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Reliability-Based Design Optimization Strategies Based on FORM: A Review

In deterministic optimization, the uncertainties of the structural system (i.e. dimension, model, material, loads, etc) are not explicitly taken into account. Hence, resulting optimal solutions may lead to reduced reliability levels. The objective of reliability based design optimization (RBDO) is to optimize structures guaranteeing that a minimum level of reliability, chosen a priori by the designer, is maintained. Since reliability analysis using the First Order Reliability Method (FORM) is an optimization procedure itself, RBDO (in its classical version) is a double-loop strategy: the reliability analysis (inner loop) and the structural optimization (outer loop). The coupling of these two loops leads to very high computational costs. To reduce the computational burden of RBDO based on FORM, several authors propose decoupling the structural optimization and the reliability analysis. These procedures may be divided in two groups: (i) serial single loop methods and (ii) uni-level methods. The basic idea of serial single loop methods is to decouple the two loops and solve them sequentially, until some convergence criterion is achieved. On the other hand, uni-level methods employ different strategies to obtain a single loop of optimization to solve the RBDO problem. This paper presents a review of such RBDO strategies. A comparison of the performance (computational cost) of the main strategies is presented for several variants of two benchmark problems from the literature and for a structure modeled using the finite element method.

Keywords: RBDO, structural reliability, structural optimization

Introduction

In deterministic design optimization, the uncertainties of the structural system (i.e. dimensions, models, materials, loads, etc) are taken into account in a subjective and indirect way, by means of partial safety factors specified in design codes. As a consequence, deterministic optimal solutions may lead to reduced reliability levels (Beck and Gomes, 2012). Reliability Based Design Optimization (RBDO) has emerged as an alternative to properly model the safety-under-uncertainty aspect of the optimization problem. With RBDO, one can ensure that a minimum (and measurable) level of safety, chosen a priori by the designer, is maintained by the optimum structure.

The RBDO problem may be stated as:

$$\begin{aligned} \text{Minimize: } & J(\mathbf{d}, \mathbf{X}) \\ \text{subject to: } & P_{f_i} = P(G_i(\mathbf{d}, \mathbf{X}) < 0) \leq P_{f_i}^{\text{allowable}} \\ & i = 1 \dots n_c \end{aligned} \quad (1)$$

where $\mathbf{d} \in \mathfrak{R}^n$ is the design vector (e.g. structural configuration and dimensions), $\mathbf{X} \in \mathfrak{R}^m$ contains all the random variables of the system under analysis (e.g. random loads, uncertain structural parameters), J is the objective function to be minimized (e.g. the structural weight, volume or manufacturing cost), P_{f_i} is the probability of failure of i^{th} constraint (G_i), and $P_{f_i}^{\text{allowable}}$ is the allowable (maximum) failure probability for the i^{th} constraint. Although the objective function is generally a random variable, it is usual to consider it as a deterministic value by calculating it using only the mean values of the random variables \mathbf{X} . Hence, from now on, the objective function is written as $J(\mathbf{d})$. The reader is referred to Mínguez and Castillo (2009) and references therein for problems dealing with probabilistic quantities in the objective function.

The failure probability P_{f_i} for each constraint may be obtained by evaluating the integral in Eq. (2), which is the fundamental expression of the structural reliability problem:

$$P_{f_i} = \int_{G_i(\mathbf{d}, \mathbf{X}) < 0} f_{\mathbf{X}}(\mathbf{x}) d\mathbf{x} \quad (2)$$

where $f_{\mathbf{X}}(\mathbf{x})$ is the joint probability density function (PDF) of random vector \mathbf{X} . In practice, it is impossible to obtain the joint PDF because of scarcity of joint observations for a large number of random variables. At best, what is known are the marginal probability distributions of each random variable and possibly correlations between pairs of random variables. Another difficulty in solving Eq. (2) is the fact that the limit state equations, G_i , are sometimes given in implicit form, as the response of finite element models (Beck and da Rosa, 2006). Such difficulties have motivated the development of various approximate reliability methods.

Intensive research has been carried out to provide methods to approximate the integral in Eq. (2). These methods may be divided into three major classes: (i) simulation methods; (ii) numerical integration and (iii) analytical methods (Lopez et al., 2011a). For a detailed review on these methods, the reader is referred to Lee and Chen (2008) and Melchers (1999).

- i. Simulations methods using sampling and estimation are well known in the literature, the most widely used being the Direct Monte Carlo simulation (MCS) method (Rubinstein, 1981). The main drawback of MCS is that it requires a huge amount of calculations. Several improvements of the MCS have been developed to reduce the computational effort, such as the quasi-MCS (Niederreiter and Spanier, 2000), directional simulation (Nie and Ellingwood, 2000) and importance sampling (Engelund and Rackwitz, 1993). In practice, MCS is considered *the reference* response and is used to validate the results of other, approximate methods.
- ii. The multi-dimensional integral to determine the probabilistic characteristics of random output is evaluated numerically (Seo and Kwak, 2002; Lee and Kwak, 2006; Rahman and Xu, 2004). Numerical integration is limited, in practice, to dimension 5 or 6.
- iii. The main approaches of this class are the First and Second Order Reliability Methods (FORM and SORM, respectively). FORM and SORM are said to be transformation methods, because the integral in Eq. (2) is not solved in the original space (\mathbf{X}), but is mapped to the Standard Gaussian space (\mathbf{U}). In this space, the Most Probable Point of failure (MPP) is also the point, from all points in the failure domain, closest to the origin. The reliability index β (Hasofer and Lind, 1974), which is a measure of the reliability, can be evaluated as the distance between the MPP point and the origin, in the transformed Standard Gaussian space. The main advantage of FORM-based

approaches is their computational cost, which is a fraction of the cost of crude Monte Carlo simulation, for instance.

Hence, FORM based approaches have been widely employed to evaluate Eq. (2) in RBDO problems. As detailed in Section 2, these approaches are optimization problems themselves and consequently, the RBDO using FORM is a double-loop strategy:

- the inner loop is the reliability analysis,
- the outer loop the structural optimization;

that is, the two optimizations are coupled. Such coupling of optimization loops: structural optimization and reliability assessment - leads to very high computational costs. To reduce the computational burden of RBDO, several authors decoupled the structural optimization and the reliability analysis. Techniques for de-coupling the optimization loops may be divided in two groups: (i) serial single loop methods and, (ii) unilevel methods.

This paper presents a review of RBDO methods based on FORM. First, the two main (coupled) FORM based approaches, Reliability Index Approach (RIA) and Performance Measure Approach (PMA) are described and compared. Then, a review on decoupling methodologies is presented, and two decoupling methodologies are described in detail. A comparison of the performance (computational cost) of the main strategies is presented for several variants of two benchmark problems from the literature in the Numerical Analysis Section. The article is finished with some concluding remarks.

It should be noted that the review presented herein does not intend to cover all the papers published on the subject, but to present the main techniques, in order to serve as a guide to those entering this exciting and challenging subject.

Coupled Form-Based Approaches

This section briefly details the approximation of Eq. (2) using RIA and PMA, then presents a review of the comparison between these two approaches. The interested reader is referred to Madsen et al. (1986), Melchers (1999) and Haldar and Mahadevan (2000) for more details on the RIA, and to Tu et al. (1999) for a full description of the PMA.

The iso-probabilistic transformation

In order to approximate the integral in Eq. (2), it is usual to introduce a vector of normalized and statistically independent random variables $\mathbf{U} \in \mathfrak{R}^m$ and a transformation T (Fig. 1), so that $\mathbf{U} = T(\mathbf{d}, \mathbf{X})$. The most common transformations are the Rosenblatt and the Nataf ones (Lemaire et al., 2005; Melchers, 1999). The mapping T transforms every realization \mathbf{x} of \mathbf{X} in the physical space into a realization \mathbf{u} in the normalized space. Note that it also holds for the constraints:

$$G_i(\mathbf{d}, \mathbf{X}) = G_i(\mathbf{d}, T^{-1}(\mathbf{d}, \mathbf{U})) = g_i(\mathbf{d}, \mathbf{U}), \quad (3)$$

where g_i is the i^{th} constraint in the normalized space.

The main advantage in the use of this transformation is that the probability distribution on the resulting space depends only on its norm. This fact is illustrated in Fig. 1 by the circular reliability levels of the normalized space. It must be highlighted that this transformation is the first approximation proposed to solve Eq. (2): it only is exact when \mathbf{X} is comprised of independent Gaussian random variables. However, even with this simplification, it is still not an easy task to evaluate Eq. (2). The FORM approximation is used to simplify this evaluation.

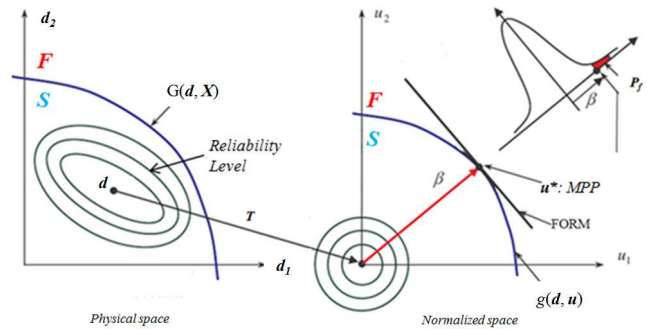


Figure 1. Transformation T and the First Order Reliability Method (FORM).

Reliability Index Approach (RIA) and Performance Measure Approach (PMA)

The main idea of the FORM is simple: it consists in replacing the limit state function G_i by a tangent hyper-plane at the most probable point of failure (MPP). Figures 1 and 2 illustrate the approximation made by this hyper-plane. The FORM approximates the probability of failure and the allowable failure probability for the i^{th} constraint by:

$$P_{f_i} \approx \Phi(-\beta_i) \text{ and } P_{f_i}^{\text{allowable}} \approx \Phi(-\beta_i^{\text{target}}), \quad (4)$$

where Φ is the standard Gaussian cumulative distribution function (CDF) and β_i^{target} is the target reliability index for the i^{th} constraint. This is the second approximation proposed to solve Eq. (2) and it only provides an exact result when the constraint is linear.

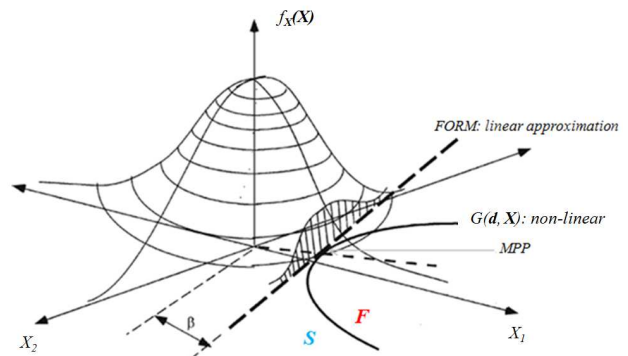


Figure 2. Illustration of the FORM approximation.

Now, recall that in order to evaluate β_i , one first needs to obtain the MPP: the point in the failure domain closest to the origin of the normalized space. The difference between the RIA and the PMA is the manner in which the MPP is calculated. The following optimization problems detail this difference:

RIA	PMA
for a given design \mathbf{d}	for a given design \mathbf{d}
finds \mathbf{u}_{RIA}^* which	finds \mathbf{u}_{PMA}^* which
minimizes: $\ \mathbf{u}\ = \beta_i$	minimizes: $g_i(\mathbf{d}, \mathbf{u})$
subject to: $g_i(\mathbf{d}, \mathbf{u}) = 0$	subject to: $\ \mathbf{u}\ = \beta_i^{\text{target}}$

The optimal solution \mathbf{u}_{RIA}^* of the RIA yields the reliability index $\beta_i = \|\mathbf{u}_{RIA}^*\|$ of the i^{th} constraint on the current design \mathbf{x} . On the other

hand, the optimal solution $\mathbf{u}_{PMA_i}^*$ of the PMA is the minimum performance target point (MPTP) on the target reliability sphere (defined by $\|\mathbf{u}\| = \beta_i^{\text{target}}$) and it provides the so-called performance measure $p_{m_i} = g_i(\mathbf{d}, \mathbf{u}_{PMA_i}^*)$ of the i^{th} constraint on the current design \mathbf{d} . The performance measure p_{m_i} is related to the reliability index β_i by the following relation:

$$p_{m_i} = F_{G_i}^{-1}(\Phi(-\beta_i)) \quad (5)$$

where F_{G_i} is CDF of the i^{th} constraint. It is important to note that $\mathbf{u}_{RIA_i}^*$ and $\mathbf{u}_{PMA_i}^*$ will be equal only when the reliability constraint is active, e.g., at the final design of a RBDO problem. At any other point, $\mathbf{u}_{PMA_i}^*$ only represents the point of minimal performance on the target reliability sphere.

RIA versus PMA

RIA corresponds to minimization of a quadratic functional (the norm) under non-linear equality constraints, for which efficient methods exist. Moreover, the equality constraint could be replaced by an inequality constraint, which simplifies the RIA solution. The efficiency of such alternatives is yet to be explored.

In the paper that introduced the PMA, Tu et al. (1999) showed that PMA is inherently robust and yields a higher overall RBDO rate of convergence when compared to a conventional RIA. Youn et al. (2003), although reaching the same conclusions, showed that the PMA is far more effective when the probabilistic constraint is either very feasible or very infeasible. In a different paper, Youn and Choi (2004a) concluded that the PMA is quite attractive when compared to other probabilistic approaches in RBDO, such as the RIA and the approximate moment approach (Lee et al., 2002).

The first main difference between the RIA and the PMA is the type of optimization problem which is solved in each case. It is easier to minimize a complicated function subject to a simple constraint (PMA) than to minimize a simple function subject to a complicated constraint (RIA). Different from the RIA, in the PMA only the direction vector needs to be determined by taking advantage of the spherical equality constraint $\|\mathbf{u}\| = \beta_i^{\text{target}}$ to find the

MPP $\mathbf{u}_{PMA_i}^*$.

Still regarding the type of optimization problem, the conceptual iteration history during the search facilitates the PMA. Usually, the RIA search requires several iterations to reach the failure surface given by $g_i(\mathbf{d}, \mathbf{u}) = 0$, while the PMA search immediately lies on the $\|\mathbf{u}\| = \beta_i^{\text{target}}$ sphere; in other words, the number of iterations of RIA increases with the reliability index while the PMA search is independent of the target performance (Lee et al., 2002). A second consequence is that, in the case of non-activated constraints, the PMA becomes even more effective.

Regarding non-linearities in the RBDO problem (i.e. use of non-normal random variables), Youn and Choi (2004b) showed that PMA is more stable, efficient and has a lower dependence on the distribution of the random variables, since it introduces small non-linearities in the space-transformations. PMA can thus handle a variety of distributions without significantly increasing the number of function evaluations. Furthermore, RIA diverged when uniform or Gumble random variables were employed. The former divergence was due to the fundamental nature of the uniform distribution and the latter was due to numerical difficulties when dealing either with

a nonlinear failure surface or with a failure surface away from the design point.

Classical coupled approaches

As the reliability analysis is an optimization procedure by itself, RBDO, in its classical version, is a double-loop strategy: the inner loop is the reliability analysis and the outer loop is the structural optimization. Thus, the two optimization loops are coupled:

for $k = 1, 2,$

...

a) structural optimization:

$$\begin{aligned} \text{minimize:} \quad & J(\mathbf{d}^{(k)}) \\ & \beta_i(\mathbf{d}^{(k-1)}) + \left(\nabla_{\mathbf{d}} \beta_i(\mathbf{d}^{(k-1)})\right)^T (\mathbf{d}^{(k)} - \mathbf{d}^{(k-1)}) \leq 0 \\ & \text{(RIA),} \\ \text{subject to:} \quad & p_{m_i}(\mathbf{d}^{(k-1)}) + \left(\nabla_{\mathbf{d}} p_{m_i}(\mathbf{d}^{(k-1)})\right)^T (\mathbf{d}^{(k)} - \mathbf{d}^{(k-1)}) \leq 0 \\ & \text{(PMA),} \\ & i = 1 \dots n_c, \\ & \mathbf{d}_l \leq \mathbf{d}^{(k)} \leq \mathbf{d}_u \end{aligned}$$

where, at each step k , given current design $\mathbf{d}^{(k-1)}$, the reliability routine is called:

b) reliability analysis:

	RIA	PMA
	finds $\mathbf{u}_{RIA}^{*(k-1)}$ which	finds $\mathbf{u}_{PMA_i}^{*(k-1)}$ which
minimizes:	$\ \mathbf{u}\ = \beta_i$	$g_i(\mathbf{d}^{(k-1)}, \mathbf{u})$
subject to:	$g_i(\mathbf{d}^{(k-1)}, \mathbf{u}) = 0$	$\ \mathbf{u}\ = \beta_i^{\text{target}}$

At the end of each reliability analysis, a sensitivity analysis of the design variables with respect to the reliability index is pursued to obtain $\nabla_{\mathbf{d}} \beta_i(\mathbf{d}^{(k-1)})$ or $\nabla_{\mathbf{d}} p_{m_i}(\mathbf{d}^{(k-1)})$. This procedure is repeated until some convergence criterion is achieved and, of course, it leads to very high computational costs. A review of techniques developed to de-couple the RBDO problem, in order to reduce the computational burden, is presented in the next section.

Decoupling Strategies

De-coupling the two optimization problems means not to have to call the reliability analysis routine at every step k of the structural optimization. In the sequel, the serial single loop and unilevel decoupling methods are reviewed.

Serial single loop methods

The basic idea of the serial single loop methods is to decouple the structural optimization (outer loop) and the reliability optimization (inner loop). Each method of single-loop decoupling employs a specific strategy to decouple the loops and then solves them sequentially until some convergence criterion is achieved. Among these methods, the following may be cited: Traditional Approximation Method (TAM) of Torng and Yang (1993), Single Loop Single Variable (SLSV) of Chen et al. (1997), Sequential Optimization and Reliability Assessment (SORA) of Du and Chen (2004) and Safety Factor Approach (SFA) of Wu et al. (2001).

Yang and Gu (2004) compared these four single-loop decoupling RBDO methods. Four different examples were solved including a vehicle side impact and a multidisciplinary optimization problem. According to their results, SLSV was the most effective method, converging nicely and requiring the fewest number of

function evaluations. The other methods also showed promising results when compared to the classical approach. In a second paper, the authors (Yang et al., 2005) investigated an engineering problem with a large number of constraints (144) and with many local minima. In addition to the four single-loop decoupling RBDO strategies, the Mean Value Method (Gu and Yang, 2003) was also studied. Results showed that the number of function evaluations depends on the RBDO method, optimization algorithm and implementation. Furthermore, algorithms with good active-constraint handling tended to perform better (e.g. SORA/SFA). Moreover, SORA/SFA and TAM have advantages over the other methods, as the target reliability is obtained at the end. Regarding the local minima, different methods and different starting points yielded different final results, since only local optimizers were used by the authors.

Unilevel methods

The central idea of unilevel methods is to replace the reliability analysis by some optimality criteria on the optimum (i.e. imposing it as a constraint in the outer loop). Thus, there is a concurrent convergence of the design optimization and reliability calculation or, in other words, they are sought simultaneously and independently.

Kuschel and Rackwitz (2000) formulated a unilevel method based on replacing the inner loop of the classical approach (FORM analysis using RIA) by the first order Karush-Kuhn-Tucker (KKT) optimality conditions of the first-order reliability problem. In other words, the KKT optimality conditions of the RIA search are imposed as constraints in the outer loop of the RBDO. As already commented, the RIA may be ill-conditioned when the probability of failure given by a constraint is zero and it is not computationally efficient when the reliability index is large. With this in mind, Agarwal et al. (2007) proposed a unilevel RBDO method which introduced the first order KKT necessary optimality conditions of PMA as constraints in the outer loop, eliminating the costly reliability analysis (inner loop) of RBDO.

Cheng et al. (2006) proposed a unilevel strategy based on the Sequential Approximate Programming (SAP) concept which was successfully applied in structural optimization. In the SAP approach, the original optimization problem is decomposed into a sequence of sub-optimization problems. Each sub-optimization problem consists of an approximate objective function subject to a set of approximate constraint functions. A SAP strategy for RBDO using the PMA to approximate the reliability constraints was also developed (Yi et al., 2008; Yi and Cheng, 2008).

Studies comparing the different unilevel methods have not been found in the literature. Yi and Cheng (2008) compared the SAP based on RIA and SAP based on PMA with SORA and SLSV methods. Several examples, including the 144 constraint problem, were solved. Based on the results, the authors concluded that SAP based on PMA achieved better results than the other methods. This result does not imply that SAP-PMA is the most effective method in all cases; but it is, at least, one of the most powerful algorithms in RBDO. As a general review of de-coupling approaches has been presented, one method of each approach is described in detail in the sequence in order to highlight the differences of each class of decoupling scheme.

Sequential Optimization and Reliability Assessment (SORA)

The SORA (Du and Chen, 2004) method is based on the strategy of serial single loops decoupling the structural optimization and the reliability analysis. At each iteration of the method, the reliability analysis is only conducted after convergence of several

loops of the structural optimization. SORA makes use of three main artifices to increase the algorithm’s performance:

- i. reliability is evaluated only at the desired level: it means that SORA is based on the PMA having all the advantages when compared to RIA-based methods;
- ii. Using an efficient and robust MPTP search algorithm (Du et al., 2003);
- iii. Employing sequential cycles of optimization and reliability.

The key concept of the method is to shift the boundaries of the violated equivalent deterministic constraints to the feasible direction based on the reliability information obtained in the previous cycle, which makes the reliability constraints improve progressively and the cost of the search for the MPP be reduced.

Figure 3 shows an example of the SORA boundary shifting procedure. The equivalent deterministic constraint at iteration k and the probabilistic constraint that has to be fulfilled are represented. Based on the percentile information obtained through the reliability analysis, the shifting value s is found and the equivalent deterministic constraint is shifted towards the probabilistic constraint (dashed line). The reliability constraint is fulfilled when s is equal to zero, in other words, when the dashed line coincides with the probabilistic constraint.

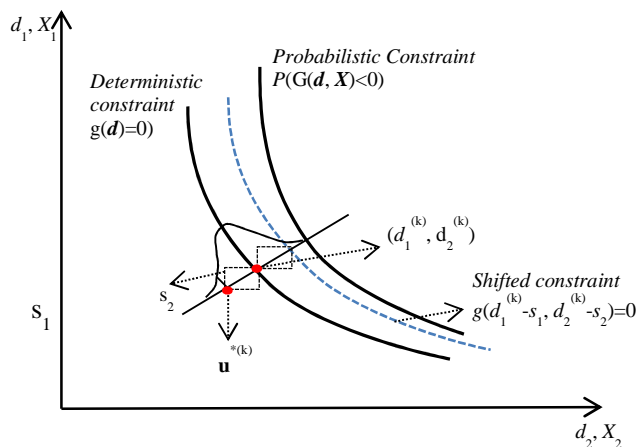


Figure 3. Shifting the boundaries of the violated deterministic constraints in SORA (Du and Chen, 2004).

Thus, the RBDO-SORA algorithm works in the following manner:

for $k = 1, 2, \dots$

a) deterministic optimization:

$$\begin{aligned} &\text{minimize: } J(\mathbf{d}^{(k)}) \\ &\text{subject to: } g_i(\mathbf{d}^{(k)}, \mathbf{u} - \mathbf{s}_i^{(k-1)}) = 0, i = 1 \dots n_c \\ &\mathbf{d}_l \leq \mathbf{d}^{(k)} \leq \mathbf{d}_u \end{aligned}$$

where, at the end of each deterministic optimization k , the PMA routine is called:

b) PMA: given current optimal design $\mathbf{d}^{(k)*}$ find $\mathbf{u}^{*(k)}_{PMA}$ which

$$\begin{aligned} &\text{minimizes: } g_i(\mathbf{d}^{(k)*}, \mathbf{u}) \\ &\text{subject to: } \|\mathbf{u}\| = \beta_i^{\text{target}} \end{aligned}$$

With the information provided by the MPTP, the shifting vector \mathbf{s} is updated. Notice that the optimization loops are no longer coupled, i.e., an entire deterministic optimization is performed and then, the reliability analysis routine is called. This procedure is repeated until some convergence criterion is achieved. Hence, the optimization problems are performed sequentially and by this reason the name of this class of de-coupling methods is *serial single loop methods*.

Sequential Approximate Programming (SAP)

Cheng et al. (2006) proposed a strategy based on the sequential approximate programming (SAP) concept that was successfully applied in structural optimization. In the SAP approach, the original optimization problem is decomposed into a sequence of sub-optimization problems. Each sub-optimization problem consists of an approximate objective function subjected to a set of approximate constraint functions. Thus, the SAP method makes use of an approximation of the reliability constraint, where a linear Taylor approximation of the approximate reliability index is used instead of the accurate one, reducing the computational effort by eliminating the need for the reliability analysis. Such approximate reliability index and its sensitivity are determined from a recurrence formula that is derived from the optimality conditions of the PMA and end up being the iterative formula of the Advanced Mean Value (AMV) method (Wu et al., 1990). The SAP-PMA algorithm is, then:

$$\begin{aligned} \text{for } k = 1, 2, \dots \\ \text{minimize: } & J(\mathbf{d}^{(k)}) \\ \text{subject to: } & \hat{p}_{m_i}(\mathbf{d}^{(k-1)*}) + \left(\nabla_{\mathbf{d}} \hat{p}_{m_i}(\mathbf{d}^{(k-1)*}) \right)^T (\mathbf{d}^{(k)} - \mathbf{d}^{(k-1)*}) \leq 0 \\ & \mathbf{d}_l \leq \mathbf{d}_i^{(k)} \leq \mathbf{d} \leq \mathbf{d}_u^{(k)} \leq \mathbf{d}_u \end{aligned} \quad , i = 1 \dots n_c$$

where k is the sub-optimization problem number, $\hat{p}_{m_i}(\mathbf{d}^{(k-1)*})$ is the approximate performance measure of the optimal design $\mathbf{d}^{(k-1)}$ of the previous sub-optimization problem, $\mathbf{d}_l^{(k)}$ and $\mathbf{d}_u^{(k)}$ are the lower and upper bounds of $\mathbf{d}^{(k)}$, respectively. Notice that at the end of each sub-optimization problem k , the reliability analysis routine is **not** called. Instead, the approximate performance measure and the approximate MPTP of the i^{th} constraint are just updated by the following relations:

$$\begin{aligned} \mathbf{u}_i^{(k)} &= \frac{\nabla_{\mathbf{u}} g_i(\mathbf{d}^{(k)*}, \mathbf{u}_i^{(k-1)})}{\left\| \nabla_{\mathbf{u}} g_i(\mathbf{d}^{(k)*}, \mathbf{u}_i^{(k-1)}) \right\|} \\ \hat{p}_{m_i}(\mathbf{d}^{(k)*}) &= g_i(\mathbf{d}^{(k)*}, \mathbf{u}_i^{(k)}) \end{aligned} \quad (6)$$

It should be noted that, at the end of this process, the optimal solution of the original problem is found and that there is a concurrent convergence of the design optimization and reliability calculation; in other words, they are sought simultaneously and independently. It is this fact that gives name to this class of de-coupling methods: *unilevel methods*.

Further comments

Some aspects should be stressed based on the papers cited in this section:

- i. the observation that a given method’s effectiveness depends on several factors suggests that different RDBO strategies may be better for different problems, and this, in turn, indicates that more benchmark studies need to be performed in order to

identify which RBDO strategy is best for each type of problem (e.g., type of objective function and constraints);

- ii. this dependence also indicates that more robust methods (e.g. those less sensible to variance of parameters) should be preferable;
- iii. the presence of many local minima, which is normal in complex engineering problems, indicates the need of using global optimization algorithms in the solution of RBDO problems (Lopez et al., 2011b; Torii et al., 2012).

Recently, a benchmark study on several RBDO methods based on FORM was presented by Aoues and Chateaneuf (2010), comparing the performance and robustness of these methods. In the next section, the classical approaches based on RIA and PMA, as well as the SORA and the SAP-PMA, are compared in the RBDO of two classical examples taken from the literature and a structure modeled using the finite element method.

Numerical Analysis

Example 1: multiple limit states

A classical problem in the RBDO literature is analyzed in this section. The problem is comprised of three probabilistic constraints and the design variables are the means of the two random variables of the problem ($\mathbf{d} = (\mu_{x_1}, \mu_{x_2})$):

$$\begin{aligned} \text{Minimize: } & J(\mathbf{d}) = \mu_{x_1} + \mu_{x_2} \\ \text{subject to: } & P_{f_i} = P(G_i(\mathbf{d}, \mathbf{X}) < 0) \leq P_{f_i}^{\text{allowable}} \quad i = 1 \dots 3 \\ & 0 \leq d \leq 10 \end{aligned}$$

where

$$\begin{aligned} G_1(\mathbf{X}) &= X_1^2 X_2 / 20 - 1, \\ G_2(\mathbf{X}) &= (X_1 + X_2 - 5)^2 / 30 + (X_1 - X_2 - 12)^2 / 120 - 1, \\ G_3(\mathbf{X}) &= 80 / (X_1^2 + 8X_2 + 5) - 1. \end{aligned} \quad (7)$$

Table 1. Comparison of the computational cost for solutions of Example 1.

	β_i^{target}	RIA	PMA	SORA	SAP-PMA	Reference
Normal	2	639 (180)	508 (172)	345 (265)	120 (108)	Yi and Cheng (2008)
	3	494 (636)	556 (-)	372 (455)	168 (-)	Yang and Gu (2004)
Normal	2	712 (504)	472 (540)	342 (255)	168 (180)	Aoues and Chateaneuf (2010)
	3	741 (531)	500 (540)	387 (264)	144 (198)	
	4	748 (531)	572 (747)	444 (348)	144 (234)	
Uniform	2	nc (nc)	643 (604)	697 (645)	144 (132)	Yi and Cheng (2008)

Each P_{f_i} is approximated using the FORM, using Eq. (4). The results obtained using our implementation of the four RBDO methods are compared to results of the literature, such as Aoues and Chateaneuf (2010), who employed the standard deviation (SD) of the random variables of 0.3, Yi and Cheng (2008), and Yang and Gu (2004), who employed SD equal to 0.6. For this purpose, the algorithms are tested for different types of random variables and values of the reliability index. The results are summarized in Table 1.

The results from the literature are in parenthesis and the source is indicated in the β^{target} column.

In comparison to the standard, most expensive approach (RIA), the methods PMA, SORA and SAP-PMA lead to average reductions of 20%, 42% and 77%, respectively, on the computational cost of the solution. The SAP-PMA approach significantly out-performed the other methods in solution of this particular RBDO problem.

The number of limit state function calls in our implementation of the algorithms RIA, PMA and SORA, is 1.71, 1.38 and 1.24 times higher, on average, than the references Yi and Cheng (2008) and Yang and Gu (2004). On the other hand, our implementation of SAP-PMA required 0.85 times the number of limit state function calls of the other algorithms. When compared to the implementation of Aoues and Chateaufneuf (2010), our implementation of PMA and SAP-PMA approaches achieved better results. Regarding the uniform distribution, the results obtained were quite close to the ones in Yi and Cheng (2008). This should be no surprise as these numbers are largely influenced by the parameters and other programming details adopted in each implementation.

Example 2: short column design

The short column design is also a classical problem from the literature. It consists in the minimization of the column cross sectional area having as design variables the mean value of its random dimensions: $\mathbf{d} = (\mu_{D_1}, \mu_{D_2})$. The column is subjected to two random moments M_1 and M_2 and to a random force F . The constraint of this structure is given by the limit of elastic behaviour (σ_y), which is also a random variable. Thus, the RBDO problem is given by:

$$\begin{aligned} \text{Minimize: } & J(\mathbf{d}) = \mu_{D_1} \cdot \mu_{D_2} \\ \text{subject to: } & P(G_{\text{column}}(\mathbf{d}, \mathbf{X}) < 0) \leq P_f^{\text{allowable}} \\ & 0 \leq \mu_{D_1} \\ & 0.5d_2 \leq \mu_{D_1} / \mu_{D_2} \leq 2 \end{aligned}$$

where

$$G_{\text{column}}(\mathbf{d}, \mathbf{X}) = 1 - \frac{4M_1}{D_1 D_2^2 \sigma_y} - \frac{4M_2}{D_1^2 D_2 \sigma_y} - \frac{F^2}{(D_1 D_2 \sigma_y)^2} \tag{8}$$

The coefficient of variation (C.O.V.) of all variables, in the reference solution, is 0.05.

The probabilistic constraint was approximated using FORM (Eq. 4) and the optimization was performed for different values of β^{target} . The RBDO of the column was performed by four different methods: classical RBDO based on RIA and on PMA, SORA and SAP-PMA. Each method was run four times, for different initial points, yielding the computational costs shown in Table 2. The same problem was solved by Aoues and Chateaufneuf (2010) and a comparison to results therein also is presented.

In comparison to the standard, most expensive approach (RIA), the methods PMA, SORA and SAP-PMA lead to average reductions of 43%, 66% and 86%, respectively, on the computational cost of the solution.

Table 2. Comparison of the computational cost for solutions of Example 2.

	β^{target}	RIA	PMA	SORA	SAP-PMA	Reference
Normal	2	975/448/677/477	310/264/288/272	112/178/182/156	70/84/84/84	This work
	3	975/381/543/491	340/296/304/349	120/201/210/175	70/72/96/108	This work
	3	(600/525/630/945)	(450/651/570/684)	(119/196/252/346)	(56/80/nc/nc)	Aoues and Chateaufneuf (2010)
	4	975/390/543/449	289/304/437/365	117/222/264/196	70/84/84/120	This work
Lognormal	2	846/685/959/825	346/288/317/317	136/157/169/275	70/84/84/84	This work
	3	726/618/886/818	357/344/389/397	171/194/206/322	60/96/96/96	This work
	4	616/508/758/712	533/480/501/485	236/278/293/450	70/96/96/108	This work

*nc = no convergence

Table 3. Influence of q on the computational cost of the SAP-PMA approach.

$\beta^{\text{target}} = 4$	Normal		Lognormal	
	$q/\text{C.O.V.}$	0.05	0.15	0.05
0.04	144	240	156	180
0.06	108	168	120	132
0.08	96	144	108	120
0.10	84	120	96	108

From the numerical analysis, it was realized that the computational cost and convergence for SAP-PMA highly depends on the size of each sub-problem k : $\mathbf{d}_l \leq \mathbf{d}_l^{(k)} \leq \mathbf{d} \leq \mathbf{d}_u^{(k)} \leq \mathbf{d}_u$. Hence, a strategy was adopted where the size of each problem is considered a function of the starting point for each subproblem: $\mathbf{d}_l^{(k)} = (1-q) \mathbf{d}_0^{(k)}$ and $\mathbf{d}_u^{(k)} = (1+q) \mathbf{d}_0^{(k)}$, where q is a constant. The computational costs for different values of q and of the C.O.V. are shown in Table 3. The results in Table 2 were obtained for $q = 0.1$.

The computational cost of the SORA approach depends on the maximum number of iterations of each deterministic optimization k . In the first version of the algorithm, Du and Chen (2004) employed the full optimization of the deterministic step. Here, the SORA approach is also tested having as constraint for each deterministic optimization k the maximum number of iterations (it_{out}). Results are presented in Table 4.

Table 4. Influence of it_{out} on the computational cost of the SORA approach.

$\beta^{\text{target}} = 4$	C.O.V.	
	it_{out}	0.05
3	228/235/249	243/302/1085
5	165/175/196	190/280/336
20	171/175/193	192/282/909

Example 3: square plate modelled by finite elements

The case of a 2D square plate with a quarter of circle retired from a corner (Fig. 4), modelled using finite elements, is considered. The plate is made of steel with Young modulus $E = 200$ GPa and yield stress $\sigma_y = 200$ MPa. The plate is clamped at its lower boundary and loaded at its left boundary with a distributed load, with total magnitude of 800 N. Uncertainties are considered on the plate thickness and on the radius, which are modelled as random variables: $H \sim \Gamma(h, s_h)$ and $R \sim \Gamma(r, s_r)$, respectively (where Γ and s_i are the probability distribution and the standard deviation of the random variable). These random variables are grouped into the random vector $\mathbf{X} = (H, R)$.

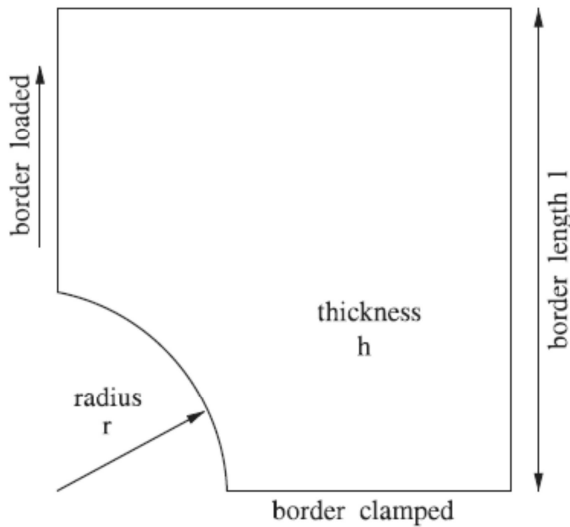


Figure 4. Square plate with a quarter of circle retired from a corner.

Design variables of the optimization problem are the mean values h and r , which are grouped into the design vector $\mathbf{d} = (\mu_h, \mu_r)$. The plate is optimized in order to minimize its volume under the constraint of remaining in the elastic domain; hence, the maximum stress must remain below the yield limit:

$$G(\mathbf{d}, \mathbf{X}) = \sigma_y - \sigma_{MAX}(\mathbf{d}, \mathbf{X}), \tag{9}$$

where σ_{MAX} is the maximum stress on the structure. Thus, the RBDO problem may be stated as:

Minimize: $J(\mathbf{d}) = \left(l^2 - \frac{\pi \cdot r^2}{4} \right) \cdot h$

subject to: $P_f = P \left(G(\mathbf{d}, \mathbf{X}) = 1 - \frac{\sigma_{MAX}}{\sigma_y} < 0 \right) \leq P_f^{allowable}$

$1mm \leq h$

$20mm \leq r \leq 60mm$

Notice that deterministic bounds were imposed on the design variables.

Two distributions are tested in this example: normal and lognormal. For both cases, the standard deviations are $s_h = 0.1$ mm and $s_r = 4.0$ mm. The length of the plate's border is fixed to $l = 1$ m. A convergence study leads to a mesh with 1352 elements and 1458 nodes. Stresses are evaluated on Gauss integration points. The normal stress in the s-direction is used for the evaluation of the limit state function G (Fig. 5).

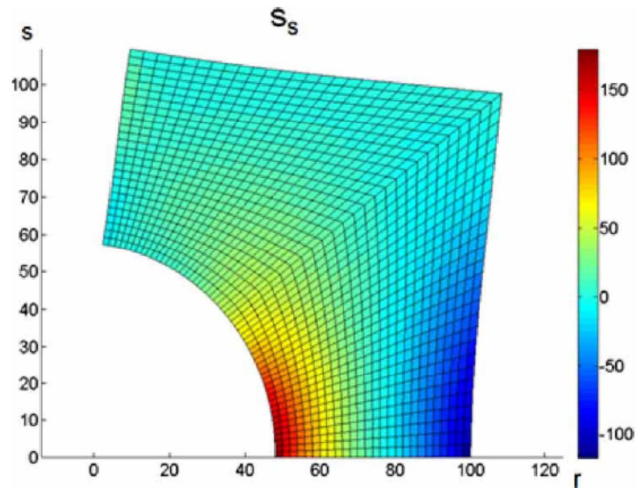


Figure 5. Stress distribution (MPa) in the s-direction.

The probabilistic constraint was approximated using FORM (Eq. (4)), and the optimization was performed for different values of β^{target} . All the final designs are presented in Table 5. The computational cost is evaluated in terms of the number of calls to the finite element code.

Table 5. Comparison of the computational cost (number of calls to the finite element code) of RBDO based methods.

Distribution Γ	β^{target}	RIA	PMA	SORA	SAP-PMA	\mathbf{x}^*
Normal	2	372	197	192	48	(1.00, 39.4)
	3	356	181	216	48	(1.00, 34.6)
	4	464	201	244	56	(1.00, 29.1)
Lognormal	2	500	311	196	64	(1.00, 39.5)
	3	624	398	229	72	(1.00, 34.9)
	4	816	455	290	96	(1.00, 29.9)

In comparison to the standard, most expensive approach (RIA), the methods PMA, SORA and SAP-PMA lead to average reductions of 67%, 56% and 88%, respectively, on the computational cost of the solution. The SAP-PMA approach significantly out-performed the other methods in solution of this particular RBDO problem.

The reduction in computational cost is very significant in this example due to use of finite element modeling for the mechanical problem. In the case of Lognormal random variables with $\beta^{\text{target}} = 4$, for example, the RIA method needed 816 finite element calls (about 3h20min) to obtain the optimal design, whereas the SAP-PMA method required only 96 finite element calls (about 27 minutes).

Surely, other benchmark comparisons are required in order to make more definite conclusions. This will be the subject of future work.

Concluding Remarks

The main goal of this paper was to review the main Reliability-Based Design Optimization (RBDO) methods based on the First-Order Reliability Method (FORM). A review and a comparison between the two main coupled FORM approaches, the Reliability Index Approach (RIA) and the Performance Measure Approach (PMA) were presented. The coupled approach to solving RBDO problems was presented in detail and its high computational cost was highlighted. A general review of de-coupling techniques was also presented, and two of the main de-coupling methods were presented in detail, the Sequential Optimization and Reliability Approach (SORA) and the Sequential Approximate Programming (SAP-PMA). The review presented herein suggests that SORA and SAP-PMA should be the methods of better performance and robustness, in comparison to the other methods described herein. Finally, this review presented the main techniques and references on the subject, and should serve as a guide to those entering this exciting and challenging subject.

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