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Entropy variation in isothermal fluid flow considering real gas effects

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***Abstract**: The present paper concerns on the estimative of the pressure loss and entropy variation in an isothermal fluid flow, considering real gas effects. The 1D formulation is based on the isothermal compressibility module and on the thermal expansion coefficient in order to be applicable for both gas and liquid as pure substances. It is emphasized on the simple methodology description, which establishes a relationship between the formulation adopted for ideal gas and another considering real gas effects. A computational procedure has been developed, which can be used to determine the flow properties in duct with a variable area, where real gas behavior is significant. In order to obtain quantitative results, three virial coefficients for Helium equation of state are employed to determine the percentage difference in pressure and entropy obtained from different formulations. Results are presented graphically in the form of real gas correction factors, which can be applied to perfect gas calculations.*

***Keywords**: Isothermal flow, Real gas, Entropy, Petroleum engineering.*

LIST OF SYMBOLS

a	Sound velocity	m/s
A	Cross section	m^2
A_W	Wet area	m^2
B	Second virial coefficient	$cm^3/mole$
C	Third virial coefficient	$cm^3/mole^2$
Cp	Specific heat at constant pressure	J/kgK
Cv	Specific heat at constant volume	J/kgK
f	Friction coefficient	-
h	Enthalpy	J/kg
M	Mach number	-
$\dot{m}$	Mass flow rate	kg/s
P	Pressure	Pa
p_W	Wet perimeter	m
$\dot{q}_c$	Heat exchange parameter	W/m^2K
R	Gas constant	J/kgK
$\tilde{R}$	Parameter of Equation (37)	$m^3/Pa/kg/K$
S	Entropy	J/kgK
t	Time	s
T	Temperature	K
u	Velocity	m/s
U	Internal energy	J/kg
v	Specific volume	m^3/kg
W_{He}	Molecular weight of helium	mole
x	Longitudinal coordinate	m
Z	Compressibility factor	-
α	Angle between the horizontal and the direction of flow	rad
α_P	Thermal expansion module	K^{-1}
β_T	Isothermal compressibility module	m^2/N
ε_P	Pressure correction factor	-
ε_S	Entropy correction factor	Pa^2
γ	Specific heat ratio	-
ρ	Density	kg/m^3
μ	Viscosity	kg/ms
τ_W	Shear stress	N/m^2

INTRODUCTION

The term "isothermal process" describes a thermodynamic process that occurs at a constant temperature. There are a lot of examples of technical analysis using isothermal process. Isothermal compression is an example to illustrate the position that isothermal process takes along other thermodynamic processes. In a real engine (compressor), the isothermicity condition cannot be fulfilled and every real thermodynamic process demands more energy than the isothermal compression. In this context, the practical importance of isothermal compression lies in its use, as a reference process to evaluate the actual compression (Oldrich and Malijesvsky, 1992). The adiabatic frictional flow assumption is appropriate to high speed flow in short ducts. For flow in long ducts, such as natural gas pipelines, the gas state approximates more closely to the isothermal one. By doing this approach, it is possible to establish a relationship among all thermodynamic quantities (White, 2005).

In the flow field thermodynamic calculation of aeronautical devices, the real process is approximated to a suitable idealized one that can be mathematically

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described. Usually, the choice lies among isentropic, polytropic, and isothermal processes. These thermodynamic processes are also used as reference ones in the internal flow field calculation of aeronautical devices. The isentropic and polytropic processes have been discussed by many authors, including their applications to real gases (Oldrich and Malijesvsky, 1992). In contrast, the isothermal process has not received such attention. Attempts at refining the results by inserting values calculated from a virial equation of state for real gas have led to some improvement. However, these efforts particularize the solution of the flow field issue.

The present paper is concerned over the estimative of the pressure loss and entropy variation in an isothermal fluid flow considering real gas effects. The formulation is based on the isothermal compressibility module and on the coefficient of thermal expansion, in order to be applicable for both gas and liquid as pure substances. The description of the simple methodology, which establishes a relationship between the formulation adopted for ideal gas and that considers the real gas effects, will be emphasized. A computational procedure has been developed, and it can be used to determine the flow properties in duct with a variable area where real gas behavior is significant. In order to obtain quantitative results, three virial coefficients for helium equation of state (Miller and Wilder, 1968; Schneider and Duffie, 1949) are employed to determine the percentage difference in pressure, and entropy between the different formulations. Results are graphically presented in the form of real gas correction factors, which can be applied to perfect gas calculations. This paper is part of a continuing effort that is being carried out at the Institute of Aeronautics and Space from the Brazilian Aerospace Technology and Science Department (DCTA/IAE, acronyms in Portuguese) to develop flow analysis methods, and design of aeronautical devices in a conceptual context.

MATHEMATICAL FORMULATION

Isothermal flow is a model of compressible fluid flow whereby the flow remains at the same temperature while flowing in a conduit. In the kind of flow, heat transferred through the walls of the conduit is offset by frictional heating back into the flow. Although the flow temperature remains constant, a change in stagnation temperature occurs because of a change in the velocity. From this approximation, it is possible to demonstrate that, for ideal gas formulation, the flow is choked at mach number (M) given by $M=\sqrt{1/\gamma}$, and not at mach number equal to one as in the case of many other models, such as Fanno flow (John and Keith, 2006). This analysis is applied to the 1D fluid flow (Fig. 1). It is assumed that the fluid undergoes an isothermal process during this process. In other words, its total energy remains unchanged by the flow. The generalized form of this process includes the possibility of the presence and effect of viscosity (through μ and τ_W), gravity (through α– the angle between the horizontal and the direction of flow), pressure (P), heat exchange ($\dot{q}_c$) and thermal conducting terms (k). τ_W is the shear stress due to wall friction. Equations 1, 2 and 3 show the generalized mathematical model.

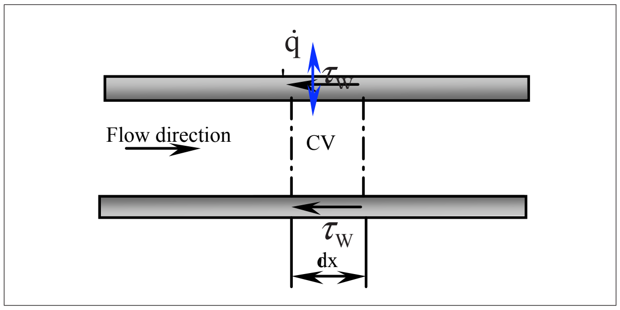

Figure 1: Differential control volume (CV).

(i) Continuity Equation

$$\frac{\partial \rho A}{\partial t}+\frac{\partial \rho u A}{\partial x}=0 \; ; \tag{1}$$

(ii) Momentum Equation

$$\frac{\partial \rho u A}{\partial t}+\frac{\partial \rho u A u}{\partial x}=-A\frac{\partial P}{\partial x}+\frac{4}{3}\frac{\partial}{\partial x}\left(\mu A\frac{\partial u}{\partial x}\right)-\tau_W p_W \; ; \tag{2}$$

(iii) Energy Equation:

$$\frac{\partial \rho h A}{\partial t}+\frac{\partial \rho u A h}{\partial x}=A\frac{\partial P}{\partial t}+Au\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left(kA\frac{\partial T}{\partial x}\right)+$$
$$\frac{4}{3}\mu A\left(\frac{\partial u}{\partial x}\right)^2+|u|\tau_W p_W+\frac{\dot{q}p_W}{\cos\alpha}. \tag{3}$$

In order to create practical relations for engineering use, steady state flow hypothesis is adopted, and the thermal conduction and viscosity terms are neglected. This approach gives the following results:

(i) Continuity Equation

$$\frac{d\,\rho u A}{dx}=0\,; \tag{4}$$

(ii) Momentum Equation

$$\frac{d\,\rho uAu}{dx} = -A\frac{dP}{dx} - \tau_W p_W\,; \tag{5}$$

(iii) Energy Equation:

$$\frac{d\,\rho uAh}{dx} = Au\frac{dP}{dx} + \left|u\right|\tau_W A_W + \frac{\dot{q}p_W}{\cos\alpha} \tag{6}$$

The strategy chosen in this development is to convert the dependent variable (*ρ, u, h*) in terms of pressure (*P*) and temperature (*T*). Thus, the equation system is reduced to two equations, whose dependent variables are pressure and temperature, and the independent one is the longitudinal coordinate (*x*). It is important to highlight that during this development, any constitutive relation, such as virial equation of state, was adopted, in order to generalize the applicability of the method to all pure substances.

Energy Equation

Since $h = h[P(x), T(x)]$, it can be written as:

$$dh = \left(\frac{\partial h}{\partial T}\right)_P dT + \left(\frac{\partial h}{\partial P}\right)_T dP = Cp\left(P,T\right)dT + \left(\frac{\partial h}{\partial P}\right)_T dP \tag{7}$$

By definition:

$h = U + Pv$, so

$dh = dU + Pdv + vdP$.

Considering an isothermal process:

$$\left(\frac{\partial h}{\partial P}\right)_T = \left(\frac{\partial U}{\partial P}\right)_T + P\left(\frac{\partial v}{\partial P}\right)_T + v \tag{8}$$

The second law applied to the reversible process is given by: $\frac{\delta Q}{T} = dS$. Since $\delta Q = dU + Pdv$, the first law becomes: $TdS = dU + Pdv$. In this expression, the isothermal condition is used: $T\left(\frac{\partial S}{\partial P}\right)_T = \left(\frac{\partial U}{\partial P}\right)_T + P\left(\frac{\partial v}{\partial P}\right)_T$.

Therefore, using Eq. 8, and, by the Maxwell relation (Wyllen and Sonntag, 1987), the result is the following (Eq. 9):

$$\left(\frac{\partial h}{\partial P}\right)_T = -T\left(\frac{\partial v}{\partial T}\right)_P + v \tag{9}$$

Substituting Eq. 9 in Eq. 7, it is obtained Eq. 10:

$$dh = Cp\left(P,T\right)dT + \left[-T\left(\frac{\partial v}{\partial T}\right)_P + v\right]dP\quad. \tag{10}$$

Now, take into consideration the momentum Eq. 5 in the form $Au\frac{dP}{dx} = -\left[\frac{d\,\rho uAu}{dx} + \tau_w p_w\right]$, and substitute it in energy equation, considering the continuity equation, which will result in Eq. 11:

$$\frac{d\,\rho uAh}{dx} + u\dot{m}^2\frac{d\,\frac{v}{A}}{dx} = \frac{\dot{q}p_w}{\cos\alpha} \tag{11}$$

Note that:

$$\frac{d\,\rho uAh}{dx} = \rho uA\frac{dh}{dx} + h\frac{d\,\rho uA}{dx} = \rho uA\frac{dh}{dx} = \dot{m}\frac{dh}{dx}$$

Thus, from Eq. 11,

$$\dot{m}Cp\left(P,T\right)\frac{dT}{dx} + \dot{m}\left[-T\left(\frac{\partial v}{\partial T}\right)_P + v\right]\frac{dP}{dx} + \frac{\dot{m}v}{A}\dot{m}^2\left(\frac{A\frac{dv}{dx} - v\frac{dA}{dx}}{A^2}\right) = \dot{Q},$$

where : $\dot{Q} = \frac{\dot{q}p_W}{\cos\alpha}$, $\frac{dv}{dx} = \left(\frac{\partial v}{\partial T}\right)_P\frac{dT}{dx} + \left(\frac{\partial v}{\partial P}\right)_T\frac{dP}{dx}$,

and $u = \frac{\dot{m}v}{A}$.

From algebraic manipulation of the previous equation, the energy equation is obtained with pressure and temperature as dependent variables, which is:

$$G\left(P,T\right)\frac{dP}{dx} + H\left(P,T\right)\frac{dT}{dx} = I\left(P,T\right), \tag{12}$$

where: $G = \left(\frac{\dot{m}}{A}\right)^2 v\left(\frac{\partial v}{\partial P}\right)_T - T\left(\frac{\partial v}{\partial T}\right)_P + v,$

$H = Cp\left(P,T\right) + \left(\frac{\dot{m}}{A}\right)^2 v\left(\frac{\partial v}{\partial T}\right)_P$, and

$$I = \frac{\dot{Q}}{\dot{m}} + \left(\frac{\dot{m}}{A}\right)^2 v^2\frac{d\ln A}{dx}$$

Momentum Equation

Since the cross section of the duct is a function of the longitudinal coordinate, A=A(x), it can be written:

$$\frac{du}{dx} = \frac{d}{dx}\left(\frac{\dot{m}v}{A}\right) = \dot{m}\frac{d}{dx}\left(\frac{v}{A}\right)\;;$$

thus, $\dfrac{d\,\rho u A u}{dx} = \dot{m}^2 \dfrac{d}{dx}\left(\dfrac{v}{A}\right)$.

By substituting the last equation into momentum Eq. 5 and dividing it by A, one obtains the representative equation of momentum conservation in terms of pressure and temperature, which can be seen in Eq. 13:

$$J(P,T)\frac{dP}{dx} + K(P,T)\frac{dT}{dx} = M(P,T)\,, \tag{13}$$

where: $J = \left(\dfrac{\dot{m}}{A}\right)^2 \left(\dfrac{\partial v}{\partial P}\right)_T + 1$,

$K = \left(\dfrac{\dot{m}}{A}\right)^2 \left(\dfrac{\partial v}{\partial T}\right)_P$, and

$M = \left(\dfrac{\dot{m}}{A}\right)^2 v \dfrac{d \ln A}{dx} - \dfrac{\tau_W p_W}{A}$.

Generalized flow formulation

Equations 12 and 13 represent the generalized formulation written in terms of pressure and temperature, which include heat exchange, flow with friction (wall friction), and flow in variable-area ducts influence. It is noteworthy that, up to this point, any approximation for equation of state was not used. In order to extend the applicability of the mathematical model, it is interesting to write the parameters G, H, I, J, K and M in terms of isothermal compressibility module (β_T) and thermal expansion module (α_P). In gas dynamics, compressibility is a measure of the relative volume change of a fluid or solid as a response to a pressure (or mean stress) change. Since the compressibility depends strongly on whether the process is adiabatic or isothermal, it is usually defined as:

$$\beta_T = -\frac{1}{v}\left(\frac{\partial v}{\partial P}\right)_T \tag{14}$$

The thermal expansion coefficient describes how the size of an object changes with a change in temperature. Specifically, it measures the fractional change in size per degree change in temperature at a constant pressure. In the general case of a gas, liquid, or solid, the volumetric coefficient of thermal expansion is given by:

$$\alpha_P = \frac{1}{v}\left(\frac{\partial v}{\partial T}\right)_P \tag{15}$$

Therefore, by considering definitions of isothermal compressibility module and thermal expansion coefficients, it is possible to write the general equations as:

$$\begin{cases} G(P,T)\dfrac{dP}{dx} + H(P,T)\dfrac{dT}{dx} = I(P,T) \\ J(P,T)\dfrac{dP}{dx} + K(P,T)\dfrac{dT}{dx} = M(P,T) \end{cases}, \tag{16}$$

where,

$G = v\left[1 - K\dfrac{\beta_T}{\alpha_P} - T\alpha_P\right]$,

$H = Cp(P,T) + vK$,

$J = v\left(\dfrac{\dot{m}}{A}\right)^2 \beta_T + 1$, and

$K = \dfrac{u^2}{v}\alpha_P$.

Critical Conditions for Generalized Flow

Applying Kramer rule in the system of equations (Eq. 16), it will provide: $\dfrac{dP}{dx} = \dfrac{\begin{vmatrix} H & I \\ K & M \end{vmatrix}}{\begin{vmatrix} H & G \\ K & J \end{vmatrix}} = \dfrac{HM - IK}{HJ - GK}$.

It is desirable, as in the isentropic case, to investigate this relationship by the choice of a convenient reference state. Since stagnation conditions are not constant, the stagnation state is not suitable for this purpose. However, the state corresponding to unity mach number ("critical condition") is suitable, because, as Eq. 16 shows, condition there is constant for a given flow. In this case, the critical condition is obtained from the expression:

$$HJ - GK = 0\,. \tag{17}$$

Since:

$$HJ = Cp(P,T)\left(\frac{\dot{m}}{A}\right)^2\left(\frac{\partial v}{\partial P}\right)_T + Cp(P,T) +$$

$$+\left(\frac{\dot{m}}{A}\right)^4 v\left(\frac{\partial v}{\partial T}\right)_P\left(\frac{\partial v}{\partial P}\right)_T + \left(\frac{\dot{m}}{A}\right)^2 v\left(\frac{\partial v}{\partial T}\right)_P,$$

and

$$GK = \left(\frac{\dot{m}}{A}\right)^4 v\left(\frac{\partial v}{\partial T}\right)_P\left(\frac{\partial v}{\partial P}\right)_T - T\left(\frac{\dot{m}}{A}\right)^2\left(\frac{\partial v}{\partial T}\right)_P^2 + v\left(\frac{\dot{m}}{A}\right)^2\left(\frac{\partial v}{\partial T}\right)_P,$$

by substituting it in Eq. 17, it is obtained:

$$u = v\sqrt{\frac{-Cp(P,T)}{\left[Cp(P,T)\left(\frac{\partial v}{\partial P}\right)_T + T\left(\frac{\partial v}{\partial T}\right)_P^2\right]}} \tag{18}$$

It should be noted that the relationship between the specific heats at pressure and volume constant is given by Van Wyllen and Sonntag (1987, p. 284):

$$Cp(P,T) - Cv(P,T) = -T\left(\frac{\partial v}{\partial T}\right)_P^2\left(\frac{\partial v}{\partial P}\right)_T$$

The sound velocity (a), in an isothermal process, can be written as (John and Keith, 2006, p. 45): $a = \sqrt{\frac{\gamma v}{\beta_T}}$.

Substituting the relations above in Eq. 18, the following equation is obtained by:

$$u = v\sqrt{\frac{-Cp(P,T)}{-\left(\frac{\partial v}{\partial P}\right)_T Cv(P,T)}} = \sqrt{\frac{Cp(P,T)v^2}{Cv(P,T)\left(\frac{\partial v}{\partial P}\right)_T}} = a$$

In other words, the critical condition in a generalized flow field, considering real gas effects, is given by the mach number equals to one. This value is in accordance with technical literature.

Isothermal flow formulation

Isothermal flow can be characterized by the relation (Eq. 19):

$$\frac{dT}{T} = 0 \tag{19}$$

Considering Eq. 19 and the general system (Eq. 16), it is obvious the relation:

$$\frac{dP}{dx} = \frac{I}{G} = \frac{M}{J} \tag{20}$$

From this expression, it is possible to characterize isothermal flow in another, but similar, manner: *IJ* = *MG*.

Using the previous definitions for parameters *J* and *G*, it can be concluded that:

$$\frac{I}{Mv} = 1 - \frac{1}{1-\left(\frac{\dot{m}}{A}\right)^2 v\beta_T}\alpha_P T = 1 - \frac{1}{1-\gamma_{\text{Real}} M^2}\alpha_P T \tag{21}$$

Equation 21 presents an interesting mathematical format. It is possible to distinguish the "driver" terms (left side of the equation), in a dimensionless form, which represents the physical conditions necessary to obtain an isothermal flow. This term is built with the contributions of heat exchange, wall friction, area variation, and mass flow. The right side of the Eq. 21 represents the thermodynamic conditions obtained in an isothermal flow, with the "driver" conditions defined by the left side. Another important thing to note about Eq. 21 is that the variation of the parameter I/Mv, at constant temperature, can be found from the equation of state and mass flow rate. This information can be used in the control system design based on isothermal fluid flow. The simplicity of this formulation is a great attractive for conceptual design and engineering analysis.

Critical conditions for isothermal flow

Equation 21 shows that the critical condition can be obtained from the expression:

$$1-\left(\frac{\dot{m}}{A}\right)^2 v\beta_T = 0 \tag{22}$$

Considering the concepts of sound velocity and isothermal compressibility module, the second term of the left side becomes: $\left(\frac{\dot{m}}{A}\right)^2 v\beta_T = \frac{u^2\beta_T}{v} = \gamma \mathrm{M}^2$. Then, $1-\left(\frac{\dot{m}}{A}\right)^2 v\beta_T = 1-\gamma_{\mathrm{Re}al}\mathrm{M}^2 = 0$. From that, it is obtained:

$$\mathrm{M} = \sqrt{\frac{1}{\gamma_{\mathrm{Re}al}}} \tag{23}$$

Analogously to the ideal formulation (John and Keith, 2006, p. 366), Eq. 23 shows that the critical mach number for isothermal flow is not subcritical flow. It follows that because the adiabatic speed of sound is greater than the isothermal speed of sound, isothermal flow may be supercritical without being necessarily supersonic. It is important to highlight that constitutive relations are not used in this development. Thus, it is assumed that Eq. 23 is valid for all real gas or liquid formulation, for the flow process does not involve change of physical state.

Pressure drop in isothermal flow

The most effective approach to flow problems of this type is to express ratios of the gas properties and flow parameters between any two points in the flow stream, as function of the mach number and specific heat ratio of the gas. Using these ratios, a reference state is defined, and the ratios of the variables at any mach number to those at the reference state are tabulated. Following this reasoning, consider the case of duct with constant cross section, by applying Eq. 20:

$$\frac{dP}{dx}=\frac{M}{J}=\frac{\dfrac{-\tau_W P_W}{A}}{1-\gamma_{Real}\mathrm{M}^2} \qquad (24)$$

It is a common practice to assume (John and Keith, 2006, p. 291):

$$\tau_W=\frac{1}{8}\rho u^2 f \qquad (25)$$

Thus,

$$\frac{\tau_W P_W}{A}=\frac{\tau_W}{A}\frac{4A}{D}=\frac{4\tau_W}{D}=\frac{1}{2}\rho u^2\frac{f}{D}.$$

By substituting it in Eq. 24, it is obtained:

$$dP=-\frac{1}{2}\frac{u^2}{v}\frac{1}{1-\gamma_{Real}\mathrm{M}^2}\frac{fdx}{D}.$$

Since

$$\frac{1}{2}\frac{u^2}{v}=\frac{1}{2v}\frac{\gamma_{Real}\mathrm{M}^2 v}{\beta_T}=\frac{1}{2}\frac{\gamma_{Real}\mathrm{M}^2}{\beta_T},$$

It is demonstrated that:

$$dP=-\frac{1}{\beta_T}\left(\frac{\dfrac{\gamma_{Real}\mathrm{M}^2}{2}}{1-\gamma_{Real}\mathrm{M}^2}\frac{fdx}{D}\right) \qquad (26)$$

Equation 26 establishes the relationship between the pressure drop and the parameters ‘mach number’, ‘wall friction’, and ‘thermodynamic properties of the gas’, when the longitudinal direction considered. Note that for ideal gas:

$$\beta_T=-\frac{1}{v}\left(\frac{\partial v}{\partial P}\right)_T=-\frac{1}{v}\frac{-v}{P}=\frac{1}{P} \qquad (27)$$

Substituting Eq. 27 into 26, the pressure drop formulation developed for ideal gas (John and Keith, 2006, p. 366) is recovered, which is:

$$\left(\frac{dP}{P}\right)_{Ideal}=-\frac{\dfrac{\gamma_{Real}\mathrm{M}^2}{2}}{1-\gamma_{Real}\mathrm{M}^2}\frac{fdx}{D} \qquad (28)$$

In other words, it can be concluded that:

$$\beta_T dP=\left(\frac{dP}{P}\right)_{Ideal} \qquad (29)$$

The right side of the Eq. 29 can be related to the mach number. Using the definition of β_T and an isothermal process, we will have:

$$dv=\left(\frac{\partial v}{\partial T}\right)_P dT+\left(\frac{\partial v}{\partial P}\right)_T dP=\left(\frac{\partial v}{\partial P}\right)_T dP, \qquad (30)$$

The expression (Eq. 26) can be written as:

$$\frac{dv}{v}=\frac{\dfrac{\gamma_{Real}\mathrm{M}^2}{2}}{1-\gamma_{Real}\mathrm{M}^2}\frac{fdx}{D} \qquad (31)$$

However, from continuity equation, one has:

$$\frac{dv}{v}=\frac{du}{u}=\frac{d\mathrm{M}}{\mathrm{M}} \qquad (32)$$

Thus:

$$\beta_T dP=\left(\frac{dP}{P}\right)_{Ideal}=-\left(\frac{d\mathrm{M}}{\mathrm{M}}\right)_{Ideal} \qquad (33)$$

It is easy to demonstrate that Eq. 33 can be used for flow in ducts with variable cross section.

Entropy variation in isothermal flow

Since $S=S(P,T)$, it can be written:

$$dS=\left(\frac{\partial S}{\partial T}\right)_P dT+\left(\frac{\partial S}{\partial P}\right)_T dP \qquad (34)$$

Using the concepts of specific heat at constant pressure and Maxwell relation:

$$dS=Cp(P,T)\frac{dT}{T}-\left(\frac{\partial v}{\partial T}\right)_P dP \qquad (35)$$

Considering an isothermal process and the definition of α_P, it is obtained:

$$dS = -\alpha_P v dP \quad . \tag{36}$$

Substituting Eq. 26 in 36, it is obtained:

$$\frac{dS}{\tilde{R}} = \frac{\frac{\gamma_{\mathrm{Re}al}\mathrm{M}^2}{2}}{1-\gamma_{\mathrm{Re}al}\mathrm{M}^2}\frac{fdx}{D} = \left(\frac{d\mathrm{M}}{\mathrm{M}}\right)_{Ideal} , \tag{37}$$

where: $\tilde{R} = \dfrac{\alpha_P v}{\beta_T}$

By definition:

$$\tilde{R} = \frac{-v\left(\frac{\partial v}{\partial T}\right)_P}{\left(\frac{\partial v}{\partial P}\right)_T} \tag{38}$$

Since

$$\left(\frac{\partial v}{\partial P}\right)_T\left(\frac{\partial P}{\partial T}\right)_v\left(\frac{\partial T}{\partial v}\right)_P = -1,$$

or

$$\left(\frac{\partial P}{\partial T}\right)_v = -\frac{\left(\frac{\partial v}{\partial T}\right)_P}{\left(\frac{\partial v}{\partial P}\right)_T} , \tag{39}$$

the value of $\tilde{R}$ can be written as:

$$\tilde{R} = v\left(\frac{\partial P}{\partial T}\right)_v \tag{40}$$

Note that for ideal gas:

$$\tilde{R} = v\frac{R}{v} = R , \tag{41}$$

which is in accordance with results from technical literature (John and Keith, 2006, p. 367).

EXAMPLE OF APPLICATION

From Eq. 29 and 37, it is possible to define multiplicative correction factors that can be used in an ideal gas formulation in order to solve problems associated with the real gas formulation. However, it is important to define the concept of specific heat ratio in terms of $\tilde{R}$. According to Wyllen and Sonntag (1987, p. 285):

$$Cp\left(P,T\right) - Cv\left(P,T\right) = \alpha_P T\tilde{R} \; ; \tag{42}$$

thus, the expression for specific heat ratio is given by:

$$\gamma_{\mathrm{Re}al} = 1 + \frac{\alpha_P T\tilde{R}}{Cv\left(P,T\right)} \tag{43}$$

This value is used in Eq. 44, which is:

$$\left(\frac{dP}{P}\right)_{Ideal} = -\frac{\frac{\gamma_{\mathrm{Re}al}\mathrm{M}^2}{2}}{1-\gamma_{\mathrm{Re}al}\mathrm{M}^2}\frac{fdx}{D} = -\left(\frac{d\mathrm{M}}{\mathrm{M}}\right)_{Ideal} \tag{44}$$

Equation 44 can be integrated in terms of mach number, considering the friction factor f constant in the segment dx. From this result, it is possible to evaluate the pressure drop for ideal gas formulation. Thus, applying the multiplicative factor β_T on the last result, the drop pressure considering real gas effects is obtained. Regarding to the entropy variation, it is possible to write from Eq. 37 and 44:

$$\frac{dS}{\tilde{R}} = \left(\frac{dP}{P}\right)_{Ideal} , \tag{45}$$

CORRECTION FACTORS

In this section, the pressure correction factor (ε_P) and entropy correction factor (ε_S) are defined. The parameter ε_P is calculated from the use of Eq. 33 for ideal and real gases. By substituting Eq. 14 in Eq. 33, the following result is given:

$$\left(\frac{dP}{P}\right)_{\mathrm{Re}al} = \frac{\rho}{P}\left(\frac{\partial P}{\partial \rho}\right)_T\left(\frac{dP}{P}\right)_{Ideal} = \varepsilon_P\left(\frac{dP}{P}\right)_{Ideal} , \tag{46}$$

where the correction factor ε_P is given by:

$$\varepsilon_P = \frac{1}{\beta_T P} , \tag{47}$$

In analogous fashion, it is defined the entropy correction factor. In accordance with Eq. 37 and 44, the entropy correction factor is given by:

$$\varepsilon_S = \tilde{R} / R , \tag{48}$$

In order to obtain quantitative results, an equation of state for helium, based on a three virial coefficient (Miller and Wilder, 1968; Schneider and Duffie, 1949), is employed to determine the correction factor for pressure and entropy variation. The equation used to represent the pressure-density-temperature relationship of real helium gas is the virial equation of state:

$$P = Z\rho RT = \rho RT\left[1+\rho B(T)+\rho^2 C(T)\right] \qquad (49)$$

A relatively large amount of experimental data on the second virial coefficient, *B(T)*, exist for low and moderate temperatures (up to about 1,000 K). Experimental results for the third virial coefficient, *C(T)*, are few and show a great deal of scatter. At the conditions dealt within this paper, the contribution of *C(T)* to the state equation is relatively small. In this context, the expression derived for the virial coefficients is given by:

$$B(T) = b_0 + b_1 T^{-1/4} + b_2 T^{-3/4} + b_3 T^{-5/4} + b_4 T^{-7/4}, \qquad (50)$$

and

$$C(T) = c_0 + c_1 T^{-1/4} + c_2 T^{-3/4} + c_3 T^{-5/4} \qquad (51)$$

The coefficients of virial equation are given in Tables 1 and 2. The units of *T*, *B(T)* and *C(T)* are K, cm^3/mole and (cm^3/mole)^2, respectively.

Table 1: Second virial coefficient – *B(T)*

Coefficient	T < 1300 K	T > 1300 K
b_0	-13.4067	1.178236
b_1	165.4459	-7.57134
b_2	-1357.92	5225.701
b_3	5959.061	-188923
b_4	-12340.8	2460461

Table 2: Third virial coefficient – *C(T)*

Coefficient	T
c_0	-13.7898
c_1	139.7339
c_2	8114.259
c_3	-17456.9

Pressure correction factor (ε_P)

From the Eq. 49, it is obtained:

$$\left(\frac{\partial P}{\partial T}\right)_\rho = \rho R\left\{1+\rho\left[B(T)+T\frac{dB(T)}{dT}\right]+\rho^2\left[C(T)+T\frac{dC(T)}{dT}\right]\right\}, \qquad (52)$$

and

$$\left(\frac{\partial P}{\partial \rho}\right)_T = RT\left[1+2\rho B(T)+3\rho^2 C(T)\right] \qquad (53)$$

Considering the virial equation of state, Eq.49 and 47, the pressure correction factor (ε_P) is given by:

$$\varepsilon_P = \frac{1}{Z}\left[1+2\rho B(T)+3\rho^2 C(T)\right] \qquad (54)$$

Entropy correction factor (ε_S)

Using Eq. 40 and 48, it is demonstrated that:

$$\varepsilon_S = 1+\rho\left[B(T)+T\frac{dB(T)}{dT}\right]+\rho^2\left[C(T)+T\frac{dC(T)}{dT}\right]. \qquad (55)$$

RESULTS

The values of *B(T)* and *C(T)* computed from Eq. 2 and 3, for real helium gas, are plotted in Figs. 2 and 3. Note that the virial coefficients are plotted in terms of [cm^3/mole], where [cm^3/mole] = W_{He}/1000 (m^3/kg). The accuracy of these coefficients was checked by comparing the values of the resulting compressibility coefficient, Z, with those given in the tabulation of helium properties prepared by the International Union of Pure and Applied Chemistry (IUPAC). Figure 4 shows the compressibility coefficient Z, which was in very close agreement with IUPAC for all densities.

Figure 5 shows the multiplicative correction factor for pressure drop. It is a common practice the use of relation (Eq. 26), in a context of ideal gas, in order to obtain the solution for isothermal flow of real gas by only using a real specific heat ratio. However, it can be noted that the specific heat ratio is not the only parameter of influence. In fact, the correction factor ε_P must also be considered in this procedure. In this context, although the first methodology is easy to conceive, the latter, more complete, requires the knowledge of more details about the physical properties of the pure substance. Another interesting aspect of this result is related to the value of ε_P. When a gas undergoes a reversible process, in which there is heat transfer, the process frequently takes place in such a manner that a plot of log *P versus v* is a straight line. This is called polytropic process (Wyllen and Sonntag, 1987, p. 167), in other words:

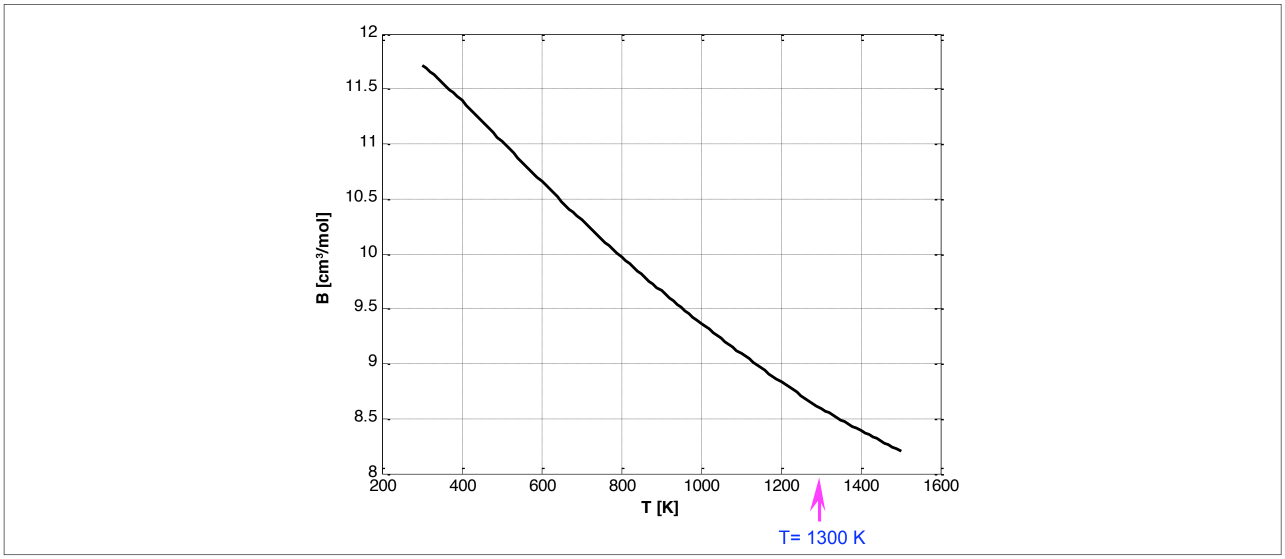

Figure 2: Second virial coefficient – *B(T)*.

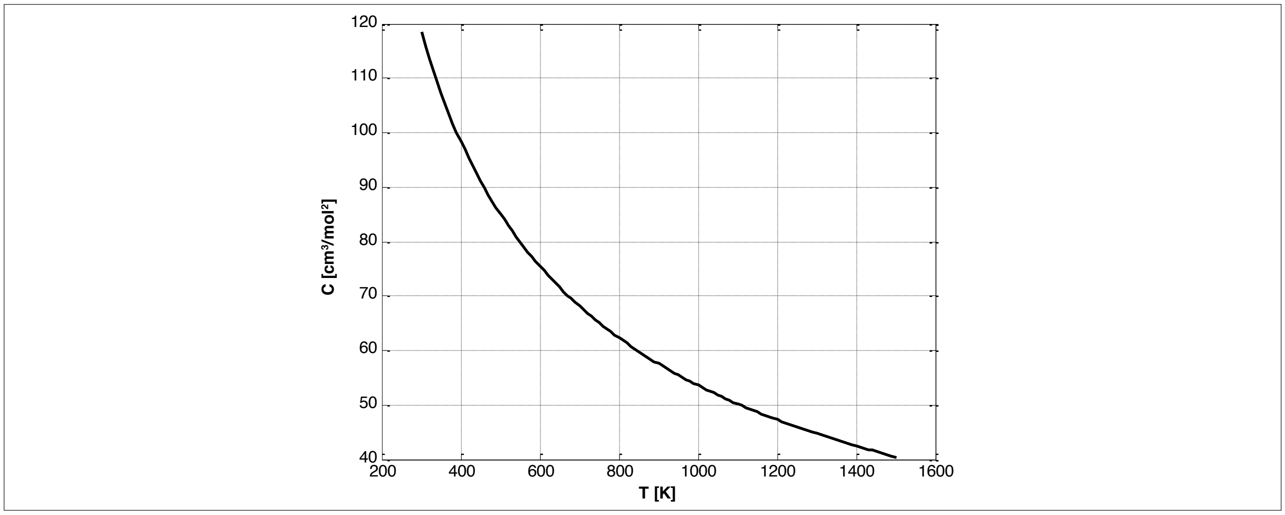

Figure 3: Third virial coefficient – *C(T)*.

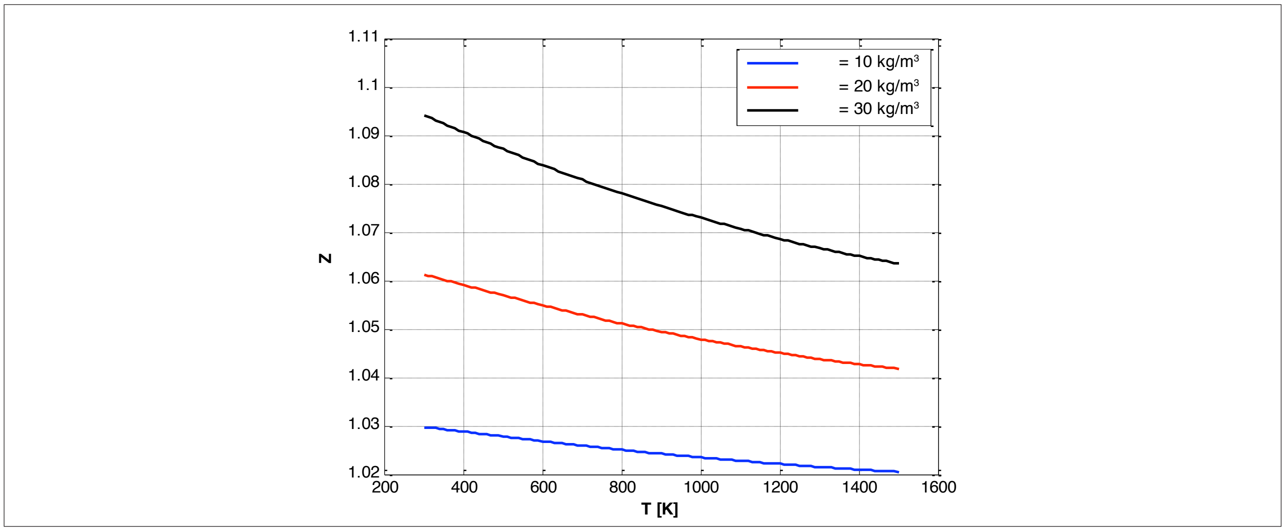

Figure 4: Compressibility coefficient – Z.

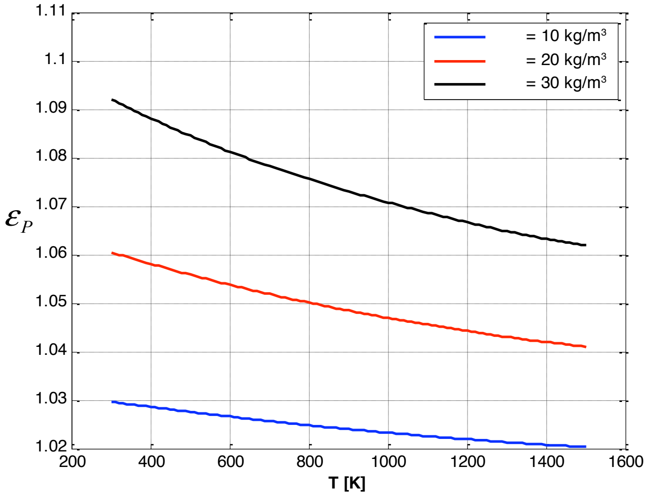

Figure 5: Correction factor for pressure drop.

$$\left[\frac{\frac{dP}{P}}{\frac{dv}{v}}\right]_{Ideal} = -n, \tag{56}$$

where, $n = 1$ (one) is an ideal isothermal process.

From Eq. 32, 33 and 52, it is possible to note that:

$$\left[\frac{\frac{dP}{P}}{\frac{dv}{v}}\right]_{\mathrm{Re}al} = -\frac{1}{\beta_T P} = -\varepsilon_P \tag{57}$$

Thus, the pressure corrector factor ε_P establishes a direct comparison of ideal and real isothermal processes. Results presented in Fig. 5 can be used as a good indicative of the tolerance that must be adopted in a system specification, based on isothermal fluid flow results.

Figure 6 shows the entropy correction factor, ε_S. It is clear that ε_S is more temperature sensitive than ε_P. Internal energy and entropy are not directly physically measurable, whereas certain of the intensive variables (e.g. T,P) are. Thus, the entropy formulation presented in this paper is an important mathematical model to know the accuracy of entropy correction factor, obtained indirectly from the 1D analysis using ideal gas formulation. Another aspect that must be considered during flow analysis of helium gas is related to the discontinuity observed at $T = 1300$ K. Although it is not clear in *B(T)* variation (Fig. 2), this discontinuity is probably associated with the model adopted for gas equation. The fit curve coefficients for high temperature (Table 1) are not consistent when it is considered high variation in density.

COMENTS AND CONCLUSION

The primary purpose of this investigation was to develop a method required to study the behavior of real pure substance in an isothermal fluid flow. Particular emphasis was given to develop useful procedures and techniques in order to study the general types of gas, which are encountered in aeronautical applications, such as, wind tunnel, combustors, and so on. More complicated systems can be studied, in a 1D context, with little additional difficulty. From this research, it is possible to draw several conclusions:

- The mathematical formulations developed for pressure and entropy variation, Eq. 26 and 36, respectively, can be used for different pure substances. the pressure correction factors (ε_P) that must be adopted in an isothermal gas flow are a function of isothermal compressibility module, β_T, and static pressure. These factors establish a direct comparison of ideal and real isothermal processes (Eq. 47).

- The entropy correction factor (ε_S) is a function of thermal expansion module (α_P), isothermal compressibility module (β_T), and specific volume (v). It is given by the Eq. 48. Similar to the pressure correction factor, it depends on the state equation.

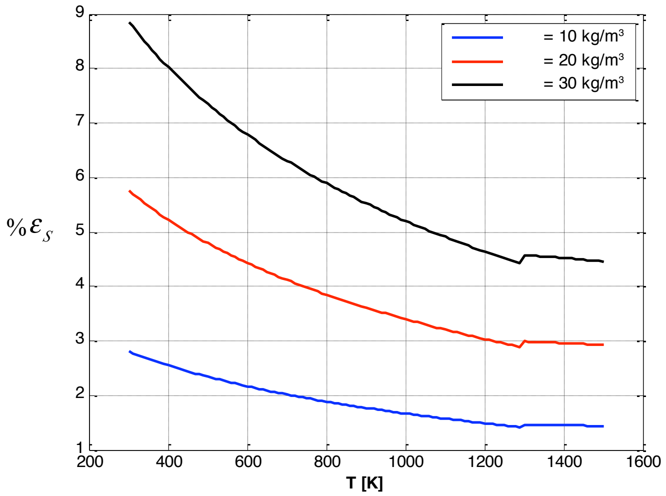

Figure 6: Correction factor for entropy variation.

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