Solving Probabilistic Tasks in Geometrical Context by Primary School Students

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ABSTRACT – Solving Probabilistic Tasks in Geometrical Context by Primary School Students[1]. We present an exploratory study of solving probabilistic tasks proposed to a sample of 55 primary school 6th grade Costa Rican children on comparison of probabilities and the construction of the sample space, analysing their strategies and errors. Comparing the results with previous investigations, an improvement is observed in the item in which the comparison of favorable and possible cases can be applied, and where the comparison of areas is necessary; however, there were no differences in the item in which the order in which the favorable cases are located is introduced as a distractor. The sample space is generally correctly built in the cases of possible and equiprobable event, but not in those of impossible or certain events.
Keywords: **Comparison of Probability. Sample Space. Geometric Context. Primary Education.**

RESUMEN – Resolución de Tareas Probabilísticas en Contexto Geométrico por Estudiantes de Educación Primaria. Se presenta un estudio exploratorio de resolución de tareas probabilísticas de comparación de probabilidades y construcción del espacio muestral por parte de una muestra de 55 niños costarricenses de 6º curso de primaria, analizando sus estrategias y errores. Comparando con investigaciones previas, se observa un mejor desempeño en el ítem en que se puede comparar casos favorables y posibles y en el que es necesaria la comparación de áreas, pero no hay diferencias en aquel en que se introduce como distractor el orden de colocación de los casos favorables. Se observa que, en general, se construye correctamente el espacio muestral en los casos de suceso posible y equiprobable, no así en los de sucesos imposible o seguro.
Palabras-Clave: **Comparación de Probabilidades. Espacio Muestral. Contexto Geométrico. Educación Primaria.**

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Introduction

Elementary knowledge of probability is nowadays a basic requirement for the citizen, due to the many random situations in everyday life; it also plays an important role in the further study of statistics (Batanero; Chernoff; Engel; Lee; Sánchez, 2016; Gal, 2005). It also take an important part in the development of scientific thought, where many processes are described by random laws (Bryant; Nunes, 2012). This need has led many developed countries to introduce the teaching of probability from primary school onwards (e.g., Australian Curriculum, Assessment and Reporting Authority, 2013; Common Core State Standards Initiative, 2010, Ministerio de Educación, Cultura y Deporte, 2014; National Council of Teachers of Mathematics, 2000).

In Costa Rica, new mathematics curricula were approved in 2012, in which the area of statistics and probability was given a much higher profile than in previous national curricula, as well as an orientation closer to the nature of this discipline. These programmes include probability content throughout primary education (MEP, 2012); more specifically, children are expected to attain the following general skills by the end of each cycle of education, which are comprised of several grades each:

First cycle (1st to 3rd grades): Identify random and certain situations in everyday life and events associated with them. Classify random events as more or less likely for particular situations or experiments. Identify events according to simple outcomes that are linked to them (p. 147).

Second cycle (grades 4th -6th): Identify more probable, less probable or equally probable events according to the number of simple outcomes belonging to each event. Determine elementary probabilities associated with particular events. Pose and solve problems related to random situations (p. 247).

When including a new curricular content, it is important to ensure that students have the necessary competences to deal with it successfully, an information that can usually be acquired from didactic research. However, due to the low relevance given to probability in the curricula prior to 2012, there has been a lack of investigation on the teaching and learning of probability in Costa Rica. Moreover, previous studies on the topic with children of this age in other countries have been conducted when they had not been taught probability.

Our research aims to provide original information about Costa Rican students who have started probability learning, which can be compared with that of previous studies. More specifically, we focus on children in the 6th grade of primary education (11 and 12 years old). In addition, we centre on probability in geometric context, where previous research is scarce.

The choice of this educational level is due to two reasons: this grade marks the end of primary education, and this group of students had followed the first two years of the second cycle with the current Mathematics curriculum, which was approved in 2012 and was fully implemented at all educational levels from 2015 onwards. It is important to note that at the time this research was carried out, the participants in the sample were just starting the 6th grade, so they had not yet studied the contents of probability corresponding to that school year.

Specifically, the aim of the research is to explore and describe the skills and reasoning of a sample of children, when solving simple probability comparison problems in a geometric context, compare the results with previous research, and to evaluate the children's skills in the construction of sample spaces in this context.

Theoretical Framework

The study of children's ability to compare probabilities in simple random experiments begins with Piaget and Inhelder (1951), who built on their constructivist theory that assumes that learning arises from experience, activity and prior knowledge. According to the authors, when facing a problem, the child uses the knowledge he or she already possesses, and if he or she is not able to deal with it, a cognitive conflict arises, which is solved through the processes of assimilation and accommodation. Assimilation is the incorporation (acceptance) by the subject of new data or ideas, and accommodation consists in changing or restructuring of existing ones. The authors suggest that knowledge progresses in developmental stages, which have an established order, although the age at which a child reaches one of these stages may vary.

To study the children's ability and reasoning in comparing simple probabilities, Piaget and Inhelder (1951) used white tokens marked or unmarked with a cross, placing a small number of tokens of each type in transparent boxes. They asked the subjects in their study to choose between two such boxes, asking them which of the boxes they preferred to choose, in a game where the winner has to obtain a marked token. The authors changed the number of white tokens (unfavourable cases) and marked tokens (favourable cases) in the two boxes and conducted interviews using this game with boys and girls from the age of three and a half to 13-14 years. By comparing similar responses from groups of subjects of the same age, they obtained a description of three development stages of their reasoning about this type of problem.

The first stage (I) is divided into two levels. Level IA is characterised by a lack of those logical schemas that allow understanding the inclusion of the part in a whole, the disjunction between two types of elements and the conservation of quantities (e.g., when the tokens are moved from a place to another). Therefore, these subjects can only deal with problems in which there is double impossibility (all the tokens are white in both boxes), double certainty (all of them are marked) or certainty-impossibility (one box with white tokens and one with printed tokens). Not all the possible cases are considered and only the favourable cases are counted. At level IB, the subjects consider only one type of tokens (favourable or unfavourable), and are neither yet able to conceive the favourable cases as part of the possible cases (part-whole comparison); nor are they able to compare the favourable cases with the unfavourable cases (part-part comparison). However, they begin to understand that probability depends on the number of favourable cases.

The second stage (II) starts around the age of 7 and is also divided into two sub-levels. At level IIA, problems involving a single variable can be worked out, i.e., when only favourable or unfavourable cases need to be taken into account. Additive operations (e.g., subtracting the number of favourable from the unfavourable cases or vice versa, in each box and comparing the differences) are used. Disjunction begins to be understood (each case is either favourable or unfavourable), but there is a systematic failure in cases where the composition of favourable and unfavourable cases in both boxes is proportional, since the idea of fraction or proportion has not been acquired. At level IIB, the child begins to solve the problem by correspondence, when the composition of the boxes is proportional (for example, two favourable cases for each unfavourable one, in both boxes).

At stage III, the subject is able to solve the proportionality case easily and can think of a general solution if the number of favourable and unfavourable cases is small and the ratio between them is simple (e.g., double, triple, etc.); this solution becomes more general with age.

Successful probability comparison, in the general case, involves proportional reasoning, the developmental stages of which have been analysed by several authors (Karplus; Peterson, 1970; Noelting, 1980a; 1980b) and summarised in Behr et al. (1992) and Ben-Chaim, Keret & Ilany (2012). The most relevant author for our work is Noelting (1980a; 1980b), who, through a problem of comparing two mixtures (water and orange juice), subdivided Piaget and Inhelder' (1951) stages, and determined the approximate ages at which each stage is reached, which are: IA: 4 years, IB: 7 years, IIA: 8 years, IIB: 11 years and III: 12-13 years.

Finally, our work also builds on the understanding of the sample space (set of possible events in an experiment). Such understanding is a prerequisite for the child to be able to compare probabilities, as this problem requires thinking about the set of favourable and unfavourable cases as a whole set of possible cases (Bryant; Nunes, 2012). Probability estimation or comparison begins by enumerating, or imagining, the set of elements in the sample space, the correct determination of which is an essential part of solving the problem (Chernoff, 2009). However, little research has focused on children's construction of the sample space for a simple experiment. An exception is the work by Abrahamson (2006), who asked the children to write down all the possibilities for an experiment consisting in obtaining four coloured balls from a set of two-coloured balls. This corresponds to a compound experiment, a task in which most students have difficulties. In this article, we deal with a simpler experience, working only with simple experiments and in the same context.

Previous research

Piaget and Inhelder's research inspired a series of papers on children's probabilistic reasoning, which are described in detail in Bryant and Nunes (2012), Jones, Langrall and Mooney (2007) and Langrall and Mooney (2005). Those most relevant to our work are summarised below.

Falk, Falk and Levin (1980) asked to 4 to 11 years old children to compare probabilities by varying the number of favourable and possible cases, and using two contexts: urns with balls and roulettes. In the case of roulettes, they used two-coloured sectors with different numbers of sectors of each colour. From the age of 6 years onwards, the subjects in their sample presented some correct ideas, when the problem was simple. The authors observed that children did not always use the same strategy to compare probabilities, but that the strategy depended on the values assigned to favourable and possible cases. In the case of roulettes, many children compared the number of sectors, rather than using areas. Finally, some subjects had irrelevant ideas (such as favourite colour) that they used instead of analysing the data to compare probabilities. As Pratt (2000) pointed out, the strategies used to make probabilistic judgements or comparisons are often subject to systematic biases.

Truran (1994) conducted research with 32 children aged 8 to 15 years on comparing probabilities in urns. As a result, he identified new strategies that extend those described in Piaget and Inhelder's research, such as describing the contents of the urn without making a choice, giving a correct answer without justification, using different strategies to estimate the probability in each urn, preference for the smallest total number of balls, comparison with known simple proportions and comparison between odds ratios for and against a given event.

The study most related to this topic was carried out by Green (1983), who assessed probabilistic reasoning in English students aged 11 to 16 with a test that reproduced on paper and pencil Piaget and Inhelder's experiments. Some of the items involved comparison of probabilities in the context of urns and roulettes. The strategies he found in the comparison of probabilities in urns were: a) choosing the urn with the highest number of possible cases; b) selecting the one with the highest number of favourable cases; c) choosing the highest difference between favourable and unfavourable cases; d) preferring the highest proportion between favourable and unfavourable cases.

In the context of roulettes, Green identified the following types of strategies: a) comparing the areas of the parts into which the roulette is divided; b) analysing the number of favourable or unfavourable sectors, regardless of the area; c) comparing the number of favourable or unfavourable sectors, or both; d) using ratios of favourable or unfavourable cases; e) other strategies, such as considering the separation or continuity of favourable or unfavourable sectors in the roulette.

Cañizares (1997) carried out a study with 320 Spanish children aged 10 to 14 years and, among other problems, proposed to the children comparison of probabilities in urns and roulettes. The author described the strategies used by the students, which were classified into one and two variable strategies. The one-variable strategies consisted of comparing only favourable, unfavourable or possible cases; and the two-variable strategies consisted of comparing favourable and possible cases in an additive or multiplicative way. Cañizares deduced that the most frequent level of reasoning of the children in her sample was IB to IIB, according to Noelting's categorisation (1980a; 1980b), with few subjects reaching the level IIIB. Some variables that influenced the response were the composition of the urns (number of favourable and possible cases) and the existence of possible biases in the context (for example, beliefs in a favourite number, equiprobability or possibility to control the situation).

In his work, Green (1983) also tested the comprehension of probability language in his sample, by including the expressions *impossible*, *possible*, and *equal chance*. In the 6th graders, he obtained 84% correct answers for the meaning of *impossible*, 73% for *possible* and 18% for *equal possibility*. Cañizares (1997), with the same tasks and also in 6th grade boys and girls, obtained 81.3% of correct answers in the meaning of *impossible*, 68.1% for *possible* and 42.9% for *equal possibility*. These results support Pratt's (1998) assertion that probability is possibly the branch of mathematics with the greatest distance between everyday application and formal understanding of the concepts and that, in fact, mathematical discourse on the subject is often different from everyday language.

In this paper we propose three probability comparison items based on roulettes (adapted from Green's, 1983 items 3 and 19) and spinners (modified from Green's item 17), which also were used by Cañizares (1997), as well as another item related to the ideas of certain, possible, equiprobable and impossible events to a sample of Costa Rican children who, unlike children in the studies described, had received instruction in probability throughout primary school.

Metodology

The sample consisted in 55 boys and girls in primary school 6th grade; 40 students aged 11 years and 15 students aged 12 years, of whom 29 were studying in a private institution and 26 in a public (state-funded) school in the province of Cartago, Costa Rica. Although the institutions are located closer than two kilometres apart, the students in the private school come from different districts of the province, while 90% of the students in the public school live in the school district.

Both institutions follow the Mathematics syllabus of the Costa Rican Ministry of Public Education – MEP (2012). Although theoretically, in the public school there are 8 Mathematics lessons per week, and in the private school only 5 (one of them specifically dedicated to Statistics and Probability), the number of lessons per year in a public institution is much lower than in the private institution, due to different causes (such as extracurricular activities or training, among others).

From the interviews with the teachers in charge of teaching Mathematics, it is known that the children studied probability according to the MEP syllabus (2012) from 2016. Consultations with teachers indicate that the study of probability was based on the textbook, with no evidence of activities involving experiments. In the review of the texts used, we only found one exercise in the context of roulettes, with no comparison of probabilities.

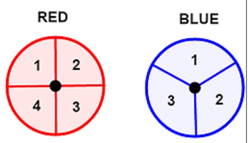
The sample was given a questionnaire with four items, all of them related to probability in a geometric context. The first three items were taken from Cañizares (1997), who translated them from similar items by Green (1983), in which the probabilities of a given event should be compared in two roulettes or two spinners and the answer justified.

The roulettes reproduced in item 1 (Figure 1) are divided into equal parts, with four and three sectors, respectively, corresponding to equiprobable areas. The number of favourable cases (getting number 1) is the same in both roulettes, while the number of unfavourable cases is lower in the blue roulette (2) than in the red one (3), so that the probability of getting 1 is 1/3 and 1/4, respectively. The children could compare the unfavourable cases in the two roulettes to give the correct solution, without resorting to fractions, which would be a strategy of reasoning level IIA, according to Piaget and Inhelder (1951).

Figure 1 – Item 1

In the figure there are two disks (roulettes) with pointers which after spinning stop in a number (look to the figure):
In which disk it is easier to get a 3? Mark the correct response:

A. It is easier to obtain 3 in the red disk.
B. It is easier to obtain 3 in the blue disk.
C. Both disks give the same chance to obtain a 3.
D. I do not know.



Why did you choose this response?

Source: Adapted from Green (1983).

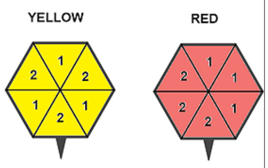
In item 2 (Figure 2), two spinners are shown with the top area forming a regular hexagon and divided into six equal triangles (same area in each of them), three of which correspond to the number 1 (unfavourable cases) and three to the number 2 (favourable cases). Therefore, the probability of getting a 2 is the same in both spinners. The task could be solved by establishing a correspondence between the favourable and unfavourable cases, which is typical of reasoning level IIB, in Piaget and Inhelder's theory (1951). Additionally, the order in which the triangles numbered 1 and 2 are placed in each spinner is different: in the yellow spinner the numbers alternate, while in the red one they are consecutive. This placement of the numbers in the spinner is a distractor that can affect the child when comparing probabilities (Maury, 1984).

Figure 2 – Item 2

Two six-sided spinners are marked with 1 and 2 as shown in the figure:
Which spinner gives you the better chance of landing on a 2 when it spins? or do they give you the same chance?

A. Yellow is better for getting a 2.
B. Red is better for getting a 2.
C. Both spinners give you the same chance for 2.
D. I do not know.

Why?



Source: Adapted from Green (1983).

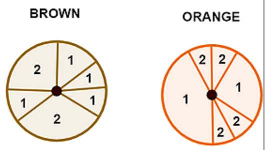
In item 3 (Figure 3) there are two roulettes with 6 sectors of different amplitude. In the brown roulette there are four sectors numbered with number 1 (favourable cases) and two sectors with number 2 (unfavourable cases). The number of favourable cases is higher in the brown roulette than in the orange (4 vs. 2), but the amplitude and surface occupied by the favourable cases is higher in the orange roulette. If we compare the ratio between favourable and unfavourable cases, we would choose the brown roulette, which is an incorrect answer, as the orange roulette has a larger surface area favourable to number 1.

Figure 3 – Item 3

Two disks (roulettes), one orange and one brown are marked with numbers (look to the figure).
Each disk has a pointer which spins round. If you want to get a 1, is one of the disks better than the other, or do they both give the same chance?

A. Brown is better for getting a 1.
B. Orange is better for getting a 1.
C. Both disks gave the same chance
D. No one can say.

Why did you choose this answer?

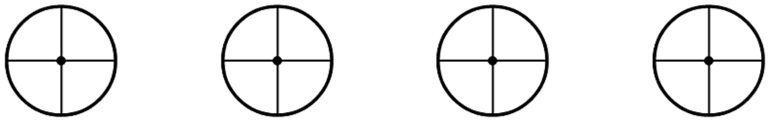


Source: Adapted from Green (1983).

A fourth item (Figure 4) of our own elaboration was included, which children are requested to create a sample space such that, in a hypothetical game, the event Mary wins the game (if she gets the number 1) is certain, possible, equiprobable or impossible. The purpose of this item is to assess the sample subjects' understanding of this type of event.

Figure 4 – Item 4

Maria and Esteban are playing with a roulette. Maria wins a candy if the spinning needle lands on 1 and Esteban wins a candy if it lands on 2. Place the numbers in the following roulette wheels to make it happen:



(A) María wins

(B) It's possible than María wins

(C) María and Esteban have same chances to win

(D) It's impossible than Maria wins

Source: Own elaboration created for the research.

The random generator used in the game described in item 4 is a roulette divided into 4 equal parts, so that the child can create a sample space of up to 4 different events (4 different numbers), but if any of the numbers is repeated (e.g., constructing a roulette with sectors numbered 1, 1, 2, 3), a sample space is obtained where event 1 is twice as likely as the others, and therefore Mary would be twice as likely to win as Esteban. To achieve the certain event, the number 1 must be repeated four times (so that Mary always wins), while to achieve the impossible event, the number 1 must be excluded. In summary, solving the task requires intuition about the random experiment and its outcomes, as well as about the different types of events.

Results

Comparing Geometrical Probabilities

Firstly, the correctness of the response to each of the first three items was analysed in order to compare the results with those obtained in previous research.

Table 1 shows the response choices for item 1, where the majority of the sample chose the correct option, with a percentage (80%) somewhat higher than that obtained in students of the same age by Green (1983), 71%, and Cañizares (1997), 79.1%. Therefore, this item was very easy for both the children in the sample, who show the IIA level of reasoning, according to Piaget and Inhelder's (1951) classification. The best result in the present research is attributed to the teaching received by the children along primary education.

Table 1 – Frequency and percentage of responses to item 1

Response	Frequency	Percentage
It is easier to obtain 3 in the red disk	4	7.3
It is easier to obtain 3 in the blue disk (correct)	44	80.0
Both disks give the same chance to obtain a 3	7	12.7

Source: Own elaboration created for the research.

Table 2 reproduces the results of the second item, where more difficulty is observed, although 50.9 % of the children still answered correctly.

Table 2 – Frequency and percentage of responses to item 2

Response	Frequency	Percentage
Yellow is better for getting a 2	12	21.8
Red is better for getting a 2	14	25.5
Both spinners give you the same chance for 2 (correct)	28	50.9
I do not know	1	1.8

Source: Own elaboration created for the research.

The results in this item were close to those by Green (49%) and Cañizares (51.6%), with students of the same age. Therefore, the bias introduced by the location of the favourable cases in the spinners did not seem to have been overcome by the teaching. Moreover, according to Piaget and Inhelder (1951), the comparison of probabilities in equiprobable events corresponds to a higher level of reasoning (this item is placed at level IIB).

In Table 3 the results in the third item are presented. In this item Laplace's rule or the comparison of favourable or possible cases cannot be applied, as the areas of each roulette sectors are different. Even so, a large part of the sample managed to solve the problem correctly (63.6%), while in Green's research only 43% of the subjects of the same age and 46.2% in Cañizares provided correct solutions, a difference which again we explain by the teaching received in our study.

Table 3 – Frequency and percentage of responses to item 3

Response	Frequency	Percentage
Brown is better	15	27.3
Orange is better (correct)	35	63.6
Both disks gave the same chance	5	9.1

Source: Own elaboration created for the research.

Table 4 shows a better performance, in general, in the private school, since comparing with Green (1983) and Cañizares (1997) results, in the first two items, the public school had lower results, while the private school remained above and increases its difference. In item 3, both schools were above the percentages of previous research.

Table 4 – Percentage of correct responses to items 1, 2 y 3, according to school type in relation to results in previous research

Item Nº	Type of school		Cañiza-res (1997)	Green (1983)
	Public	Private		
1	65.4	93.1	79.1	71.0
2	34.6	65.5	51.6	49.0
3	57.7	69.0	46.2	43.0

Source: Own elaboration created for the research.

Arguments used to Justify the Comparison of Probabilities

In order to better understand the reasons that led the sample subjects to choose their answers, the arguments provided by them were analysed. These arguments were classified, by means of a qualitative analysis, according to the categories described by Cañizares (1997) and Cañizares and Batanero (1997), which are as follows:

1. Comparing the number of favourable or possible cases. According to Piaget and Inhelder (1951), this strategy corresponds to the use of one variable in the task and is typical of developmental level IIA. The subject compares the number of sectors in both roulettes favourable to the requested event or the total number of sectors in the roulettes. Unfavourable cases are not considered or at least no explicit reference is made to them. The strategy works in item 2 (the number of unfavourable cases is the same and all the sectors have the same area) and in item 1 (the roulettes are divided into equal parts, with the same number of favourable cases but different number of possible cases); but not in item 3, because the width of each sector of the roulette is not taken into account. Some examples are the following:

E1: Because there are fewer numbers and the blue does not have four (option B, item 1).

E2: Because in both of them there are three twos (option C, item 2).

E3: There are more ones (option A, item 3).

2. Explicit comparison of the number of unfavourable cases. When the unfavourable cases are explicitly counted and compared to justify the answer, in choosing the roulette with the lowest number of unfavourable cases. It also corresponds to a one-variable strategy, but it is more advanced than comparing favourable or possible cases. It gives a correct solution when the number of favourable cases is equal in both roulettes. It is more elaborate than previous strategy since it takes into account the complementary of the requested event, which implies understanding the idea of disjunction. Some examples are reproduced below:

E4: Because in the blue disk there are 2 chances of loose and in the red disk there are 3 (option B, item 1).

E5: Because 4 is not in the blue disk (option B, item 1).

3. Explicit comparison of the number of favourable and unfavourable cases. According to Cañizares and Batanero (1997), this is a two-variable strategy, where all the problem data are used. The favourable and unfavourable cases are compared in an additive way, as shown in E6, or a multiplicative comparison is made, as in E7. Piaget and Inhelder (1951) warn that these strategies do not serve, in general, to solve any problem (as in item 3, which leads to error because the area of each sector is not taken into account), although they can lead to success depending on the data (as in item 2).

E6: Because in the orange there are more 1 than 2 (option B, item 3).

E7: There are 3 times 2 and 3 times 1 (option C, item 2).

4. Comparing the area occupied by the intended number. This is a strategy that can only be applied in a geometric probability context and can be used successfully in all the proposed items. In this paper it is employed in item 3, where the comparison of favourable or unfavourable cases does not work and the student should resort to analysing the total area covered (extent of the surfaces), for example:

E8: Because 1 has more space and it is probable to obtain 1 (option B, item 3).

E9: Because 1 has the widest space (option B, item 3).

5. Compares areas and number of unfavourable or favourable cases. This is a combination of the two previous strategies, which involves a higher level of reasoning, for example:

E10: I chose that answer because there are fewer numbers and the spaces are bigger (option B, item 3).

E11: There are less numbers and 3 has a bigger space (option B, item 1).

6. Equiprobability bias. Some children refer to chance (also described as luck) to deduce that any event is equally likely, regardless of the area occupied or the number of favourable or possible cases. These responses (E12) are typical from the equiprobability bias, described by Lecoutre (1992), consisting of equating randomness and equiprobability; they also appeared in Cañizares (1997).

E12: They vary in the number 4, but can have same likelihood (option C, item 1).

7. Physical considerations. Other arguments based on the placement of the numbers on the spinner, the force given to the needle or similar considerations were found in some students. E13 and E17 based on their subjective belief that the order in which the numbers are placed on the roulette influences the probability, even when the areas corresponding to the favourable and possible cases are identical (case E13). Other answers reveal a preference for the possible outcome, depending on the location (in the corners, E14), the initial position of the needle (E15) or even the force applied to the spinner (E16, E18).

E13: Yellow, since it is more distributed (option A, item 2).

E14: Because it sure falls on the corners (option A, item 2).

E15: Because on the blue disk the needle points to two, and the next number will be three, while on the red disk it points to one and the next number will be two (option B, item 3).

E16: It's depended on the strength (option C, item 1).

E17: Because both are close to 1 (option C, item 3).

E18: If you spin it carefully you get 3 (option A, item 1).

Table 5 presents the results concerning the arguments identified for each item, where comparing favourable, unfavourable or possible cases are correct in the first two items (the number of favourable cases is equal), and comparison of areas only works in the third item.

Item 1 is characterized by the comparison of possible cases, which disguises the comparison of unfavourable cases as there is only one favourable case. Thus, implicitly, disjunction is used and the sample space is conceived as a union of favourable and unfavourable cases. That is, if the subject indicates that in a roulette there is more probability because there are fewer numbers, he implicitly refers to the number of unfavourable cases. In Green's research (1983) only 28% of the children of the same age as those participating in the present study used this argument and 30.8% in Cañizares' (1997). Therefore, the results in our sample outperformed those of the aforementioned studies.

Table 5 – Frequency and percentage of strategies in the items

Strategy	Item 1		Item 2		Item 3	
	N	%	N	%	N	%
1. Comparing favourable or possible cases	27	49.1*	7	12.7*	13	23.6
2. Comparing unfavourable cases	5	9.1*				
3. Comparing favourable and unfavourable cases			14	25.5*	1	1.8
4. Comparing areas	6	10.9*			27	49.3*
5. Areas and favourable and unfavourable cases	5	9.1*			4	7.3*
6. Equiprobability bias	6	10.9	3	5.5	2	3.6
7. Physical considerations	6	10.9	30	54.5	7	12.7
Does not know			1	1.8	1	1.8

*Correct argument in this item.
Source: Own elaboration created for the research.

In item 2, both the comparison of possible or favourable cases and the comparison of favourable and unfavourable cases are correct, and the use of both together is close to 40%. In Green's case, 35% of the subjects compared favourable and possible cases or favourable and unfavourable cases, and in Cañizares' research, around 25%, so that, once again, the results of the current study are superior. However, in this study there was a very high percentage of children who used irrelevant physical considerations (54.5%), while in Cañizares' work there were only 21%, being in her study the reference to luck or equiprobability somewhat higher than in this research.

The third item is dominated by comparison of areas, sometimes combined with comparison of favourable or unfavourable cases. The proportion of correct arguments presented in this article is higher than in Green and in Cañizares with boys and girls of the same age (41% and 41.8% respectively), while the comparison of favourable and unfavourable cases (incorrect strategy) is lower than in these authors.

According to the type of school (Table 6), the results were generally similar except for item 2, where 41.4% of students in private schools opted for strategy 3, compared to 7.7% of students in public schools. It is important to note that, according to Cañizares and Batanero (1997), this strategy is more sophisticated when considering two variables. This item, in general, was more complex for public school children, where less than 27% had correct arguments, and the argument based on physical considerations had a rate of 65.4%.

Table 6 – Percentage of strategies in items 1 to 3 by school type

Strategy	Item 1		Item 2		Item 3	
	Priv.	Pub.	Priv.	Pub.	Priv.	Pub.
1. Comparing favourable or possible cases	55.2*	42.3*	10.3*	19.2*	17.2	30.8
2. Comparing unfavourable cases	10.3*	7.7*				
3. Comparing favourable and unfavourable cases			41.4*	7.7*	3.4	
4. Comparing areas	10.3*	11.5*			55.2*	46.2*
5. Areas and favourable and unfavourable cases	10.3*	7.7*			6.9*	3.8*
6. Equiprobability bias	6.9	15.4		3.8	3.4	3.8
7. Physical considerations	6.9	15.4	48.3	65.4	10.3	15.4
Does not know				3.8	3.4	

*Correct argument in this item.
Source: Own elaboration created for the research.

Building the sample space

In Table 7 the frequency and percentage of the type of sample space the students constructed is presented, whether it is a certain event, very probable (probability equal to or greater than 0.75), possible (probability between 0.25 and 0.75), equiprobable, unlikely (probability equal to or less than 0.25) and impossible, depending on what is requested in each item question (certain, possible, equiprobable or impossible).

Table 7 – Frequency and percentage of response categories in item 4

Sample space bult corresponds to …	Type of event requested: Certain		Possible		Equiprobable		Impossible	
	N	%	N	%	N	%	N	%
Certain event	19	34.5*					1	1.8
Very likely event	23	41.8	20	36.4*			10	18.2
Equiprobable event	6	10.9	27	49.1*	50	90.9*	9	16.4
Unlikely event	5	9.1	6	10.9*	3	5.5	19	34.5
Impossible event							14	25.5*
No response	2	3.6	2	3.6	2	3.6	2	3.6

*Correct in this item.

Source: Own elaboration created for the research.

Few subjects did not answer the questions, and it was easiest to understand the idea of possible event, correctly answered by the majority of the sample (96.4%). Almost half of them constructed a sample space corresponding to the equiprobable event, which may be associated with the belief that in a random event all outcomes are equiprobable, (equiprobability bias, Lecoutre, 1992), although in this case the argument is valid. It was also easy to identify the equiprobable event, as a large majority (90.9%) answered correctly. Regarding the certain event, more than a third of the sample constructed it correctly (34.5%), but also a high frequency of students interpreted certain as very probable (41.8%). It can be observed that the greatest difficulty was in the impossible event, which was generally considered unlikely (34.5%) and only a quarter of the sample built it correctly.

When comparing with Green (1983) and Cañizares (1997), the present results were better. The authors obtained a 16% and 26.4% success rate, respectively in the interpretation of the certain event in 6th graders, 18% and 42.9%, respectively in the equiprobable event and 73% and 68.1%, respectively in the possible event. The only results of the present study that were worse was the identification of the impossible event, where the mentioned authors obtained correct response rates of 84% and 81.3%, respectively. It should be noted, however, that the task posed in the present study is more difficult than those proposed by the Green and Cañizares, who asked the children to propose synonyms or write sentences with the terms analysed, while in this research the children are asked to construct the sample space of a possible experiment.

Regarding the type of school (Table 8), the results are very similar, although it should be noted that public school participants had all correct answers in the equiprobable event; and the private school participants performed slightly better than public school participants in the impossible event.

Table 8 – Frequency and percentage of response categories in item 4 by school type

Sample space built	Event requested							
	Certain		Possible		Equipro-bable		Impossible	
	Priv.	Pub.	Priv.	Pub.	Priv.	Pub.	Priv.	Pub.
Corresponds to the certain event	34.5*	34.6*						3.8
Corresponds to the very likely event	41.4	42.3	31.0*	42.3*			10.3	26.9
Corresponds to the equiprobable event	6.9	15.4	51.7*	46.2*	82.8*	100.0*	13.8	19.2
Corresponds to the unlikely event	10.3	7.7	10.3*	11.5*	10.3		37.9	30.8
Corresponds to the impossible event							31.0*	19.2*
No response	6.9		6.9		6.9		6.9	

*Correct in this item.

Source: Own elaboration created for the research.

Relating the Comparison of Probabilities and the Construction of the Sample Space

Table 9 reproduces the joint distribution of the number of correct answers in the probability comparison tasks (items 1 to 3) and sample space construction, according to different types of events (item 4). All percentages are computed with respect to the row total, except for the percentages of the total number of children getting 0, 1, 2, 3 or 4 correct sample spaces, which is computed with respect to the total sample.

Only two children incorrectly constructed all four requested sample spaces, most frequently getting two correct solutions (56.4%) followed by four correct (21.8%) and three correct (14.5%); only two children responded correctly to a single construction of the requested sample space. Thus, the construction of the sample space was a complex task, with just over a third of the responses having at least three correctly constructed sample spaces. We explain these results by the general confusion between impossible and unlikely events or between certain and very likely events. Moreover, only two subjects failed in all probability comparisons, the most frequent case being to answer correctly to two of them (36.4%) or to all three (30.9%).

We also observe the relationship between the number of correct answers in probability comparison and in the construction of the sample space. Thus, the two children who failed all the probability comparison questions only managed to complete two sample spaces adequately; and the percentage of those who constructed all four sample spaces correctly increases to 35.3% of the children who correctly solved all the probability comparison tasks.

Table 9 – Frequency and percentage of correct responses in comparing probabilities and constructing the sample space

Correct comparison of probabilities		Correct sample spaces 0	1	2	3	4	Total	% Total
0	Frequency			2			2	3.6
	Row %			100.0				
1	Frequency		1	9	3	3	16	29.1
	Row %		6.3	56.3	18.8	18.8		
2	Frequency	1	1	13	2	3	20	36.4
	Row %	5.0	5.0	65.0	10.0	15.0		
3	Frequency	1		7	3	6	17	30.9
	Row %	5.9		41.2	17.6	35.3		
Total	Frequency	2	2	31	8	12	55	100.0
	Row %	3.6	3.6	56.4	14.5	21.8	100.0	100.0

Source: Own elaboration created for the research.

Discussion and Conclusions

The responses in the comparison of probabilities in roulettes and spinners indicates better results in the present study than those obtained with boys and girls of the same age in previous research, which was carried out in a period when no elementary probability teaching was given in schools. These differences are found mainly in the first item, which can be solved by comparing favourable and unfavourable cases, and in the third item, whose solution is based on the comparison of areas. This better reasoning is attributed to the current teaching of children, which confirms Fischbein's (1975) theories on the importance of building on the intuitive basis with probability and providing instruction in probability, supported as far as possible by manipulative material.

However, teaching does not seem to have influenced the answers to the second item, as the results were similar to those obtained in Green's (1983) and Cañizares's (1997) work. The arrangement of the numbers on the spinners was taken into account as a subjective and distracting element in the comparison of probabilities. In this sense, it is considered necessary for students to carry out experiments with physical devices similar to those shown in the items so that they can gradually correct inadequate reasoning biases. According to Pratt (2000), many materials can serve as resources to support the construction of correct intuitions about chance. This recommendation follows the principle that knowledge is actively constructed by the subject and not passively received from the environment (Piaget; Inhelder, 1951); hence the importance of active teaching also in the field of probability.

It was easy to construct the sample space associated with a possible or equiprobable event in the task of constructing sample spaces, but there were many errors in the case of certain or impossible events, reinforcing the results of research such as those by Green (1983) and Cañizares (1997), who pointed out to the children's difficulty with probability language.

Understanding this language and the sample space associated with simple experiments was related to performance in the probability comparison tasks in this work, which suggests the need to reinforce the students' training in such tasks. In fact, some researchers also support working with probability from early childhood to introduce children to the language of chance. Alsina (2012) analyses the possibilities of commonly played games in childhood (e.g., with dice). In children's conversations, games, stories and songs, references to chance can often be observed. For example, children use songs to draw lots in hide-and-seek or rescue games, they use board games with dice or roulettes. These games can be used for educational purposes to reinforce the children's probabilistic intuition and their differentiation of certain, impossible and possible events.

It is clear that good probability teaching requires well prepared teachers who are enthusiastic about the subject. However, some teachers and student teachers in primary education may feel insecure about teaching probability because they have not received sufficient training in probability education or have no experience in teaching probability (Alpízar et al., 2012; Alpízar; Chavarría; Oviedo, 2015). This may lead them to reduce or omit this teaching, in providing reasons such as lack of time due to tight schedules or insecurity in teaching the topic.

A contribution of this paper is the material presented, which includes the items proposed to the children, the detailed analysis of the children's responses and their classification. All of this can be used to organise teacher training courses in which teachers analyse the items and the prototypical responses of children at different levels of cognitive development. It is important that they reflect on the cognitive demands of the tasks posed to the children, their ways of reasoning, possible biases and the best ways to overcome them, and finally that they design similar tasks that they can develop with their students.

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