COMPUTATIONAL & APPLIED MATHEMATICS

Volume 30, N. 2, pp. 247–265, 2011
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ISSN 0101-8205
www.scielo.br/cam

A study of axi-symmetric waves through an isotropic thermoelastic diffusive medium

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Abstract. The paper deals with the propagation of axial symmetric cylindrical surface waves in a cylindrical bore through a homogeneous isotropic thermoelastic diffusive medium of infinite extent. The three theories of thermoelasticity namely, Coupled Thermoelasticity (CT), Lord and Shulman (L-S) and Green and Lindsay (G-L) are used to study the problem. The frequency equations, connecting the phase velocity with wave number, radius of bore and other material parameters, for empty and liquid filled bore are derived. The numerical results obtained have been illustrated graphically to understand the behaviour of phase velocity and attenuation coefficient versus wave number of a wave. A particular case of interest has also been deduced from the present investigation.

Mathematical subject classification: 74B, 74F, 74J, 74L, 80A.

Key words: cylindrical bore, diffusion, generalized thermoelasticity, phase velocity, attenuation coefficient.

1 Introduction

The classical theory of thermoelasticity (Coupled Thermoelasticity, (Biot, 1956)) is based on the Fourier's heat conduction equation. The Fourier's heat conduction theory assumes that the thermal disturbances propagate at infinite speed. This prediction is unrealistic from the physical point of view, particularly in situations like those involving very short transient durations, sudden

#CAM-50/09. Received: 02/II/09. Accepted: 09/III/11.

high heat flux exposures, and at very low temperatures near the absolute zero. During the last three decades, non-classical theories of thermoelasticity so called 'Generalized thermoelasticity' have been developed in order to remove the paradox of physically impossible phenomenon of infinite velocity of thermal signals in the conventional coupled thermoelasticity. Lord-Shulman (1967) theory and Green-Lindsay (1972) theory are important generalized theories of thermoelasticity that become centre of interest of recent research in this area. In Lord-Shulman (1967) theory, a flux rate term into the Fourier's law of heat conduction is incorporated (with one relaxation time) and formulated a generalized theory admitting finite speed for thermal signals. Green-Lindsay (1972) theory called as temperature rate-dependent is included among the constitutive variables with two constants that act as two relaxation times, which does not violate the classical Fourier law of heat conduction when the body under consideration has a center of symmetry. The Lord-Shulman (1967) theory of generalized thermoelasticity was further extended to homogeneous anisotropic heat conducting materials recommended by Dhaliwal and Sherief (1980). Chanderashekhariah (1986) refers to this wave like thermal disturbance as second sound. A survey article of various representative theories in the range of generalized thermoelasticity have been brought out by Hetnarski and Ignaczak (1999).

These days, oil companies are interested in the process of thermodiffusion for more efficient extraction of oil from oil deposits. The spontaneous movement of the particles from high concentration region to the low concentration region is defined as diffusion and it occurs in response to a concentration gradient expressed as the change in the concentration due to change in position. The thermodiffusion in elastic solids is due to coupling of fields of temperature, mass diffusion and that of strain in addition to heat and mass exchange with environment.

Nowacki (1974a, b, c, 1976) developed the theory of thermoelastic diffusion by using coupled termoelastic model. Dudziak and Kowalski (1989) and Olesiak and Pyryev (1995), respectively, discussed the theory of thermodiffusion and coupled quasi-stationary problems of thermal diffusion in an elastic layer. They studied the influence of cross effects arising from the coupling of the fields of temperature, mass diffusion and strain due to which the thermal excitation results in additional mass concentration and that generates additional fields of temperature.

Sherief et al. (2004, 2005) developed the generalized theory of thermoelastic diffusion with one relaxation time, which allows the finite speeds of propagation of waves. Singh (2005, 2006) discussed the reflection phenomenon of waves from free surface of an elastic solid with generalized thermodiffusion. Aouadi (2006a, b, 2007a, b, 2008) investigated the different types of problems in thrmoelastic diffusion. Sharma (2007, 2008) discussed the plane harmonic generalized thermoelastic diffusive waves and elasto-thermodiffusive surface waves in heat conducting solids. Recently Kumar et al. (2008) derived the basic equations for generalized thermoelastic diffusion for GL-theory and discussed the propagation of Lamb waves.

The problem of propagation of waves along a cylindrical bore embedded in an infinite isotropic thermoelastic diffusive medium is of great importance due to its manifold applications. In practice, the cylindrical bore may be realized by a borehole or a mine gallery. Borehole studies are of great help in exploration seismology, e.g. in the exploration of oils, gases and hydrocarbons etc. In oil industry, acoustic borehole logging is commonly practiced. A borehole is drilled in a potential hydrocarbon reservoir and then probed with an acoustic tool. Almost all oil companies relay on seismic interpretation for selecting the sites for exploratory oil wells. Seismic wave methods also have higher accuracy, higher resolution and are more economic as compared to drilling which is costly and time consuming.

Tomar and Kumar (1999), Deswal, Tomar and Kumar (2000) and Kumar, Deswal and Choudhary (2001) studied problems of wave propagation through cylindrical bore in micropolar elastic medium with stretch and micropolar elastic medium.

In the present paper we have discussed the propagation of surface waves near a cylindrical bore hole through homogeneous isotropic thermoelastic diffusive medium of infinite extent. Frequency equations relating the phase velocity and wave number are derived for empty as well as for liquid filled bore. The dispersion curves giving the phase velocity and attenuation coefficient as functions of wave number are plotted for empty as well as for liquid filled bore.

2 Basic equations

The basic governing equations of homogeneous isotropic thermoelastic diffusive medium in the absence of body forces, heat sources and diffusive mass sources are

$$(\lambda+\mu)\nabla(\nabla.\overrightarrow{u})+\mu\nabla^2\overrightarrow{u}-\beta_1\nabla(T+\tau_1\dot{T})-\beta_2\nabla(C+\tau^1\dot{C})=\rho\ddot{u} \quad (1)$$

$$K\nabla^2 T-\rho C_E(\dot{T}+\tau_0\ddot{T})=\beta_1 T_0(\dot{e}+\epsilon\tau_0\ddot{e})+a^*T_0(\dot{C}+\gamma\ddot{C}) \quad (2)$$

$$D\beta_2\nabla^2 e+Da^*\nabla^2(T+\tau_1\dot{T})+(\dot{C}+\epsilon\tau^0\ddot{C})-Db\nabla^2(C+\tau^1\dot{C})=0 \quad (3)$$

where $\overrightarrow{u}$ is the displacement vector, T is the temperature change, T_0 is the reference temperature assumed to be such that $|T/T_0|<<1$, C is the concentration, C_E is the specific heat at constant strain, K is thermal conductivity, e is dilatation, ρ is the density assumed to be independent of time, λ, μ are Lame's parameters,

$$\beta_1=(3\lambda+2\mu)\alpha_t \quad \text{and} \quad \beta_2=(3\lambda+2\mu)\alpha_c,$$

where α_t is the coefficient of linear thermal expansion and α_c is the coefficient of linear diffusive expansion, a^*, b are respectively, the coefficients describing the measure of thermoelastic diffusion effects and diffusion effects, D is thermoelastic diffusion constant, τ^0, τ^1 are diffusion relaxation times with $\tau^1 \geq \tau_0 \geq 0$ and τ_0, τ_1 are thermal relaxation times with $\tau_1 \geq \tau_0 \geq 0$. Here $\epsilon=\tau_0=\tau^0=\tau_1=\tau^1=\gamma=0$ for Coupled Thermolastiicity (CT), $\tau_1=\tau^1=0, \epsilon=1, \gamma=\tau_0$ for Lord-Shulman (L-S) theory and $\epsilon=0, \gamma=\tau^0$ for Green-Lindsay (G-L) theory.

3 Problem formulation and its solution

We consider a cylindrical bore of radius 'a' having circular cross section in a generalized thermoelastic medium of infinite extent. We use cylindrical polar coordinates (r, θ, z) with z-axis pointing upwards along axis of cylinder. The propagation of axial symmetric waves is considered near the bore hole and these waves are the analogue of Rayleigh waves propagating at a traction free boundary of a generalized thermoelastic medium. This section deals with the situation when bore does not contain any fluid. We are discussing a two dimensional problem with symmetry about z-axis, so all partial derivatives with respect to the variable θ would be zero. Therefore, we take $\overrightarrow{u} = (u_r, 0, u_z)$ and $\partial/\partial\theta = 0$, so that the field equations and constitutive relations in cylindrical polar coordinates reduce to

$$(\lambda+\mu)\frac{\partial e}{\partial r} + \mu\left(\nabla^2 u_r - \frac{u_r}{r^2}\right) - \beta_1\left(1+\tau_1\frac{\partial}{\partial t}\right)\frac{\partial T}{\partial r} - \beta_2\left(1+\tau^1\frac{\partial}{\partial t}\right)\frac{\partial C}{\partial r} = \rho\frac{\partial^2 u_r}{\partial t^2}, \tag{4}$$

$$(\lambda+\mu)\frac{\partial e}{\partial z} + \mu\nabla^2 u_z - \beta_1\left(1+\tau_1\frac{\partial}{\partial t}\right)\frac{\partial T}{\partial z} - \beta_2\left(1+\tau^1\frac{\partial}{\partial t}\right)\frac{\partial C}{\partial z} = \rho\frac{\partial^2 u_z}{\partial t^2}, \tag{5}$$

$$K\nabla^2 T - \rho C_E\left(1+\tau_0\frac{\partial}{\partial t}\right)\frac{\partial T}{\partial t} = \beta_1 T_0\left(1+\epsilon\tau_0\frac{\partial}{\partial t}\right)\frac{\partial e}{\partial t} + a^* T_0\left(1+\gamma\frac{\partial}{\partial t}\right)\frac{\partial C}{\partial t}, \tag{6}$$

$$D\beta_2\nabla^2 e + Da^*\left(1+\tau_1\frac{\partial}{\partial t}\right)\nabla^2 T + \left(1+\epsilon\tau^0\frac{\partial}{\partial t}\right)\dot{C} - Db\left(1+\tau^1\frac{\partial}{\partial t}\right)\nabla^2 C = 0, \tag{7}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

Let us now introduce the dimensionless quantities defined by

$$\left\{r', z', u_r', u_z'\right\} = \frac{1}{a}\left\{r, z, u_r, u_z\right\}, \quad \left\{t', \tau_0', \tau_1', \tau^{0'}, \tau^{1'}\right\} = \frac{c_1}{a}\left\{t, \tau_0, \tau_1, \tau^0, \tau^1\right\},$$

$$t_{rr}' = \frac{t_{rr}}{\beta_1 T_0}, \quad t_{rz}' = \frac{t_{rr}}{\beta_1 T_0}, \quad T' = \frac{\beta_1 T}{\rho c_1^2}, \quad C' = \frac{\beta_2 C}{\rho c_1^2}, \quad c_1^2 = \frac{\lambda+2\mu}{\rho}. \tag{8}$$

Introducing the quantities defined above, into the equations (4)-(7), after suppressing the dashes, the field equations reduce to

$$\left(\frac{\partial^2 u_r}{\partial r^2}+\frac{1}{r}\frac{\partial u_r}{\partial r}-\frac{u_r}{r^2}\right)+\delta_1\frac{\partial^2 u_r}{\partial z^2}+\delta_2\frac{\partial^2 u_z}{\partial r\partial z}-\tau_T^1\frac{\partial T}{\partial r}-\tau_C^1\frac{\partial C}{\partial r}=\frac{\partial^2 u_r}{\partial t^2}, \quad (9)$$

$$\delta_1\left(\frac{\partial^2 u_z}{\partial r^2}+\frac{1}{r}\frac{\partial u_z}{\partial r}\right)+\frac{\partial^2 u_z}{\partial z^2}+\delta_2\left(\frac{\partial^2 u_r}{\partial r\partial z}+\frac{1}{r}\frac{\partial u_r}{\partial z}\right)$$
$$-\tau_T^1\frac{\partial T}{\partial z}-\tau_C^1\frac{\partial C}{\partial z}=\frac{\partial^2 u_z}{\partial t^2}, \quad (10)$$

$$\nabla^2 T-\zeta_1\tau_T^0\frac{\partial T}{\partial t}-\zeta_2\tau_e^0\frac{\partial e}{\partial t}-\zeta_3\tau_C^0\frac{\partial C}{\partial t}=0, \quad (11)$$

$$\varsigma_1\nabla^2 e+\varsigma_2\tau_T^1\nabla^2 T-\varsigma_3\tau_C^1\nabla^2 C+\tau_G^0\frac{\partial C}{\partial t}=0. \quad (12)$$

$$\delta_1=\frac{\mu}{\lambda+2\mu},\quad \delta_2=\frac{\lambda+\mu}{\lambda+2\mu},\quad \zeta_1=\frac{\rho C_E c_1 a}{K},\quad \zeta_2=\frac{\beta_1^2 a T_0}{K\rho c_1},$$
$$\zeta_3=\frac{aa^* c_1\beta_1 T_0}{K\beta_2},\quad \varsigma_1=\frac{D\beta_2^2}{a\rho c_1^3},\quad \varsigma_2=\frac{Da^*\beta_2}{a\beta_1 c_1},\quad \varsigma_3=\frac{Db}{ac_1},$$
$$\tau_T^1=1+\tau_1\frac{\partial}{\partial t},\quad \tau_C^1=1+\tau^1\frac{\partial}{\partial t},\quad \tau_T^0=1+\tau_0\frac{\partial}{\partial t},$$
$$\tau_C^0=1+\gamma\frac{\partial}{\partial t},\quad \tau_e^0=1+\epsilon\tau_0\frac{\partial}{\partial t},\quad \tau_G^0=1+\epsilon\tau^0\frac{\partial}{\partial t}. \quad (13)$$

Assuming the solutions of equations (9)-(12) for the waves propagating in the z-direction as

$$\{u_r,u_z,T,C\}=\{b_1K_1(mr),b_2K_0(mr),b_3K_0(mr),b_4K_0(mr)\}e^{\imath(kz-\omega t)}, \quad (14)$$

where $K_0()$, $K_1()$ are respectively the modified Bessel functions of order zero and one and of second kind $\omega(=kc)$ is the angular velocity of the wave, k is the wave number and c is the phase velocity.

Substituting (14) in equations (9)-(12), we obtain four homogeneous linear equations in four unknowns b_1, b_2, b_3 and b_4 which for the non-trivial solution yields

$$m^8+Am^6+Bm^4+Cm^2+D=0, \quad (15)$$

where

$$A = \frac{b_2^* - a_1^* b_1^* \omega^2 + a_2^* b_5^* + \tau_T^{11} b_8^* - \tau_C^{11} b_{11}^*}{b_1^* - \delta_1 \varsigma_1 \tau_C^{11}},$$

$$B = \frac{b_3^* - a_1^* b_2^* \omega^2 + a_2^* b_6^* + \tau_T^{11} b_9^* - \tau_C^{11} b_{12}^*}{b_1^* - \delta_1 \varsigma_1 \tau_C^{11}},$$

$$C = \frac{b_4^* - a_1^* b_3^* \omega^2 + a_2^* b_7^* + \tau_T^{11} b_{10}^* - \tau_C^{11} b_{13}^*}{b_1^* - \delta_1 \varsigma_1 \tau_C^{11}},$$

$$D = \frac{-a_1^* b_4^* \omega^2}{b_1^* - \delta_1 \varsigma_1 \tau_C^{11}},$$

$$b_1^* = \delta_1 \varsigma_3 \tau_C^{11},$$

$$b_2^* = \varsigma_3 \tau_C^{11} \big(\delta_1 \big(a_6 - 2k^2\big) + a_3 \omega^2\big) + \delta_1 \omega \big(\imath \tau_G^{10} + a_7 \varsigma_2 \tau_T^{11}\big) + k^2 \tau_C^{11},$$

$$\begin{aligned} b_3^* = {} & \varsigma_3 \tau_C^{11} \big(a_6 \big(a_3 \omega^2 - \delta_1 k^2\big) + k^2 \big(\delta_1 k^2 - 2 a_3 \omega^2\big)\big) \\ & + \imath \tau_G^{10} \omega \big(\delta_1 \big(a_6 - k^2\big) + a_3 \omega^2\big) - \omega^3 \Bigg(\big(a_3 a_7 + \imath a_5 \tau_C^{11}\big) \varsigma_2 \tau_T^{11} \\ & + \frac{\big(a_7 \varsigma_2 \tau_T^{11} + \imath \tau_G^{10}\big) \delta_1}{c^2} + \frac{\big(\imath a_5 \varsigma_3 \tau_C^{11} - a_7 \varsigma_1\big) \tau_T^{11}}{c} \Bigg) + k^2 \tau_C^{11} \big(a_6 - 2k^2\big), \end{aligned}$$

$$\begin{aligned} b_4^* = {} & a_3 \omega^2 \big(k^2 - a_6\big) \big(k^2 \varsigma_3 \tau_C^{11} - \imath \omega \tau_G^{10}\big) + \frac{\omega^5}{c^3} \tau_T^{11} \big(a_7 \big(a_3 \varsigma_2 c - \varsigma_1\big) \\ & + \imath a_5 \tau_C^{11} \big(\varsigma_2 - \varsigma_3\big)\big) + \frac{\omega^4}{c} a_5 \tau_T^{11} \tau_G^{10} + k^4 \tau_C^{11} \big(k^2 - a_6\big), \end{aligned}$$

$$b_5^* = \tau_C^{11} \big(\imath k \varsigma_1 - a_2 \varsigma_3\big),$$

$$\begin{aligned} b_6^* = {} & -\tau_C^{11} \varsigma_3 \big(a_2 \big(a_6 - 2k^2\big) + \imath k a_4 \tau_T^{11}\big) - \imath \omega a_2 \tau_G^{10} + \omega a_7 \tau_T^{11} \big(a_2 \varsigma_2 + \imath k \varsigma_1\big) \\ & + \imath k \tau_C^{11} \big(\big(a_6 - 2k^2\big) \varsigma_1 - a_4 \varsigma_2 \tau_T^{11}\big), \end{aligned}$$

$$\begin{aligned} b_7^* = {} & a_2 \big(a_6 \big(k^2 \tau_C^{11} \varsigma_3 - \imath \omega \tau_G^{10}\big) + k^2 \big(\imath \omega \tau_G^{10} - k^2 \varsigma_3 \tau_C^{11} - a_7 \omega \varsigma_2 \tau_T^{11}\big)\big) \\ & + \imath k^3 \tau_C^{11} \big(a_4 \big(\varsigma_3 + \varsigma_2\big) \tau_T^{11} + \big(k^2 - a_6\big) \varsigma_1\big) + \omega k \tau_T^{11} \big(a_4 \omega \tau_G^{10} - \imath a_7 k^2 \varsigma_1\big), \end{aligned}$$

$$b_8^* = a_4 \delta_1 \varsigma_3 \tau_C^{11},$$

$$b_9^* = \varsigma_3\omega^2\tau_C^{11}\left(a_2a_5 + a_4\left(a_3 + \frac{\delta_1}{c^2}\right)\right) + k^2\varsigma_1\big(a_4\tau_C^{11} + \imath c a_2 a_7\big)$$

$$+ \imath\omega\delta_1 a_4\tau_G^{10} + \omega\varsigma_1\big(a_7 - \imath\omega k a_5\tau_C^{11}\big),$$

$$b_{10}^* = a_2a_5\omega^2\big(\imath\omega\tau_G^{10} - k^2\varsigma_3\tau_C^{11}\big) - \omega^2k^2\left(\frac{\imath a_2a_7\varsigma_1}{c} + a_3a_4\varsigma_3\tau_C^{11}\right)$$

$$+ \omega^3\left(\imath a_3a_4\tau_G^{10} - \varsigma_1\left(\frac{ka_4\varsigma_1\tau_C^{11}}{c^3} + \frac{a_7}{c^2} - \frac{\imath k^2a_5\varsigma_1\tau_C^{11}}{c}\right)\right),$$

$$b_{11}^* = \varsigma_1\big(-\imath k a_2 + \omega^2 a_3 + \delta_1\big(a_6 - 2k^2\big)\big) - a_4\delta_1\varsigma_2\tau_T^{11},$$

$$b_{12}^* = a_2\big(\imath k\varsigma_1\big(2k^2 - a_6\big) - a_5\omega^2\varsigma_2\tau_T^{11}\big) + a_4\omega^2\varsigma_2\tau_T^{11}\left(\frac{\delta_1}{c^2} - a_3\right)$$

$$+ a_3\omega^2\varsigma_1\big(a_6 - 2k^2\big) + \varsigma_1k^2\big(\delta_1\big(k^2 - a_6\big) + \tau_T^{11}\big(a_4 - \imath\omega c a_5\big)\big),$$

$$b_{13}^* = k^2a_2\big(a_5\omega^2\varsigma_2\tau_T^{11} + \imath k\varsigma_1\big(a_6 - k^2\big)\big) + \omega^2k^2a_3\big(a_4\varsigma_2\tau_T^{11} + \varsigma_1\big(k^2 - a_6\big)\big)$$

$$+ k^4\varsigma_1\tau_T^{11}\big(\imath\omega c a_5 - a_4\big).$$

$$a_1 = \frac{\delta_1}{c^2} - 1, \quad a_2 = \imath k\delta_2, \quad a_3 = 1 - \frac{1}{c^2}, \quad a_4 = \imath\omega\zeta_2\tau_e^{10},$$

$$a_5 = \frac{\zeta_2\tau_e^{10}}{c}, \quad a_6 = \imath\omega\zeta_1\tau_T^{10}, \quad a_7 = -\imath\zeta_3\tau_C^{10},$$

$$\tau_T^{11} = 1 - \imath\omega\tau_T^1, \quad \tau_C^{11} = 1 - \imath\omega\tau_T^1, \quad \tau_T^{10} = 1 - \imath\omega\tau_T^0,$$

$$\tau_C^{10} = 1 - \imath\omega\tau_T^0, \quad \tau_e^{10} = 1 - \imath\omega\tau_e^0, \quad \tau_G^{10} = 1 - \imath\omega\tau_G^0.$$

The roots of equation (15) are complex in general. These roots are denoted by m_i^2, $i = 1, \ldots, 4$. Corresponding to these roots, the waves with amplitudes b_1, b_2, b_3 and b_4 are obtained which are designated by $b_1(i), b_2(i), b_3(i)$ and $b_4(i)$. These are given by

$$b_1(i) = \frac{\triangle_1(i)}{\triangle(i)}, \quad b_2(i) = -\frac{\triangle_2(i)}{\triangle(i)}, \quad b_3(i) = \frac{\triangle_3(i)}{\triangle(i)}, \quad b_4(i) = -\frac{\triangle_4(i)}{\triangle(i)},$$

where

$$\triangle_1(i) = b_1^*m_i^6 + b_2^*m_i^4 + b_3^*m_i^2 + b_4^*,$$

$$\triangle_2(i) = b_5^*m_i^5 + b_6^*m_i^3 + b_7^*m_i,$$

$$\triangle_3(i) = b_8^* m_i^5 + b_9^* m_i^3 + b_{10}^* m_i,$$

$$\triangle_4(i) = \delta_1 \varsigma_1 m_i^7 + b_{11}^* m_i^5 + b_{12}^* m_i^3 + b_{13}^* m_i$$

and

$$\triangle(i) = \sqrt{(\triangle_1(i))^2 + (\triangle_2(i))^2 + (\triangle_3(i))^2 + (\triangle_4(i))^2}. \tag{16}$$

Thus the appropriate solutions of (9)-(12), corresponding to the wave propagating along z-axis are

$$u_r = \sum_{i=1}^{4} f(i) b_1(i) K_1(m_i r) e^{i(kz - \omega t)},$$

$$u_z = \sum_{i=1}^{4} f(i) b_2(i) K_0(m_i r) e^{i(kz - \omega t)},$$

$$T = \sum_{i=1}^{4} f(i) b_3(i) K_1(m_i r) e^{i(kz - \omega t)},$$

$$C = \sum_{i=1}^{4} f(i) b_4(i) K_0(m_i r) e^{i(kz - \omega t)}, \tag{17}$$

where $f(i)$ are relative excitation factors.

3.1 *Derivation of frequency equation*

At the surface $r = 1$, the appropriate boundary conditions are

$$t_{rr} = (\lambda + 2\mu)\frac{\partial u_r}{\partial r} + \lambda\left(\frac{u_r}{r} + \frac{\partial u_z}{\partial z}\right) - \tau_T^1 T - \tau_C^1 = 0,$$

$$t_{rz} = \mu\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) = 0,$$

$$\frac{\partial T}{\partial r} = 0,$$

$$\frac{\partial C}{\partial r} = 0, \tag{18}$$

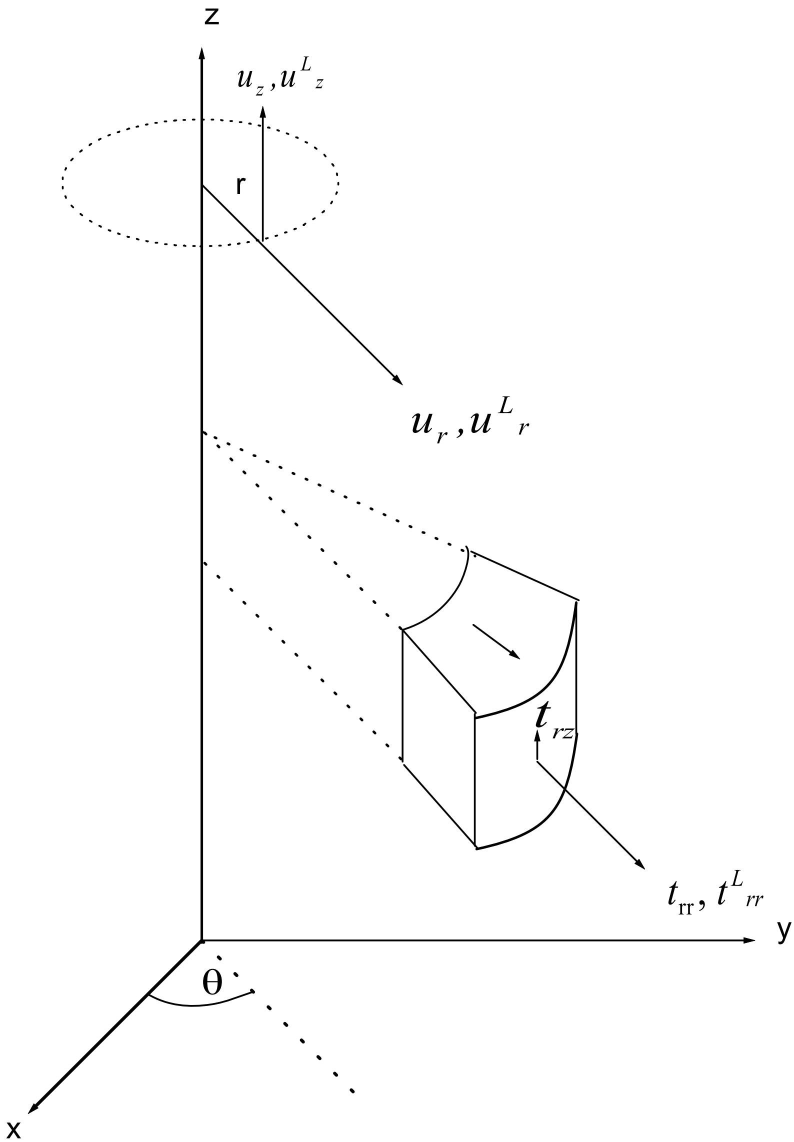

Figure 1 – Geometry of the investigated problem.

where t_{rr}, t_{rz} are the radial and tangential stress components. Making use of (17), into the boundary conditions (18), after employing dimensionless quantities defined by (8), we obtain four homogeneous equations in four unknowns $f(1)$, $f(2)$, $f(3)$ and $f(4)$. The elimination of these unknowns gives the frequency equation

$$H_1\Delta' - H_2\Delta'' + H_3\Delta''' - H_4\Delta'''' = 0, \tag{19}$$

where

$$\Delta' = P_2(S_3M_4 - S_4M_3) - P_3(S_2M_4 - S_4M_2) + P_4(S_2M_3 - S_3M_2),$$

$$\Delta'' = P_1(S_3M_4 - S_4M_3) - P_3(S_1M_4 - S_4M_1) + P_4(S_1M_3 - S_3M_1),$$

$$\Delta''' = P_1(S_2M_4 - S_4M_2) - P_2(S_1M_4 - M_1S_4) + P_4(S_1M_2 - S_2M_1),$$

$$\Delta'''' = P_1(S_2M_3 - M_2S_3) - P_2(S_1M_3 - S_3M_1) + P_3(S_1M_2 - M_1S_2).$$

$$\begin{aligned} H_i = \Big\{ &- b_1(i)m_i + \imath k(\delta_2 - \delta_1)b_2(i) - \tau_T^{11}b_3(i) - \tau_C^{11}b_4(i)\Big\} \\ &\times K_0(m_i) + \{(\delta_2 - \delta_1)b_2(i) - b_1(i)\}K_1(m_i), \end{aligned}$$

$$\begin{aligned} P_i &= \{\imath k b_1(i) - b_2(i)m_i\}K_1(m_i), \\ S_i &= -m_i b_3(i)K_1(i), \\ M_i &= -m_i b_4(i)K_1(i), \end{aligned} \tag{20}$$

Equation (19) determines the dimensionless phase velocity c of axial symmetric surface waves as a function of dimensionless wave number k and other thermomicropolar parameters of the medium.

4 Propagation of waves in a cylindrical bore filled with liquid

Here, we consider the same problem as in the previous section with the additional constraint that the borehole is filled with homogeneous inviscid liquid.

The field equation and constitutive relations for homogeneous inviscid liquid are

$$\lambda^L\nabla\big(\nabla\cdot\overrightarrow{u}^L\big) = \rho^L\frac{\partial^2\overrightarrow{u}^L}{\partial t^2}, \tag{21}$$

and

$$t_{ij}^L = \lambda^L\big(\nabla\cdot\overrightarrow{u}^L\big)\delta_{ij}, \tag{22}$$

where $\overrightarrow{u}^L$ is the displacement vector, λ^L and ρ^L are respectively the bulk modulus and density of liquid. Other symbols have their usual meaning as defined earlier.

For two dimensional problem, we take

$$\overrightarrow{u}^L = \left(u_r^L, 0, u_z^L\right), \quad \text{and} \quad \frac{\partial}{\partial\theta} = 0. \tag{23}$$

The dimensionless variables defined in this case, in addition to those defined by (8), are

$$\left\{u_r^{L'}, u_z^{L'}\right\} = \frac{1}{a}\left\{u_r^L, u_z^L\right\}, \quad t_{rr}^{L'} = \frac{t_{rr}^L}{\beta_1 T_0}. \tag{24}$$

We relate the dimensionless displacement components and potential function ϕ^L as

$$u_r^L = \frac{\partial\phi^L}{\partial r}, \quad u_z^L = \frac{\partial\phi^L}{\partial z}. \tag{25}$$

Making use of equation (25) in equations (21) and (22), with the help of equations (23) and (24), after suppressing the primes yields

$$\frac{\partial^2\phi^L}{\partial r^2} + \frac{1}{r}\frac{\partial\,\phi^L}{\partial\,r} + \frac{\partial^2\phi^L}{\partial\,z^2} = \delta_1^{*2}\frac{\partial^2\phi^L}{\partial\,t^2}, \tag{26}$$

and

$$t_{rr}^L = \frac{\lambda^L}{\beta_1 T_0}\nabla^2\phi^L, \tag{27}$$

where

$$\delta_1^* = \frac{c_1}{c^L}, \quad c^L = \sqrt{\frac{\lambda^L}{\rho^L}}. \tag{28}$$

The solution of (26) corresponding to surface waves may be written as

$$\phi^L = f(0)I_0(m_0 r)e^{\iota(kz-\omega t)}, \tag{29}$$

After some simplification, the pressure and radial displacement of liquid are given by

$$p^L = -t_{rr}^L = \frac{\lambda^L}{\beta_1 T_0}\delta_1^2\omega^2 f(0)I_0(m_0 r)e^{\iota(kz-\omega t)}, \tag{30}$$

$$u_r^L = m_0 f(0)I_1(m_0 r)e^{\iota(kz-\omega t)}, \tag{31}$$

where $I_0()$ and $I_1()$ are modified Bessel functions of first kind and of order zero and one respectively.

4.1 Derivation of frequency equation

The appropriate boundary conditions for the present situation are

$$t_{rr} = -p^L, \quad t_{rz} = 0, \quad \frac{\partial T}{\partial r} = 0, \quad \frac{\partial C}{\partial r} = 0, \quad u_r = u_r^L, \quad \text{at} \quad r = 1. \tag{32}$$

Making use of (29)-(31) and (17), in the boundary conditions (32), we obtain five homogeneous equations in five unknowns $f(0)$, $f(1)$, $f(2)$, $f(3)$ and $f(4)$. The condition for the non-trivial solution yields the required frequency equation as

$$\begin{aligned}\delta^2\delta_1^{*2}\omega^2 I_0(m_0)\big\{ &- b_1(1)K_1(m_1)\Delta' + b_1(2)K_1(m_2)\Delta'' \\ &-b_1(3)K_1(m_3)\Delta''' + b_1(4)K_1(m_4)\Delta''''\big\} \\ &-m_0 I_1(m_0)\{H_1\Delta' - H_2\Delta'' + H_3\Delta''' - H_4\Delta''''\} = 0.\end{aligned} \tag{33}$$

5 Particular case

The frequency equations in the absence of diffusion effect reduce to

(I) For empty bore:

$$\Delta^* = 0, \tag{34}$$

(II) For liquid filled bore:

$$\begin{aligned}\delta^2\delta_1^{*2}\omega^2 I_0(m_0)\big\{&b_1(1)K_1(m_1)\big(P_2S_3 - S_2P_3\big)b_1(2)K_1(m_2)\big(P_1S_3 - S_1P_3\big) \\ &-b_1(3)K_1(m_3)\big(P_1S_2 - S_1P_2\big)\big\} - m_0I_1(m_0)\Delta^* = 0,\end{aligned} \tag{35}$$

where

$$\Delta^* = H_1(P_2S_3 - S_2P_3) - H_2(P_1S_3 - S_1P_3) + H_3(P_1S_2 - S_1P_2),$$

and the expressions for H_i, P_i and S_i can be obtained from (20), substituting $b_4(i) = 0$ and $m_i^2 (i = 1, \ldots, 3)$ are the roots of

$$m^6 + A^*m^4 + B^*m^2 + C^* = 0, \tag{36}$$

where

$$A^* = \frac{-1}{\delta_1}\left\{\delta_1\left(k^2 + a_1\omega^2 + a_4\tau_T^{11} - a_6\right) + a_2^2 - a_3\omega^2\right\},$$

$$B^* = \frac{-1}{\delta_1}\left\{\left(\left(\delta_1 a_1 - a_3\right)\omega^2 + a_2^2\right)\left(a_6 - k^2\right) + \tau_T^{11}\left(\imath k\left(a_2 a_4 + a_5\omega^2\right)\right.\right.$$
$$\left.\left. - \omega^2\left(a_2 a_5 + a_3 a_4\right)\right) + a_1 a_3\omega^4\right\},$$

$$C^* = \frac{-1}{\delta_1}\left\{a_1 a_3\omega^4\left(a_6 - k^2\right) - \imath k a_1 a_5\omega^4\tau_T^{11}\right\}.$$

6 Numerical results and discussion

For numerical computation, we take the material of copper as an isotropic material. The physical data for a single crystal of copper material is given by

$$\lambda = 7.76 \times 10^{10} Kgm^{-1}s^{-2}, \quad \mu = 3.86 \times 10^{10} Kgm^{-1}s^{-2}, \quad T_0 = 0.293 \times 10^3 K,$$

$$C_E = .3831 \times 10^3 JKg^{-1}K^{-1}, \quad \alpha_t = 1.78 \times 10^{-5} K^{-1}, \quad \alpha_c = 1.98 \times 10^{-4} m^3 Kg^{-1},$$

$$a^* = 1.2 \times 10^4 m^2 s^{-2} K^{-1}, \quad b = 9 \times 10^5 Kg^{-1} m^5 s^{-2}, \quad D = 1.65 \times 10^{-8} Kgsm^{-3},$$

$$\rho = 8.954 \times 10^3 Kgm^{-3}, \quad K = 0.383 \times 10^3 Wm^{-1}K^{-1}.$$

The non-dimensional radius of bore and relaxation times are taken as

$$a = 10, \quad \tau_0 = 0.2, \quad \tau_1 = 0.6, \quad \tau^0 = 0.03, \quad \tau^1 = 0.07.$$

Equations (19) and (33) determine the phase velocity c of the axial symmetric surface waves as a function of wave number k, radius of bore and various physical parameters in complex form. If we write

$$\frac{1}{c} = \frac{1}{v} + \imath\frac{q}{\omega}, \tag{37}$$

then wave number $k = R + \imath q$, where $R = \omega/v$ and q are real numbers. This shows that v is propagation speed and q is attenuation coefficient of the surface waves.

The graphical representation for the variation of non dimensional phase velocity and attenuation coefficient of wave propagation in context of CT, L-S, and G-L theories of thermoelastic diffusion for various values of R, i.e, the real part of wave number in Figures 2 and 3 respectively. In these figures, the

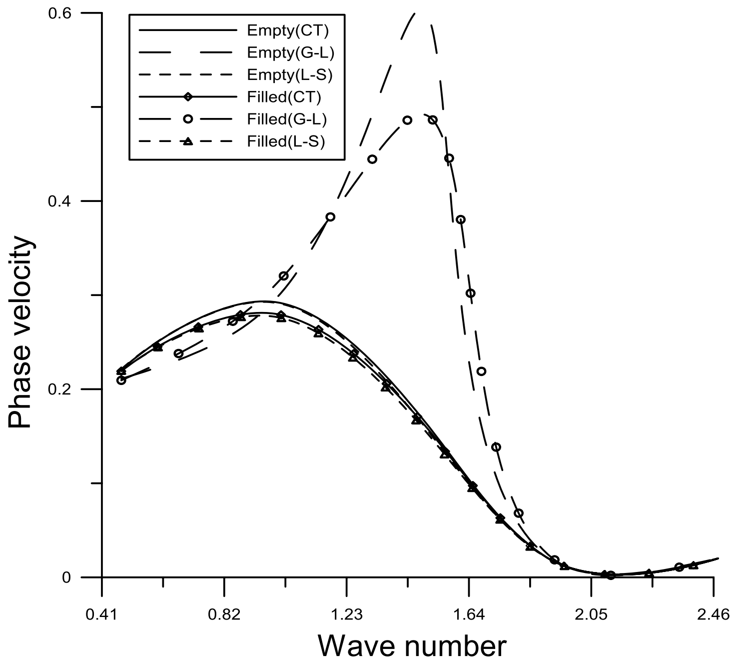

Figure 2 – Variation of phase velocity w.r.t. wave number.

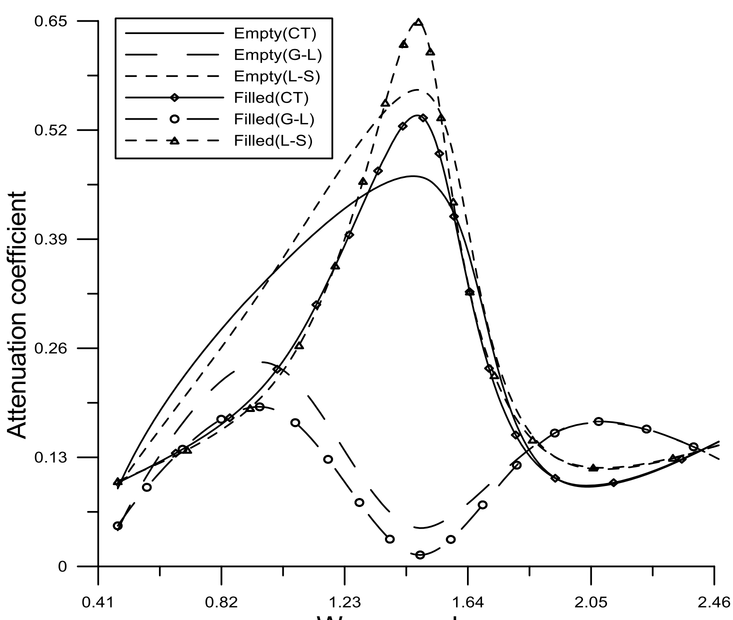

Figure 3 – Variation of attenuation coefficient w.r.t. wave number.

curves with solid lines, small dashed lines and long dashed lines without central symbol, represent variations in context of CT, L-S and G-L theories respectively in case of empty bore whereas the corresponding lines with central symbols represent the same situation in case of liquid filled bore.

6.1 Phase velocity

It is observed from Figure 2 that the phase velocity increases at initial value wave number and ultimately decreases to attain vanishingly small constant value for both empty as well as for liquid filled bore and for all the three theories of thermoelastic diffusion described above. It is observed that the behavior and trend of variation of the curves for C-T and L-S theories is almost same except for the difference in magnitude values. For C-T and L-S theories, the values of phase velocity for empty bore are higher than those for liquid filled bore. Also for empty as well as for liquid filled bore the values in case of CT theory are slightly higher than those in case of L-S theory. Within the range $R < 1.0$, the values of phase for CT and L-S theories are higher than those for G-L theory whereas the behavior is reversed after this range for both empty as well as liquid filled bore. For G-L theory, the phase velocity increases to its peak value within the range $R < 1.4$ and decreases to vanishingly small values after this range. For $R < 1.1$, the phase velocity for liquid filled bore is higher than that for empty bore, but after this range the behavior is reversed for both CT and L-S theories of thermoelastic diffusion.

6.2 Attenuation coefficient

The values of attenuation coefficient for CT and L-S theories first increase to peak values and then decrease for both empty as well as liquid filled bore and ultimately increase after the range $R > 1.84$, whereas for G-L theory, the attenuation co-efficient shows an oscillating behavior. For both CT and L-S theories and within the range $R < 1.3$, the values of attenuation coefficient for empty bore are higher than those for liquid filled bore however the behavior is reversed after this range. For G-L theory, the values of attenuation coefficient are higher for empty bore as compared to those for liquid filled bore. For empty bore and $R < 1.84$, the values of attenuation coefficient for CT and L-S theories are higher than those for G-L theory but the behavior is reversed after this range. For liquid filled bore and within the range $R < 1.0$, the curves show a mixed behavior, for $1.0 < R < 1.84$ the values for both CT and L-S theories are than those for G-L theory for both empty as well as liquid filled bore and the behavior is reversed after this range.

7 Conclusion

The propagation of waves in a homogeneous isotropic thermoelastic diffusive medium has been investigated after deriving the secular equations (19) and (33). The modified Bessel functions with complex arguments have been used to study the problem. An appreciable effect of relaxation times on the phase velocity as well as attenuation coefficient is observed. It is observed that the phase velocity for all the theories first increases and then decrease to vanishingly small constant values. The values of attenuation coefficient first increase then decrease and ultimately increase except for G-L theory for which the attenuation coefficient show an oscillating behavior within the whole range.

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