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Duality results for stationary problems of open pit mine planning in a continuous function framework

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Abstract. Open Pit Mine Planning problems are usually considered in a Mixed Integer Programming context. Characterizing each attainable profile by a continuous function yields a continuous framework. It allows for a more detailed modeling of slope constraints and other material properties of slanted layers. Although the resulting nonlinear programming problems are in general non-convex and non-differentiable, they provide certain advantages as one can directly compute sensitivities of optimal solutions w.r.t. small data perturbations. In this work duality results are derived for the stationary problems of the continuous framework employing an additional condition called convex-likeness.

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Key words: Duality, convex-likeness, continuous optimization, Mine planning, calculus of variations.

1 Introduction

In the continuous framework for Open Pit Mine planning any profile is described by a continuous function. A profile p is called feasible if it satisfies the Dirichlet boundary condition $p(x) - p_0(x) = 0$ for $x \in \partial\Omega$, the nonnegativity condition $p(x) - p_0(x) \geq 0$ for all $x \in \Omega$ and the so called slope constraint

$$\Lambda_p(x) = \limsup_{\hat{x} \to x \leftarrow \tilde{x}} \frac{|p(\hat{x}) - p(\tilde{x})|}{\|\hat{x} - \tilde{x}\|} \leq \omega(x, p(x)) \tag{1}$$

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with an upper semi continuous parameter ω. By construction, $\Lambda_p(\cdot)$ is an upper semi continuous functional ([11, Theorem 9.2]). The feasible set $\mathcal{P} \subset C(\overline{\Omega})$ contains all those profiles. The stationary Capacitated Final Open Pit Problem (CFOP) for a given effort constraint $\overline{E} \in \mathbb{R}_+$ reads

$$\begin{array}{lll} \min & -G(p) & \\ s.t. & p & \in \mathcal{P} \\ & \hat{E}(p) & \leq 0 \end{array} \tag{CFOP}$$

with $\hat{E}(p) = E(p) - \overline{E}$ and

$$\begin{aligned} G(p) &\equiv \int_\Omega \int_{p_0(x)}^{p(x)} g(x,\tau) d\tau dx \\ E(p) &\equiv \int_\Omega \int_{p_0(x)}^{p(x)} e(x,\tau) d\tau dx \end{aligned}$$

representing the gain generated by a certain profile p and the effort which is necessary to create it. Here, the densities $g, e \in L^\infty(\Omega \times Z)$ are only assumed to be essentially bounded and the effort density has to be strictly positive, i.e. $e(x,z) \geq e_0 > 0$ which is a natural assumption. For the analysis of general optimization problems in Banach spaces one normally needs at least continuous Fréchet differentiability or convexity of the objective functional and the constraint mapping [8]. Problem (CFOP) usually exhibits neither continuous Fréchet differentiability (consequence of [2, Proposition 5(ii)] as Gateaux differentiability is necessary for Fréchet differentiability) nor convexity (consequence of [2, Lemma 1] as this property is obtained only for rather artificial choices of ω).

Although certain additional assumptions on the parameters of the model ensure these properties, they are not taken into account in this work.

In block models [4] the bound on the slope ω is represented by precedence relations, which effectively restricts its value to the simple rationals m/n with m and n typically not exceeding 3. A major advantage of the continuous model is the possibility to vary ω continuously and to obtain a Lagrange multiplier in the sense of a measure of the sensitivity for the slope constraint. This is important as the parameter ω can only be derived by geostatistic tools and hence carry uncertainties about the exact value. Because the mapping representing the slope condition is expected to be neither differentiable nor convex an alternative to the concepts named above needs to be applied. As it is not even expected to be Lipschitz continuous the direct application of subdifferential calculus will not yield satisfactory results either.

The article reviews duality results for so called convex-like optimization problems [8, 9] as these cover a slightly wider class of problems than the properly convex ones. It is organized as follows.

Section 2 recalls the basic definitions needed for the analysis of convex-like optimization problems. Moreover basic duality theorems for this class will be given. It closes with the presentation of a characterization of solutions as saddle points of the Lagrange functional.

Section 3 applies the duality theory for convex-like optimization problems to the problem formulation of (CFOP) presented above with $p \in \mathcal{P}$ being considered as an implicit hard constraint, i.e. the optimization is done by only considering feasible profiles. This formulation is a continuous analog to the well known discrete ones as the values for ω are prescribed and hence equal some kind of pointwise predecessor relation. Moreover an example shows that the characteristic convex-likeness is not generally given. The section closes with a more general theorem which covers at least some instances of the type given in the example.

In Section 4 the stability constraint is included as a non-implicit constraint in the optimization process. After the definition of the corresponding range space of the constraint and the appropriate dual space we may apply again duality results for convex-like optimization problems.

2 Preliminaries

In general an optimization problem defined on a Banach space X is given by

$$\begin{aligned} \min \quad & F(x) \\ \text{s.t.} \quad & g(x) \in -C_Y \\ & x \in \hat{S} \end{aligned} \tag{P}$$

where $F : X \to \mathbb{R}$ is the objective functional, $g : X \to Y$ a vector space valued constraint mapping with Y being partially ordered by some cone C_Y and $\hat{S}$ is a nonempty subset of X. To justify the investigation of (P) the feasible set $S = \{x \in \hat{S} | g(x) \in -C_Y\}$ is assumed to be nonempty as well. Problem (P) will be referred to as *primal problem* throughout.

A more intuitive access for this problem is obtained when (P) is reformulated as a penalized optimization problem given in terms of

$$\min_{x \in \hat{S}} \sup_{y \in C_Y^*} F(x) + \langle y, g(x) \rangle. \tag{P$'$}$$

Here $C_Y^* \equiv \{y \in Y^* | \langle y, x \rangle \geq 0 \text{ for all } x \in C_Y\}$ represents the so called *dual cone* w.r.t. the duality pairing $\langle \cdot, \cdot \rangle_{Y^*, Y}$. Throughout the indices Y^*, Y will be left out as it will be clear from the context for which spaces the pairings are considered.

While if (P) and (P$'$) are not equivalent in general, it is well known that this is guaranteed if the ordering cone C_Y is closed (e.g., see [8, Lemma 6.5]). Now one introduces the *dual problem* of (P$'$) as

$$\max_{y \in C_Y^*} \inf_{x \in \hat{S}} F(x) + \langle y, g(x) \rangle. \tag{D}$$

For any feasible element $\tilde{x} \in S$ of the primal problem and any feasible element of the dual problem $\hat{y} \in C^*$ one obtains the weak duality relation

$$\inf_{x \in \hat{S}} F(x) + \langle \hat{y}, g(x) \rangle \quad \leq \quad F(\tilde{x}).$$

Consequently a lower bound for the optimal value of (P) as the relationship has to hold for the supremum of the left hand side w.r.t. all elements of the dual cone as well. Note, that so far this infimum might be $-\infty$ and hence it is not possible to obtain quantitative properties of the solution of the primal problem. As (P) is neither a convex problem nor F and g are once continuously Fréchet differentiable, one has to generalize one of this concepts to obtain characterizations of the solutions. One generalization of the well known concept of convexity is the so called *convex-like* behavior first introduced in [9]. Here not a function is assumed to be convex but a set which is constructed with the help of it.

Definition 2.1 (convex-like). Let S be a nonempty subset of a vector space X. Moreover let Y be a partially ordered vector space with ordering cone C_Y.

A mapping $\hat{g} : S \to Y$ is called **convex-like** if the set $M_g = \hat{g}(S) + C_Y$ is convex in Y.

The concept of convex-likeness indeed covers slightly more functions than the ones just being convex. For example consider the function $g : \mathbb{R} \to \mathbb{R}^2$, $g(x) = (sin(x), x)$ which is convex-like w.r.t. the positive orthant but obviously not convex. The next step for the introduction of duality results is to ensure of a constraint qualification. As it is well known, the Slater condition was originally defined for convex problems but can be generalized to convex-like problems as well.

Definition 2.2 (generalized Slater condition). Problem (P) satisfies the **generalized Slater condition** (GSC) if there exists $x \in \hat{S}$ such that $g(x) \in -int(C_Y)$.

The following duality result applies the definitions introduced above and can be found in [8, Theorem 6.7].

Theorem 2.1 (Duality Theorem applying convex-likeness). *Consider an optimization problem of form* (P)*. Moreover, let the ordering cone C_Y be closed and contain interior points, i.e. $int(C_Y) \neq \emptyset$. Furthermore, let the mapping $(F, g) : \hat{S} \to \mathbb{R} \times Y$ be convex-like w.r.t. the ordering cone $\mathbb{R}_+ \times C_Y$ in the product space $\mathbb{R} \times Y$.*

If (P) *is solvable and the generalized Slater condition holds, there exist a solution of the dual problem* (D) *as well and the extremal values of both problems coincide.*

Essentially, the proof uses the convexity of the set

$$M = (F, g)(\hat{S}) + \mathbb{R}_+ \times C_Y$$

and the fact, that due to the generalized Slater condition this set contains interior points. We may thus apply the classical Eidelheit separation theorem ([7, Theorem 1.3]) on $int(M)$ and $(F(x^*), 0_Y)$ with x^* being the optimal solution of (P).

One speaks of strong duality when (P) and (D) have solutions whose optimal values coincide. For optimal solutions of (P) the following characterization can be given where the proof is adapted from [6, Corollary 5.3].

Theorem 2.2 (characterization of solutions). *Consider a problem of form* (P).

Moreover let the composite mapping $(F, g)\colon \hat{S} \to \mathbb{R} \times Y$ *be convex-like w.r.t. the product cone* $\mathbb{R}_+ \times C_Y$*,* C_Y *be closed with* $int(C_Y) \neq \emptyset$ *and (GSC) be satisfied.*

Then the following assertions are equivalent

(i) $\bar{x}$ *is an optimal solution of* (P)

(ii) $\exists \bar{y} \in C_Y^*$ *s.t.* $(\bar{x}, \bar{y})$ *is a saddle point of the Lagrange functional*

$$L(x, y) = F(x) + \langle y, g(x) \rangle$$

in the sense of $L(\bar{x}, y) \leq L(\bar{x}, \bar{y}) \leq L(x, \bar{y})$ *for all* $x, y \in S \times C_Y^*$.

Proof. (i) $\Rightarrow$ (ii)

By Theorem 2.1 the dual problem is solvable and the extremal values coincide. Hence one has

$$\begin{aligned} \min_{x \in \hat{S}} \sup_{y \in C_Y^*} F(x) + \langle y, g(x) \rangle &\leq F(\bar{x}) + \langle \bar{y}, g(\bar{x}) \rangle \\ &\leq \max_{y \in C_Y^*} \inf_{x \in \hat{S}} F(x) + \langle y, g(x) \rangle \end{aligned}$$

By $\hat{g}(\bar{x}) \in -C_Y$ and the definition of the infimum it follows

$$L(\bar{x}, y) \leq L(\bar{x}, \bar{y}) \leq L(x, \bar{y})$$

i.e. the Lagrange functional admits a saddle point.

(ii) $\Rightarrow$ (i)

As L admits an saddle point in $(\bar{x}, \bar{y})$ it follows

$$\langle y, \hat{g}(\bar{x}) \rangle \leq \langle \bar{y}, \hat{g}(\bar{x}) \rangle \forall y \in C_Y^*.$$

Consequently, $\hat{g}(\bar{x}) \in -C_Y$ and thus $\bar{x}$ is a feasible point of (P). Now the saddle point provides the assertion by

$$L(\bar{x}, \bar{y}) \leq L(x, \bar{y}). \qquad \square$$

3 Partial dual w.r.t. capacity constraint

Obviously, (CFOP) is a problem of the general class (P). $\mathcal{P} \subset C(\overline{\Omega})$ is a non-empty subset of a vector space because at least the initial profile is an element of this set. For this profile the Dirichlet boundary and the nonnegativity condition are trivially satisfied. The slope constraint has to be satisfied as we assume the initial surface to be stable. The range space of the inequality constraint $\hat{E} : \mathcal{P} \to \mathbb{R}$ is a totally ordered vector space where the ordering is characterized by the cone $\mathbb{R}_+ = [0, \infty)$. The dual cone is $\mathbb{R}_+$ as well. The feasible set S contains all profiles in $\mathcal{P}$ satisfying the capacity constraint. Hence one has

$$S := \{p \in \mathcal{P} | \hat{E}(p) \in -\mathbb{R}_+\}.$$

By $\hat{E}(p_0) = -\overline{E}$ the set S is nonempty for any effort bound providing a well defined optimization problem, i.e. for all $\overline{E} \geq 0$. As the ordering cone $C_Y = \mathbb{R}_+$ is closed w.r.t. any norm, one can pass from (CFOP) to the equivalent penalized form

$$\min_{p \in \mathcal{P}} \sup_{y \in \mathbb{R}_+} \quad -G(p) + \langle y, \hat{E}(p) \rangle. \tag{2}$$

The corresponding dual problem

$$\max_{y \in \mathbb{R}_+} \inf_{p \in \mathcal{P}} \quad -G(p) + \langle y, \hat{E}(p) \rangle \tag{3}$$

gives at least a lower bound on the extremal value of the primal problem (CFOP). The following proposition shows, how strong duality can be achieved.

Proposition 3.1 (strong duality under additional conditions). *If the composite mapping $(-G, \hat{E})(\mathcal{P})$ is convex-like w.r.t. to the product cone $\mathbb{R}_+ \times C_Y$, then the dual problem* (3) *is solvable and the extremal values of both problems coincide.*

Proof. That (CFOP) is a problem of the form (P) has been discussed already. Obviously the ordering cone $\mathbb{R}_+$ contains interior points and is closed. By [2, Proposition 3.1] the primal problem is solvable. Hence it remains to show the existence of a profile $p \in \mathcal{P}$ such that $\hat{E}(p) \in int(R_+)$ which is synonymous to $\hat{E}(p) < 0$. Consider the initial profile p_0. Obviously this profile is an element of the set of feasible profiles $\mathcal{P}$. As $E(p_0) = 0$ holds, one has $\hat{E} < 0$ as long as the capacity of the open pit mine is greater than zero, i.e. $\overline{E} > 0$. In the case of $\overline{E}$ being equal to zero, the only feasible solution for the resulting problem is the initial profile itself. So no duality analysis has to be done and nothing is to show. Hence $\overline{E} > 0$ is a proper assumption and hence the existence of an element satisfying the (GSC) is ensured.

The application of Theorem 2.1 completes the proof. □

Hence under additional requirements strong duality can be realized in the sense of Theorem 2.1, i.e. there is no duality gap. In the case of continuous gain and effort densities the class of problems which can be considered is significantly larger than only the convex problems. Unfortunately, the set of problems which does not meet this requirement is also of significant size.

The following example describes a simple situation where the requirement of convex-like behavior of $(-G, \hat{E})$ is no longer satisfied because of a special property of the gain functional. However the investigation of convex-like problems is justified as it covers a fairly large subclass of problems. For the investigation of the example it is necessary to consider the gain-optimal combinations of the image of the composite mapping.

Definition 3.1 (gain-optimal combinations). For any $p \in \mathcal{P}$ the pair $(-G(p), \hat{E}(p))$ is called **combination** of gain and effort for the problem (CFOP).

Under all combinations for a certain effort $\tilde{E}$ a unique **gain-optimal combination** maximizes the gain, i.e. $\max\{G(p) | p \in \mathcal{P}, E(p) = \tilde{E}\}$.

Note, that the existence of the gain-optimal combinations is guaranteed analogously to the proof of existence for solutions of (CFOP).

Example 1 (not convex-like). For simplicity consider $\Omega \subset \mathbb{R}^1$. Hence all profiles are located in a rectangle $\Omega \times Z$ with the characteristics shown in Figure 1. The parameters defining the optimization problem are given as follows.

$\omega \equiv 1$ uniformly in $\overline{\Omega} \times Z$.

e $\equiv 1$ uniformly in $\overline{\Omega} \times Z$.

$$\text{g} \begin{cases} \equiv 1 & \text{areas indicated by dark gray} \\ \equiv 0 & \text{areas indicated by light gray} \end{cases}$$

The Figures 1 to 6 show profiles representing gain-optimal combinations. Figure 7 summarizes the development of the gain-optimal combinations and highlights particular combinations realized by the following profiles. The sum of this graph and the positive orthant $\mathbb{R}^2_+$ would have to be convex if $(-G, \hat{E})$ were convex-like.

W.l.o.g. gain-optimal combinations will be considered for $\overline{E} = 25$ as the shape of $(-G, \hat{E}) + \mathbb{R}^2_+$ remains constant.

Figure 1 shows the unique profile realizing an gain-optimal combination for $\tilde{E} = 0$. This combination, $(0, -25)$, is denoted by **a** in Figure 7.

Figure 2 depicts a profile representing the gain-optimal combinations for $\tilde{E} = 9$. Any feasible profile with $-G(p) = -9$ is a representative of this combination. In figure 7 it can be found at point **b**.

Figure 3 displays the unique profile yielding the gain-optimal combination for $\tilde{E} = 21$. Any profile excavating less material from the dark gray in favor of more from the light gray would generate a smaller gain and any profile excavating more of the dark gray area would violate the slope constraint.

As there is no feasible profile generating more than $G(p) = 21$ with $\tilde{E} = 22$, Figure 4 shows a realization of the gain-optimal combination $(-21, -3)$.

The excavation process along the gain-optimal profiles is continued by extending the latter profile such that as much as possible of the valuable material in layer three is excavated. Figure 5 shows an intermediate state on this excavation process.

This procedure continues until the profile which is obtained fulfills the stability condition as an equality everywhere. The corresponding profile can be seen in Figure 6.

Obviously, the set M which is generated by addition of the first quadrant to this graph is not convex as for example the line connecting the points c and f cannot be in the resulting set M.

A remedy for the lack of convex-likeness is applying the convex hull operator on the image of the composite mapping. This extends the class of problems for which strong duality can be shown. Recall, that the convex hull of a set $\mathcal{K}$ is the smallest convex set containing $\mathcal{K}$. With the help of this operator, now one is able to establish the following weakened form of Theorem 2.1.

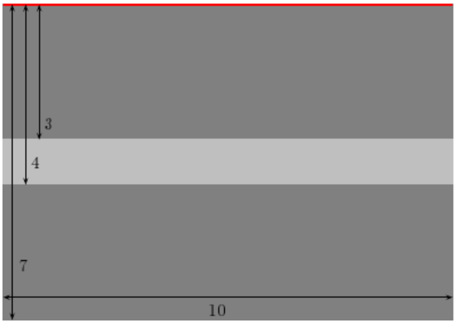

Figure 1: (a)

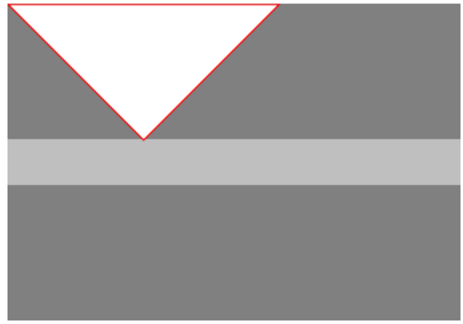

Figure 2: (b)

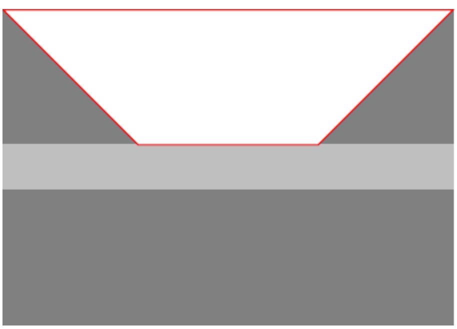

Figure 3: (c)

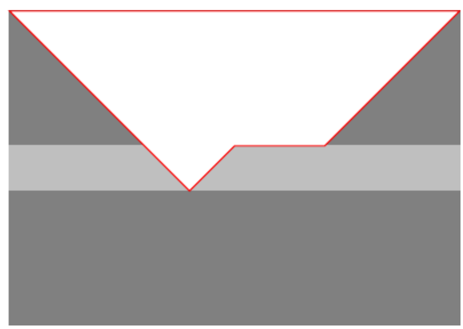

Figure 4: (d)

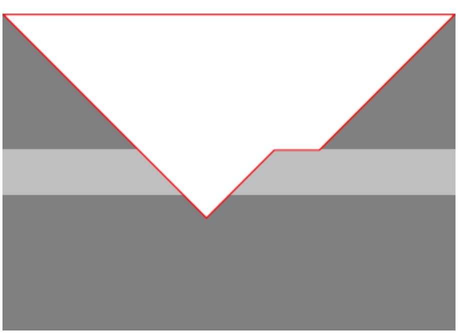

Figure 5: (e)

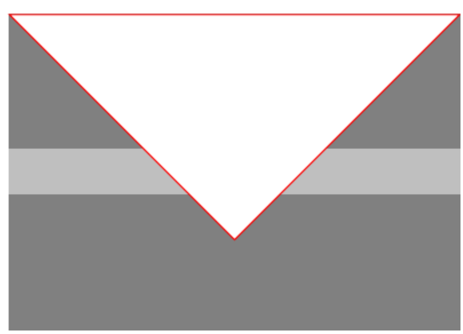

Figure 6: (f)

Theorem 3.1. *If the set* $(-G, \hat{E})(p)$ *with* $p \in \mathcal{P}$ *has a supporting tangent at* $(-G, \hat{E})(p^*)$ *where* p^* *represents the optimal solution of* (CFOP)*, then Theorem* 2.1 *remains valid without the assumption of convex-likeness on* $(-G, \hat{E})$*.*

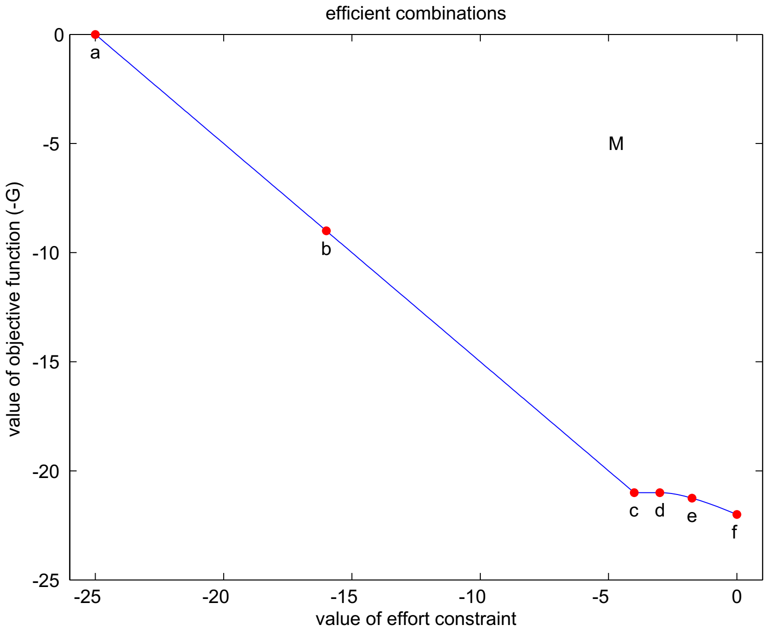

Figure 7 – Development of gain-optimal combinations.

Proof. The main argument of the proof of Theorem 2.1 consist of the fact that

$$M = (F, g)(\hat{S}) + \mathbb{R}_+ \times C_Y$$

is convex under the additional assumption of convex-likeness of the composite mapping. Then $(F(x^*), 0)$ can be separated from M.

To avoid the convex-likeness of the composite mapping, one has to ensure, that $(-G(p^*), 0)$ still can be separated from a convex set containing all combinations $(-G, \hat{E})(\mathcal{P})$. Consider the set

$$M_C = conv\Big((-G, \hat{E})(\mathcal{P})\Big) + \mathbb{R}_+ \times C_Y.$$

As a direct sum of two convex sets it is convex as well. By definition, $(-G, \hat{E})(p^*)$ is an element of the convex hull $conv\Big((-G, \hat{E})(\mathcal{P})\Big)$ and by optimality one has $(-G, \hat{E})(p^*) \notin int(M_C)$. Consequently, with

$$M := M_C + \mathbb{R}^2_+$$

one proofs the claim analogously to Theorem 2.1. $\square$

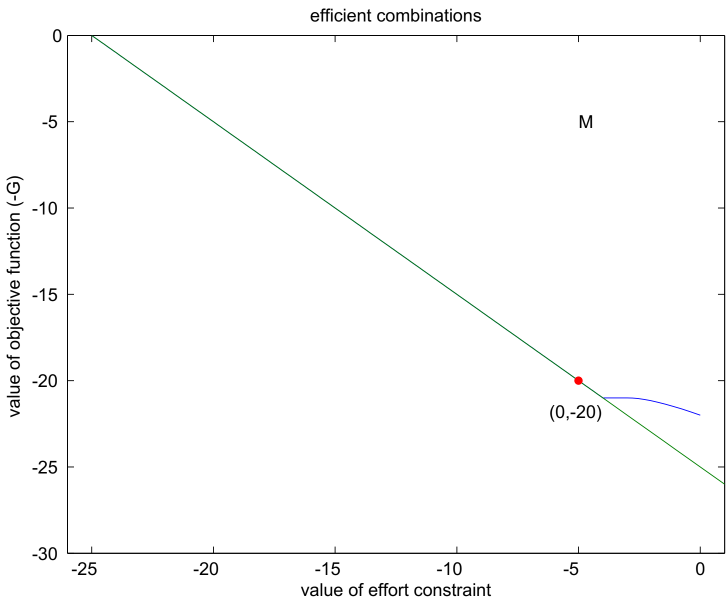

Figure 8 – Separation of (0, 20) by a linear functional.

By Example 1 it will be shown, that the given condition indeed covers a wider class of problems than the convex-like ones.

As one can observe in Figure 7 all convex combinations of the points c and f for the weights $\lambda \in (0, 1)$ are not contained in M.

If one applies Theorem 3.1 one can separate the point $(-G(p^*), 0)$ from the set $\tilde{M}$ by a linear functional as long as the upper bound on the total effort is $\overline{E} \leq 21$. In Figure 8 this can be observed for $\overline{E} = 20$.

In fact, it depends strongly on the effort bound $\bar{E}$ whether strong duality can be obtained or not. From this connection one can derive the following corollary for (CFOP). Let p_∞ denote the globally optimal profile.

Corollary 1. *If the global minimum of $-G$ within $\mathcal{P}$ is attained by a profile $p_\infty \in \mathcal{P}$ that satisfies the capacity constraint of the open pit, i.e.*

$$\min_{p\in\mathcal{P}} -G(p) = \min_{\substack{p\,\in\,\mathcal{P}\\ \hat{E}(p)\leq 0}} -G(p),$$

there is no duality gap between the primal and the dual problem.

Proof. Obviously, $(-G, \hat{E})(p_\infty)$ has to be an element on the boundary of M_C. Hence Theorem 3.1 is applicable. □

Thus a characterization of the global minimizer can be obtained if p_∞ can be reached without violating the effort constraint. In the next Corollary, p_U denotes the so called *ultimate pit* representing the maximal profile in the sense of the lattice structure of $\mathcal{P}$ (see [2, Proposition 3]) which can be reached without considering any effort constraint or gain optimality.

Corollary 2. *If $\overline{E} \geq E(p_U)$ holds, there is no duality gap between the primal and the dual problem.*

Proof. The global minimizer of the objective has to be attained by the feasible profiles satisfying the capacity constraint because p_U is an upper bound for all profiles in the optimization process. Hence one can apply Corollary 1. □

A mining engineer can be expected to define the capacity of the mine large enough to be able to excavate the global minimizer p_∞ but not the ultimate pit p_U.

All in all one concludes that the investigation of the gain-optimal profiles is one of the main challenges in the dualization theory for (CFOP). In the opinion of the authors, the approach presented by Matheron [10] provides the best framework for this task.

According to Theorem 2.2 the following saddle point property holds for (CFOP) in the case of $(-G, \hat{E})$ being convex-like.

Proposition 3.2 (Saddle Point of the Lagrangian). *If the composite mapping $(-G, \hat{E})(\mathcal{P}) \to \mathbb{R} \times \mathbb{R}$ is convex-like w.r.t. the product cone $\mathbb{R}_+ \times \mathbb{R}_+$, p^* is a solution of (CFOP) if and only if there exist a $\bar{y} \in \mathbb{R}_+$ s.t. $(p^*, \bar{y})$ is a saddle point of*

$$L(p, y) = -G(p) + \langle y, \hat{E}(p) \rangle.$$

4 Full Dual w.r.t. capacity constraint and slope constraint

So far, the slope constraint was given implicitly as this condition is included in the definition of the feasible set. Hence one only obtains a dual variable for the capacity but not for the stability constraint. A main advantage of the continuous approach is the possibility to obtain a dual variable for this one and get information about the sensitivity for this constraint.

In the following section an extended problem formulation will be analyzed. In this formulation the stability condition is not longer given implicitly but as an inequality constraint. The problem is given by

$$\begin{array}{lll} \min & -G(p) & \\ s.t. & p & \in \tilde{\mathcal{P}} \\ & \hat{\Lambda}(p(\cdot)) & \leq 0 \\ & \hat{E}(p) & \leq 0 \end{array} \tag{CFOP$'$}$$

where $\hat{\Lambda}(p(\cdot)) = \Lambda_p(\cdot) - \omega(\cdot, p(\cdot))$ represents the difference of the local slope of the profile and the value which it is allowed to be at most in a pointwise manner. Moreover, $\tilde{\mathcal{P}}$ denotes a special subset of the vector space of continuous functions. In general continuous functions do not have to admit a bounded Λ_p (e.g., $g(x) = x^{3/2}\sin(1/x)$).

If this quantity is not bounded one is not able to make any assertion on the difference $\hat{\Lambda}$. Hence one has to pass from $C(\overline{\Omega})$ to a subset of functions satisfying certain regularity conditions. This functions will be in the subspace of Lipschitz continuous functions $Lip(\overline{\Omega})$ which is dense in $C(\overline{\Omega})$ and guarantees the operator $\Lambda_p(x)$ at least to be finite for all considered profiles p and all $x \in \overline{\Omega}$. The feasible set is now

$$\tilde{\mathcal{P}} \equiv \left\{ p \in Lip(\overline{\Omega}) \mid p \text{ satisfies boundary and nonnegativity condition} \right\}$$

For the investigation of the duality properties of problem (CFOP$'$) one has to know about the range space of the constraint mapping. The first component is, as shown above, the space of real numbers $\mathbb{R}^1$. For the difference of local Lipschitz constant and ω the following Lemma answers this question.

Lemma 4.1 (range space of the slope constraint). *The difference representing the slope constraint*

$$\hat{\Lambda}(p(\cdot)) = \Lambda_p(x) - \omega(x, p(x))$$

is an element of $L^\infty(\Omega)$ *for any profile* $p \in \tilde{\mathcal{P}}$.

Proof. The proof is obvious and hence omitted. $\square$

To be able to describe the Lagrange multipliers concerning (CFOP′), the dual space of $L^\infty(\Omega)$ has to be introduced. According to Yosida and Hewitt [12] this is the space of finitely additive signed measures or shortly *ba* space which is a notation introduced in [5, IX.2.15]. Here *ba* is short for *bounded additive*. The space of the finitely additive signed measure endowed with the norm of total variation $\|\mu\|_{var}$ is a Banach space and will be referred to as $(ba(\Omega), \|\cdot\|_{var})$. To verify that it is indeed a Banach space see e.g., [1, section 4.19]. The space of bounded linear functionals on $L^\infty(\Omega)$ can be identified with this space as it can be found in [12, Theorem 2.3].

The ordering cone on the vector space $L^\infty(\Omega)$ contains all functions which are not negative almost everywhere, i.e.

$$C_{L^\infty(\Omega)} \equiv \left\{f \in L^\infty(\Omega) | f(x) \geq 0 \text{ for almost all } x \in \overline{\Omega}\right\}. \tag{4}$$

It is well known that this cone is closed and it's interior is the set of all essentially bounded functions with an essentially infimum being strictly greater than zero.

The extended problem formulation (CFOP′) is a problem which is equivalent to (P) as well. $\tilde{\mathcal{P}} \subset Lip(\overline{\Omega})$ is a nonempty subset of a vector space as at least the initial profile is contained in it. The range space of the constraint mapping $(\hat{E}, \hat{\Lambda}_p) : \tilde{\mathcal{P}} \to \mathbb{R} \times L^\infty(\Omega)$ is a totally ordered vector space with ordering cone $\mathbb{R}_+ \times C_{L^\infty(\Omega)}$. The feasible set S contains all profiles in $\tilde{\mathcal{P}}$ satisfying the capacity constraint and the slope constraint, i.e. one has

$$S = \left\{p \in \tilde{\mathcal{P}} | (\hat{E}, \hat{\Lambda})(p) \in -(\mathbb{R}_+ \times C_{L^\infty(\Omega)})\right\}$$

This set is nonempty as well as again the initial profile has to be an element of it in the case of $\overline{E} \geq 0$. To determine the dual cone of the range space of the inequality constraint recall the dual space of it.

$$\mathbb{R}_+ \times ba(\Omega)$$

The dual cone of the space of essentially bounded functions contains all finitely additive signed measures assigning any measurable subset of Ω a non negative real number, i.e.

$$C^*_{L^\infty(\Omega)} \equiv \left\{\mu \in ba(\Omega, \mathcal{B}(\Omega)) | \mu(A) \geq 0 \text{ for all } A \in \mathcal{B}(\Omega)\right\}. \tag{5}$$

Here $\mathcal{B}(\Omega)$ denotes the set of all Borel sets in Ω. The claim will be proven by contradiction. Let μ be a finitely additive signed measure in the dual cone with $\mu(A) < 0$ for at least one measurable subset $A \subset \Omega$. The indicator function χ_A of this set is an element of the ordering cone of the essentially bounded functions $C_{L^\infty(\Omega)}$ as it only attains the values 0 and 1. For this function one obtains

$$\langle \mu, \chi_A \rangle = \int_\Omega \chi_A(x) d\mu(x) = \int_A 1 d\mu(x) = \mu(A) \leq 0$$

what contradicts the definition of the ordering cone.

As the ordering cone $\mathbb{R}_+ \times C_{L^\infty(\Omega)}$ is closed one might pass from (CFOP$'$) to the equivalent penalized form

$$\min_{p \in \tilde{\mathcal{P}}} \sup_{\substack{l \in \mathbb{R}_+ \\ \mu \in C^*_{L^\infty(\Omega)}}} -G(p) + \langle l, \hat{E}(p) \rangle + \langle \mu, \hat{\Lambda}(p) \rangle. \tag{6}$$

The corresponding dual problem

$$\max_{\substack{l \in \mathbb{R}_+ \\ \mu \in C^*_{L^\infty(\Omega)}}} \inf_{p \in \tilde{\mathcal{P}}} -G(p) + \langle l, \hat{E}(p) \rangle + \langle \mu, \hat{\Lambda}(p) \rangle \tag{7}$$

gives at least a lower bound on the extremal value of problem (CFOP$'$). Moreover, under certain additional requirements, it is possible to show the validity of strong duality what is proven by the following proposition.

Proposition 4.1 (Duality and the Extended Problem). *If the composite mapping $(-G, (\hat{E}, \hat{\Lambda}))(\tilde{\mathcal{P}})$ is convex-like w.r.t. the product cone $\mathbb{R}_+ \times C_Y$ and (GSC) is satisfied the Theorem* 2.1 *is applicable.*

Hence the dual problem (7) *is solvable and the extremal values of both problems coincide.*

Proof. The proof is analogous to Theorem 2.1. □

In the setting of (CFOP$'$) the existence of a profile in the interior points of the ordering cone is a non trivial property. As the product cone $C_Y = \mathbb{R}_+ \times C_{L^\infty(\Omega)}$ is endowed with the product topology, an element lies in the interior of it if it is an element of the interior points of both original cones. As the existence of a profile p with $\hat{E}(p) \in int(\mathbb{R}_+)$ is ensured easily as seen in the preceding section, this is not clear for the slope condition. For example consider a volume with a vertical part where no slope is possible for a profile. A two dimensional sketch of this scenario can be found in Figure 9.

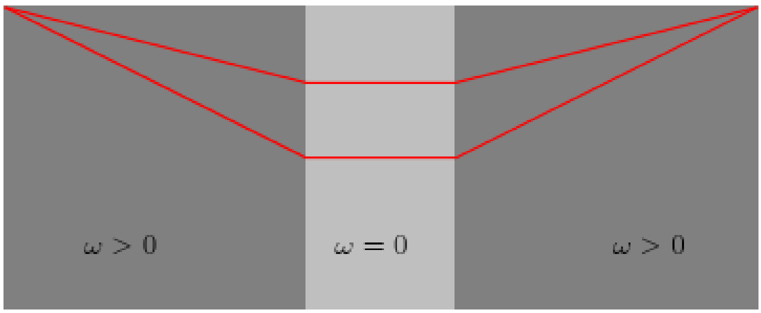

Figure 9 – Volume with vertical inclusion.

Here one can observe immediately, that **any** profile p has to satisfy

$$\hat{\Lambda}(p)(x) = 0$$

for all x with $\omega(x, \cdot) = 0$. In this case there cannot exist a profile in the interior of the negative ordering cone of $-C_{L^\infty(\Omega)}$ as these elements has to be strictly smaller than zero almost everywhere.

A possible remedy is to assume the initial profile p_0 to be an element of the feasible profiles which does strictly fulfill the slope condition anywhere according to [2, Proposition 2.3]. Then a profile in the interior of the product cone would be guaranteed.

According to Theorem 2.2 the following characterization of solutions for the extended problem formulation (CFOP$'$) in the case of $(-G, (\hat{E}, \hat{\Lambda}))$ being convex-like can be given.

Proposition 4.2 (Saddle Point property). *If the composite mapping*

$$(-G, \hat{E}, \hat{\Lambda})(\tilde{\mathcal{P}}) \to \mathbb{R} \times \mathbb{R} \times L^\infty(\Omega)$$

is convex-like w.r.t. the product cone $\mathbb{R}_+ \times \mathbb{R}_+ \times C_{L^\infty(\Omega)}$*,* p^* *is a solution of* (CFOP$'$) *if and only if there exist a* $(\bar{y}_1, \bar{y}_2) \in \mathbb{R}_+ \times ba(\Omega)$ *s.t.* $(p^*, (\bar{y}_1, \bar{y}_2))$ *is* *a saddle point of*

$$L(p, y_1, y_2) = -G(p) + \langle y_1, \hat{E}(p) \rangle + \langle y_2, \hat{\Lambda}(p) \rangle.$$

5 Conclusions

We were able to apply the duality theory for convex-like optimization problems to the stationary problem (CFOP) and the extended problem formulation (CFOP′). Correspondingly the existence of Lagrange multipliers for the effort constraint and also the slope constraint was proven.

Unfortunately this Lagrange multiplier in general only is a measure. This lack of functional regularity provides a challenge for numerical methods. Typical remedies are known, e.g. from PDE constraint optimization and can be distinguished into two main concepts. The first is to consider an a priori discretized problem as in [3]. The second one is to regularize the constraint yielding a Lagrange multiplier that is a function and can thus be conveniently represented and manipulated numerically.

Suitable numerical schemes are currently under investigation.

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