetilde{f}, \widetilde{\varphi}\}\right\rangle_{\mathcal{E}} - \left\langle \{g, \psi\}, \{\widetilde{f}, \widetilde{\varphi}\}\right\rangle_{\mathcal{F}}\end{aligned}$$

for any $\{\widetilde{f}, \widetilde{\varphi}\} \in \mathcal{F}$. By (3.5), the right-hand side of the last identity is equal to zero; that is, $\{u(T), u'(T), E(T), H(T), v(T), v'(T), \Phi(T), \Psi(T)\}$ generates the zero functional on $\mathcal{F}$.

We arrive at the following assertion.

Theorem 3.1. *Assume that $A_{ij}^{(m)}$, $B^{(m)}$, $\mu^{(m)}$, S_m and Ω satisfy the conditions of Theorem 2.3. If $T > T^*$, then for any initial data $\{f, \varphi\} \in \mathcal{F}'$ of problems* (1.2)–(1.5) *and* (1.6)–(1.9) *there exists a control $\{\mathcal{D}(x, t), \mathcal{P}(x, t)\} \in C^1(0, T; L_2(S)) \times C^0(0, T; L_2(S))$ such that the corresponding solution of problem* (1.6)–(1.9) *satisfies*

$$\{v, \frac{\partial v}{\partial t}, \Phi, \Psi\} \mid_{t=T} = 0,$$

while the vector-valued functions

$$Q(x, t) = \frac{\partial}{\partial t}\mathcal{D}(x, t), \qquad G(x, t) = [\nu, \mathcal{P}(x, t)]$$

drive the system (1.2)–(1.5) *to a state of rest at the same time T:*

$$\left\{u, \frac{\partial u}{\partial t}, E, H\right\} \mid_{t=T} = 0.$$

To prove this assertion, it suffices to construct functions $Q(x, t)$, $\mathcal{P}(x, t)$ as before, by setting

$$\mathcal{D}(x, t) = \int_0^t Q(x, \tau)d\tau + \varphi_1(x).$$

We remark that, in view of the linearity of the systems, it suffices to consider controls that reduce the systems to a state of rest.

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