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A Review on Extension of Lagrangian-Hamiltonian Mechanics

This paper presents a brief review on Lagrangian-Hamiltonian Mechanics and deals with the several developments and extensions in this area, which have been based upon the principle of D'Alambert or the other. It is not the intention of the authors to attempt to provide a detailed coverage of all the extensions of Lagrangian-Hamiltonian Mechanics, whereas detailed consideration is given to the extension of Noether's theorem for nonconservative systems only. The paper incorporates a candid commentary on various extensions including extension of Noether's theorem through differential variational principle. The paper further deals with an extended Lagrangian formulation for general class of dynamical systems with dissipative, non-potential fields with an aim to obtain invariants of motion for such systems. This new extension is based on a new concept of umbra-time, which leads to a peculiar form of equations termed as 'umbra-Lagrange's equation'. This equation leads to a simple and new fundamental view on Lagrangian Mechanics and is applied to investigate the dynamics of asymmetric and continuous systems. This will provide help to understand physical interpretations of various extensions of Lagrangian-Hamiltonian Mechanics.
Keywords: *Lagrangian-Hamiltonian Mechanics, Umbra-Lagrangian, Noether's theorem*

Introduction

From the late seventeenth century to the nineteenth century classical mechanics (Goldstein, 1980; Sudarshan and Mukunda, 1974) was one of the main driving forces in the development of physics, interacting strongly with developments in mathematics, both by borrowing and lending. In fact, mechanics and indeed all theoretical science is a game of mathematical make–believe. The topics developed by its main protagonists, Newton, Lagrange, Euler, Hamilton and Jacobi among several others form the basis of classical mechanics.

Since the last few decades, the subject of classical mechanics itself was undergoing a rebirth and expansion with strong developments in mathematics. There has been an explosion of research in the classical dynamical systems, focused on the discovery of advanced mathematics (e.g. Lie Algebra, differential geometry, etc.) (Sattinger and Weaver, 1986; Bluman and Kumei, 1989; Gilmore, 1974). The aforementioned occurrences in the second part of the 20th century have radically changed the nature of the field of classical mechanics. The first development has led to modeling and analysis of complex, multi-bodied (often elastic bodied) structures, such as satellites, robot manipulators, turbo machinery and vehicles. The second has led to the development of numerical techniques to derive the describing equations of motion of a dynamical system, integration, simulation and obtaining the response. This new computational capability has encouraged scientists and engineers to model and numerically analyze complex dynamical systems, which in past either they could not be analyzed, or were analyzed using gross simplifications.

The prospect of using computational techniques to model a dynamical system has also led dynamicists to reconsider existing methods of obtaining equations of motion. The methods of Lagrange (Lagrange, 1788) and Hamilton (Baruh, 1999) are used to carry out the primary task of deriving the equations of motion. Generalized coordinates which do not necessarily have to be physical coordinates are used as motion variables in these methods. This makes the Lagrangian-Hamiltonian approach more flexible than the Newtonian, as Newtonian approach is implemented using physical coordinates. The use of Lagrange's formulation of dynamics offers the quickest way of deriving system equations for complex physical systems, and are preferable compared to the Newtonian approach. However, the Lagrangian approach has certain limitations. The elimination of the constraint forces from the Lagrange's formulation does not allow one to directly calculate these forces. They can, however, be determined using an indirect approach. Besides this, Lagrange's equation suffers heavily in the presence of time fluctuating parameters, non-potential fields, general dissipation and gyroscopic forces. Derivation of the Lagrange's equations of motion for nonconservative and dissipative system (Rosenberg, 1977; Meirovitch, 1970; Whitaker, 1959) is essentially patchwork. This hinders the analysis of such systems, which the Lagrangian can afford. Nevertheless, the greatest advantage of Lagrangian formulation is that it brings out the connection between conservation laws and important symmetry properties of dynamical systems. Knowledge of conservation laws is of great importance in the analysis of dynamical systems as they lead to a complete integrability of dynamical system. The fundamental symmetries motivated the study of conservation laws from geometrical and group-theoretical point of view. The theorem of Emmy Noether (Noether, 1918) is one of the most fundamental justifications for conservation laws. Her theorem tells us that conservation laws follow from the symmetry property of nature. From the literature (Goldstein, 1980), (Sudarshan and Mukunda, 1974), it is found that translational symmetry implies momentum conservation, time translational symmetry implies energy conservation and rotational symmetry implies conservation of angular momentum. There exists a fundamental theorem called Noether's theorem (Noether, 1918), which shows that indeed, for every spatial continuous symmetry of a system, which can be described by a Lagrangian, some physical quantity is conserved and the theorem also allows us to find that quantity.

The objective of the paper is to present the developments in the field of Lagrangian-Hamiltonian Mechanics with particular regard to extension of Noether's theorem. In recent years, the authors have attempted to develop alternative method to the construction of the first integrals of dynamical systems by means of extended Noether's theorem. Typical contributions in this area are given in reference (Rastogi, 2005; Mukherjee et al., 2006; Mukherjee et al., 2007; Mukherjee et al., 2009). Investigating such alternatives has been applied to analyze the dynamics through invariance of the action integral for some engineering applications, which are rarely applied. Moreover, such research will open new horizons for the physics

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students, who are conversant with this theorem and its applications. A brief review on such major extensions has been presented in this paper through variational principle and group-theoretical approach or the other. This paper is divided into different sections, each dealing with the various aspects of the subjects. It begins with a summary of the evolution of classical Lagrangian-Hamiltonian Mechanics followed by a general overview of extension through variational or group theory. The fourth section of this work presents the alternative method of extension of Lagrangian-Hamiltonian Mechanics through umbra time. Few examples have been provided to elucidate the concept in brief.

Nomenclature

$F(t$ = *external force with time fluctuation*
H^* = *umbra-Hamiltonian of the system*
K = *stiffness of the spring in N/m*
L = *Lagrangian of the system*
L^* = *umbra-Lagrangian of the system*
R = *damping coefficient of the damper in N-s/m*
$\mathbf{v}$ = *infinitesimal generator of rotational SO (2) group*
$\mathbf{v}^j$ = j^{th} *infinitesimal generator of symmetric group*
$\mathbf{v}_t^j$ = *real time component of* j^{th} *infinitesimal generator*
$\mathbf{v}_\eta^j$ = *umbra time component of* j^{th} *infinitesimal generator*
V_R = *real-time potentials for resistive elements*
V^* = *total-umbra potential*
V_c^* = *umbra-potential for compliance elements*
V_p^* = *umbra-potential for external forces*
V_R^* = *umbra-potential for resistive elements*
T^* = *umbra-kinetic energy*
T_c^* = *umbra-co-kinetic energy*
e = *generalized force*
f = *generalized velocity*
m = *mass of the body in Kg*
$p(t)$ = *real-time momentum*
$p(\eta)$ = *umbra-time momentum*
$q(t)$ = *generalized displacement in real time*
$q(\eta)$ = *generalized displacement in umbra-time*
$\dot{q}(t)$ = *generalized velocity in real time*
$\dot{q}(\eta)$ = *generalized velocity in umbra-time*
$x_i(\ )$ = *linear displacements in real time or umbra-time, where* $i = 1...n$
$\dot{x}_i(\ )$ = *linear velocity in real time or umbra-time, where* $i = 1...n$
t = *real-time in s*
η = *umbra-time in s*

Evolution of Classical Lagrangian-Hamiltonian Mechanics

The major contribution in classical mechanics came from Lagrange (1788). The contributions of Lagrange put the field of analytical mechanics into a structured form now known as *Lagrangian mechanics*. In the original derivation, Lagrange's equations were written for conservative systems only, and applicable when the system is closed, constraints are integrable, and there are no gyroscopic forces. Hamilton (Baruh, 1999; Gantmachar, 1970; Calkin, 2000) has developed the most general principle of least action and showed that the Lagrangian with time integration provided the definition of action and minimization of this action integral established the Lagrange's generalized equation. The main advantage of this new formulation is that it holds for any system subject to constraints and independently of the co-ordinates, which are chosen to represent the motion. However, the problem of dissipation was handled by Rayleigh (Gantmachar, 1970; Jose and Saletan, 1998), who attempted to enlarge the scope of Lagrange's equation to incorporate dissipative forces in this generalized equation. He added velocity dependent potential through virtual work done by the dissipative elements and then re-encapsulated in an extended formula. In this formulation, the velocity's dependent potential should not be brought inside the scope of total derivative with respect to time, otherwise an unrealistic momentum and inertia would enter in the equation. This is the reason why the velocity's dependent Rayleigh potential fails in the case of gyroscopic forces.

The next problem is to deal with Nonholonomic systems in classical mechanics as to determine the equation of motion for constrained systems. When physical constraints are imposed on an unconstrained set of particles, forces of constraints are engendered, which ensure the satisfaction of the constraints. The equation of motion developed for such constrained systems is based on the principle of D'Alambert, and later elaborated by Lagrange (1788) through Lagrange multipliers. Since its initial formulation by Lagrange, the problem of constrained motion has been vigorously and continuously worked on by various scientist including Volterra, Boltzmann, Hamel, Whittaker and Synge to name a few. Gauss (1829) explained a new general principle for the motion of constrained mechanical systems referred to as Gauss's principle, by making use of acceleration. Gibbs (1879) and Appell (1899) independently discovered a new equation, which is known as the Gibbs-Appell equations of motion (Appell, 1899). Pars (1979) also referred to the Gibbs-Appell equations as the most comprehensive equations of motion so far discovered. Routh Gantmachar (1970) proposed the equations of motion in a potential field taking a part of the Lagrangian variables and a part of the Hamiltonian variables, called as Routh variables. Lie (Hassani, 1999; Olver, 1986) introduced the group theory for canonical transformations by considering infinitesimal transformations.

Extensions of Lagrangian-Hamiltonian Mechanics through Variational Principle and Group-theoretical Approach

Apparently, the first to notice the connection of conservation laws to invariance properties of dynamical systems was Jacobi (1884), who has derived the conservation law for linear and angular momentum from the Euclidean invariance of the Lagrangian. Emmy Noether (1918) formulated a theorem to find the invariants of the dynamical system and showed a relationship between symmetry aspects of conservation laws and invariance properties of space and time, i.e., their homogeneity and isotropy. These fundamental symmetries motivated the study of conservation laws from geometrical and group-theoretical point of view. Most of the results of conservation laws of classical mechanics based on Noetherian approach could be found in the research papers of Hill (1951), and Desloge and Karch (1977), where it has been applied as a reliable tool to find new conservation laws of dynamical systems.

The physics associated with the classical conservation laws widely attracted the investigations in this field, intriguing problems of classical mechanics by engineers and theoretical physicists, who

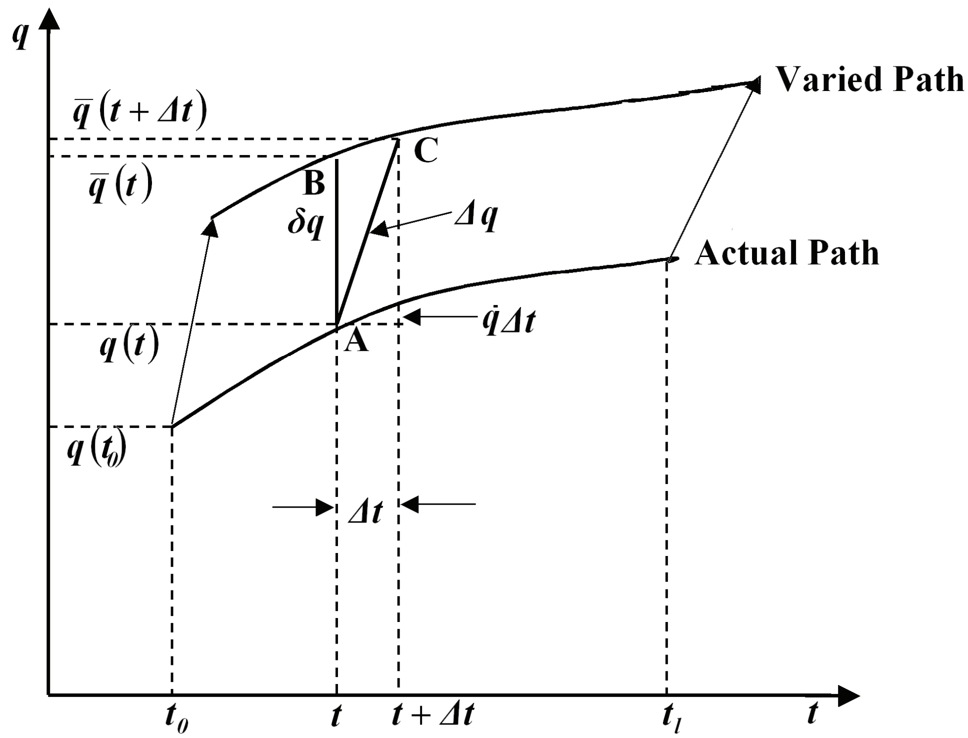

Figure 1. Non-contemporaneous Variation.

formulated newer types of constant of motion. In their several papers, Vujanovic (1970), Djukic and Vujanovic (1975) and Vujanovic (1978) have investigated this field of analytical mechanics and developed a new approach to obtain constants of motion. Vujanovic (1970) has established a group-variational procedure for finding first integral of dynamical systems. Djukic and Vujanovic (1975) have proposed a Noether's theorem for mechanical system with non-conservative forces. Primarily, this theory was based on the idea that the transformations of time and generalized coordinates together with dissipative forces determine the transformations of generalized velocities. Vujanovic (1978) has reported a method for finding the conserved quantities of non-conservative holonomic systems based on the differential variational principle of D'Alembert, which was equally valid for both conservative and non-conservative systems. His research work has shown that the existence of first integrals mainly depends on the existence of solutions of partial differential equations, known as Killing equations (Hassani, 1999; Olver, 1986).

The above procedures, however, do not have generality of the Noether's theorem, as it mainly depends on the particular structure of the special class of problems being attempted. However, our choice to relate the alternative method of umbra Lagrangian mechanics is motivated by the fact that Noether's theorem, extended by Bahar et al. (1987) tackles both the aspects which are of considerable importance in the study of conservation laws. On phenomenological level, it shows the connection of conservation laws of some non-conservative system to the symmetries of space and time. On the other hand, it also possesses a pragmatic value as it could be used in engineering applications.

The significant work in this direction was reported by Bahar and Kwatny (1987), who provided a useful method based on a differential variational principle (Vujanovic, 1978) in order to extend Noether's theorem to constrained–nonconservative dynamical systems, which includes the influence of dissipation and constraints, and thus making it suitable for use in engineering applications. The main focus of their research work was primarily concerned with the extension of the notion of variation, which also included variation in time, thus leading to non-contemporaneous variation (NCV). Here the use of NCV is limited to first order terms, and was denoted by Δ as the convention adopted in Vujanovic (1978), whereas δ is the contemporaneous variation (CV). The symbol δ also defines a simultaneous or Lagrange's variation. A representative point A that is on the actual path at time *t* and an infinitesimal point B on the varied path at the same time *t* are correlated by

$$\bar{q}(t) = q(t) + \delta q \quad , \tag{1}$$

where $\bar{q}(t)$ and $q(t)$ are coordinates of points B and A respectively. The geometrical interpretation non-contemporaneous variation (NCV) may be easily seen in Fig. 1. The following definition was used

$$\Delta q_i = \delta q_i + \dot{q}_i \Delta t \tag{2}$$

The Non-contemporaneous variation of any function $F(q_i, \dot{q}_i, t)$ is given by the expression

$$\Delta F = \frac{\partial F}{\partial q_i}\Delta q_i + \frac{\partial F}{\partial \dot{q}_i}\Delta \dot{q}_i + \frac{\partial F}{\partial t}\Delta t \ , \tag{3}$$

Putting Eq. (2) in (3) and following the procedure as given in reference (Bahar and Kwatny, 1987; Vujanovic, 1978), one may obtain

$$\frac{d}{dt}(\Delta q_i) = \Delta \dot{q}_i + \dot{q}_i \frac{d}{dt}(\Delta t) \quad (4)$$

Equation (4) demonstrates that the usual commutatively rule does not extend to NCV. For the derivation of the Noether's theorem, one may consider the variational expression as $\partial L + Q_i \partial q_i$, which defines the integrand of the action integral governing the motion of such systems. This Lagrangian can be defined up to an additive term $\frac{d}{dt}G(q_{i,}t)$ and still satisfies the Lagrangian equation of motion identically. L can be replaced by $\left(L + \overset{\bullet}{G}\right)$ to have:

$$\partial\left(L + \overset{\bullet}{G}\right) + Q_i \delta q_i \quad (5)$$

Thus expression (5) has been defined by authors (Bahar and Kwatny, 1987) as physical motivation and simply considered this equation to undergo two different transformations given as follows:

(a) Non-Contemporaneous Transformation (NCT): The variational expression given in Expression (5) may be written in the form

$$\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i + Q_i \delta q_i + \delta \dot{G} \quad (6)$$

Expression (6) in non-contemporaneous may be written as

$$\frac{\partial L}{\partial q_i}(\Delta q_i - \dot{q}_i \Delta t) + \frac{\partial L}{\partial \dot{q}_i}(\Delta \dot{q}_i - \ddot{q}_i \Delta t) + Q_i \delta q_i + \delta \dot{G}, \quad (7)$$

Expression (7) will finally reduce to

$$\Delta L - \dot{L}\Delta t + Q_i \delta q_i + \delta \dot{G}. \quad (8)$$

The last two terms of expression (8) may also be written in NCV as well.

(b) Contemporaneous Transformation (CT): Following the usual process to rewrite expression (6) as

$$\frac{\partial L}{\partial q_i}\delta q_i + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right) - \delta q_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) + Q_i \delta q_i + \delta \dot{G} \quad (9)$$

and following the procedure as in reference (Bahar and Kwatny, 1987; Vujanovic, 1978), one may obtain the following equation:

$$\begin{aligned} &\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_i}\Delta q_i + \left(L - q_i \frac{\partial L}{\partial q_i}\right)\Delta t + \Delta G\right] \\ &= Q_i(\Delta q_i - \dot{q}_i \Delta t) + \Delta L + L\frac{d}{dt}(\Delta t) + \frac{d}{dt}(\Delta G) \end{aligned} \quad (10)$$

The first integrals or conserved quantities may be obtained if right hand side of Eq. (10) can be made to vanish, then bracketed quantity under time derivative sign become a constant

$$Q_i(\Delta q_i - \dot{q}_i \Delta t) + \Delta L + L\frac{d}{dt}(\Delta t) + \frac{d}{dt}(\Delta G) = 0. \quad (11)$$

Then, one may get

$$\frac{\partial L}{\partial \dot{q}_i}\Delta q_i + \left(L - q_i \frac{\partial L}{\partial q_i}\right)\Delta t + \Delta G = Cons\tan t. \quad (12)$$

If $\Delta G = 0$, then

$$\frac{\partial L}{\partial \dot{q}_i}\Delta q_i + \left(L - q_i \frac{\partial L}{\partial q_i}\right)\Delta t = Cons\tan t. \quad (13)$$

Equation (13) is a conserved quantity. Now considering the linear infinitesimal one parameter transformation as followed in Bahar [29], one may obtain the conservation law

$$\frac{\partial L}{\partial \dot{q}_i}\xi_i \left(L - \frac{\partial L}{\partial \dot{q}_i}q_i\right)\tau + P = cons\tan t \quad (14)$$

where ξ_i, τ_i and P_i are the linear infinitesimal one-parameter transformations. This may also be obtained by following usual approach and generalized killing equations, which has been obtained in reference (Djukic and Vujanovic (1975). Many such examples of the Noether's theorem are contained in Vujanovic and Jones (1989). A variety of methods have been developed for the search of conservation laws such as methods of integrating factors, also termed as direct or ad hoc procedure as reported by Sarlet and Bahar (1980), and Djukic and Sutela (1984). Other methods were based on similarity variables (Jones and Ames, 1967) and transformation approach as presented by Crespo da Silva (1974). In this way, some procedures of group-theoretical approach with considerable generality have been established, which related the existence of first integrals to the symmetries of certain mathematical objects and served for describing the dynamical systems.

Several other studies concerned with the symmetry aspects of Lagrangian and Hamiltonian formalism have been considered in the review papers of Katzin and Levine (1976), and Fokas (1979). A generalization of Noether Theorem in classical mechanics has been attempted by Sarlet and Cantrijn (1981). Another class of methods, in the spirit of finding invariants of motion for time-dependent parameters, are primarily established by few researchers such as Lewis and Leach (1982), who have reported an approach of finding exact invariants for one-dimensional time-dependent classical Hamiltonians, and as Sarlet (1983), who has developed a methodology of finding first integrals for one-dimensional particle motion in a non-linear, time-dependent potential field. Motivated by the research works of Vujanovic (1970), Vujanovic (1978) and Djukic and Vujanovic (1975), Simic (2002) has analyzed polynomial conservation laws of one-dimensional non-autonomous Lagrangian dynamical system and demonstrated that final form of dynamical system and corresponding conservation law depends on the solution of the so-called potential equation, which will be presented as

$$L(t, x, \dot{x}) = \frac{1}{2}\dot{x}^2 - \Pi(t, x). \quad (15)$$

In Eq. (15), $\Pi(t, x)$ denotes the potential of the system and over dots denote differentiation with respect to independent variables. However, the structure of symmetry transformation, which generated particular class of conservation laws, could be prescribed independent of potential equation. In this Lagrangian function, generality of Noether's theorem is not being considered, which may be suitable to obtain invariants of any general class of systems.

Variational principles (Gelfend and Fomin, 1963) and principle of virtual work continued to attract interest of the researchers and have great importance in physics and mathematics. These principles helped in establishing connections and applications of these disciplines, and in devising diverse approximation techniques. Arizmendi et al. (2003) developed a variant of the usual Lagrangian, which describes both the equations of motion and the variational equations of the system. The required Lagrangian is defined in an extended configuration space comprising both the original configurations of the system and all virtual displacements joining any two integral curves. An extremal principal for obtaining the variational equations of a Lagrangian system is reviewed and formalized by Delgado et al. (2004) by relating the new Lagrangian function (Arizmendi et al., 2003) needed in such scheme to a prolongation (Hassani, 1999; Olver, 1986) of the original Lagrangian. In their work, they considered an *N*-degree of freedom dynamical system described by an autonomous non-singular Lagrangian function $L(q_a, \dot{q}_a, t)$, *a = 1.2…N* defined in the tangent bundle *TQ* of its configuration Manifold Q. Now, an extended configuration space *D* (D'Alambert's configuration manifold) was considered, comprising of both the original configuration of the system plus all possible "virtual displacements" joining, in a first approximation, any two of the extremal paths of the original system. With the help of *L*, they defined new Lagrangian $\gamma(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{\varepsilon}, \dot{\boldsymbol{\varepsilon}}, t)$ as

$$\gamma(\boldsymbol{q}, \dot{\boldsymbol{q}}, \boldsymbol{\varepsilon}, \dot{\boldsymbol{\varepsilon}}, t) \equiv \varepsilon^a \frac{\partial L}{\partial q^a} + \dot{\varepsilon}_a \frac{\partial L}{\partial \dot{q}^a}, \qquad (16)$$

where q and $\dot{q}$ are given configuration displacements and velocities, $\boldsymbol{\varepsilon}$ is virtual displacement and $\dot{\boldsymbol{\varepsilon}}$ are virtual velocity. It is worth mentioning that even nonconservative systems can also be handled by using a prolonged Lagrangian function and Noether's theorem in this extended space, obtained by them. It is not appropriate to provide all details of this extension, as it basically finds its applications in relativistic theories.

Alternative Method for Extending Lagrangian-Hamiltonian Mechanics

As detailed in the previous section, the procedure and methods developed by various researchers did not consider the generality of Noether's theorem, as it was mainly focused on the particular structure of the special class of dynamical problems being studied. So, it is necessary to extend the scope of Lagrangian and Noether's theorem, which includes the influence of dissipation and sometimes constraints, thus making it suitable for the larger and complex engineering applications. To overcome the limitations and enlarging the scope of Lagrangian-Hamiltonian mechanics, a new proposal of additional time like variable 'umbra-time' was made by Mukherjee (1994) and this new concept of umbra-time leads to a peculiar form of equation, which is termed as umbra-Lagrange's equation. A brief and candid commentary on idea of umbra Lagrangian is given by Brown (2007). This idea was further consolidated by presenting an important issue of invariants of motion for the general class of system by extending Noether's theorem (Mukherjee, 2001). This notion of umbra-time is again used to propose a new concept of umbra-Hamiltonian, which is used along with the extended Noether's theorem to provide an insight into the dynamics of systems with symmetries. The advantages of using such Lagrangian are many ways as one may get the both aspects of the problem. It provides a great insight of the dynamical system through extended Noether's theorem and on the other hand, it gives a pragmatic value since it could be used as a reliable tool for derivation of new conservation laws for many engineering problems, where the physicist can play a leading role. One of the most important insights gained from the umbra-Lagrangian formalism is that its underlying variational principle (Rastogi, 2005) is possible, which is based on the recursive minimization of functionals. In this direction, Rastogi (2005) also defined all these notions in an extended manifold comprising of real time, and umbra and real time displacements and velocities. The umbra Lagrangian theory has been used successfully to study invariants of motion for non-conservative mechanical and thermo-mechanical systems [48]. In another paper, the authors applied umbra Lagrangian to study dynamics of an electro-mechanical system comprising of an induction motor driving an elastic rotor (Mukherjee et al., 2009). This system was symmetric in two sets of coordinates, one set was mechanical or geometrical, and the other symmetry was in electrical domain. Recently, Mukherjee et al. (2009) presented the extension for Lagrangian-Hamiltonian Mechanics for continuous systems and investigated the dynamics of an internally damped rotor through dissipative coupling. Some basic concepts of umbra-Hamiltonian theory may be given in Appendix A for ready reference. The concept of umbra-Lagrangian may be represented as shown in Fig. 2 and briefly expressed as follows:

(a) D'Alembert's basic idea of allowing displacements, when the real time is frozen, is conveniently expressed in terms of umbra-time.
(b) Umbra-time may be viewed as the interior time of a system.
(c) Potential, kinetic and co-kinetic energies stored in storage elements like symmetric compliant and inertial fields can be expressed as functions in umbra-time (umbra-displacements and umbra-velocities).
(d) The effort of any external force, resistive element or field, gyroscopic element (treated as anti-symmetric resistive field), transformer or lever element, anti-symmetric compliant field and sensing element depends on displacements and velocities in real time. The potentials associated with them are obtained by evaluation of work-done through umbra-displacements.

In formulating the umbra-Lagrangian for a system, two classes of elements are generally required: (a) storage elements, whose energies are defined in terms of umbra-displacements and umbra-velocities, and (b) rest of the elements for which the efforts returned are evaluated entirely in terms of real time and their umbra-potential are obtained by umbra-displacement of the corresponding element. These two categories of elements can be identified through breaking the system into its basic entities or dynamical units. Bond graphs (Mukherjee and Karmakar, 2000; Karnopp et al., 1990) may be one of the tools for representing the dynamics of the system and obtaining the expressions for either the classical or the umbra-Lagrangian as provided in details in Appendix B.

The broad principle, on which the creation of umbra-Lagrangian and other relevant energies (Mukherjee, 2001; Rastogi, 2005; Mukherjee et al., 2009) are based, can be summarized as follows:

(a) All temporal fluctuations of parameters are in real time.
(b) The co-kinetic and potential energies would normally be evaluated taking generalized velocities and displacements as function of umbra time and it is assumed that there are *n* generalized co-ordinates. All definitions are separately given in Nomenclature.

The umbra-potential (for potential forces only) is defined as

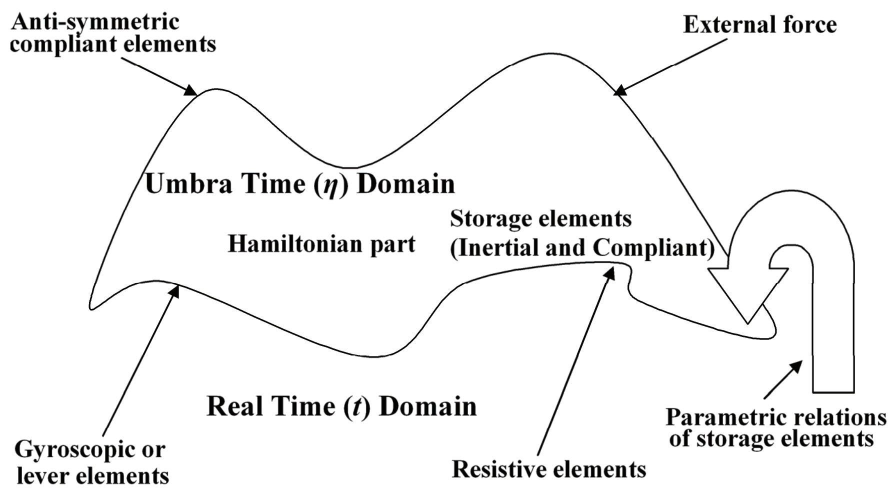

Figure 2. Iconization of umbra theory.

$$V_c^*(t,\boldsymbol{q}(\eta)) = \int_0^{\boldsymbol{q}(\eta)} e(t,\boldsymbol{q}(\eta))\, d\,\boldsymbol{q}(\eta), \qquad (17)$$

where a bold face letter represents a vector quantity. As an example, the umbra-potential energy for a spring with time varying stiffness can be written as

$$V_c^*(t,x(\eta)) = \int_0^{x(\eta)} K(t)\; x(\xi)\; dx(\xi) = \frac{1}{2}K(t)x^2(\eta). \quad (18)$$

Likewise, the umbra-kinetic energy is defined as

$$T^*(t,\boldsymbol{q}(\eta),\ \boldsymbol{p}(\eta)) = \int_0^{\boldsymbol{p}(\eta)} f(t,\boldsymbol{q}(\eta),\boldsymbol{p}(\eta))\, d\,\boldsymbol{p}(\eta), \qquad (19)$$

and the umbra co-kinetic energy as

$$T_c^*(t,\dot{\boldsymbol{q}}(\eta)) = \dot{\boldsymbol{q}}(\eta)\boldsymbol{p}(\eta) - T^*(t,\boldsymbol{q}(\eta),\ \boldsymbol{p}(\eta)). \qquad (20)$$

For instance, the umbra co-kinetic energy for a time varying mass can be represented as

$$T_c^*(t,\dot{x}(\eta)) = \int_0^{\dot{x}(\eta)} m(t)\dot{x}(\xi)d\dot{x}(\xi) = \frac{1}{2}m(t)\dot{x}^2(\eta). \qquad (21)$$

(c) The umbra-potential associated with generalized resistive fields is evaluated, which is based on the philosophy that resistive fields open the system, and thus they observe the states of motion in real time as an external observer. The force generated by them does work on the system through umbra generalized displacements

$$V_R^* = \int_0^{\boldsymbol{q}(\eta)} e(t,\boldsymbol{q}(t),\dot{\boldsymbol{q}}(t))d\boldsymbol{q}(\eta). \qquad (22)$$

For example, umbra-potential for a damper with time varying damping coefficient may be written as

$$V_R^* = \int_0^{x(\eta)} R(t)\dot{x}(t)dx(\xi) = R(t)\dot{x}(t)x(\eta). \qquad (23)$$

It is significant to note that in classical approach, one may incorporate dissipative forces through Rayleigh potentials, which in linear case can be defined as

$$V_R = \frac{1}{2}\dot{\boldsymbol{q}}^{\boldsymbol{T}}(t)[R]\dot{\boldsymbol{q}}(t). \qquad (24)$$

In such case, the anti-symmetric part of [R] in Eq. (24), if present, has no contribution to V_R. In classical approach, such anti-symmetric part is identified as gyroscopic force and subjected to a set of alternative treatment. However, in present approach, as considered in Eq. (22), it can include both the dissipative (symmetric part) as well as the gyroscopic effects (anti-symmetric part) through the resistive field, for which the corresponding umbra-potential becomes

$$V_R^* = \dot{\boldsymbol{q}}^{\boldsymbol{T}}(t)[R]\boldsymbol{q}(\eta). \qquad (25)$$

(d) The umbra-potential associated with external generalized forces may be incorporated as

$$V_p^*(t,\boldsymbol{q}(\eta)) = -\int_0^{\boldsymbol{q}(\eta)} F(t,\boldsymbol{q}(t),\dot{\boldsymbol{q}}(t))d\,\boldsymbol{q}(\eta). \qquad (26)$$

To illustrate, one may find the umbra-potential for any external force $F(t)$ as

$$V_p^*(t,x(\eta)) = -\int_0^{x(\eta)} F(t)\, dx(\xi) = -F(t)x(\eta). \qquad (27)$$

The total umbra-potential may be obtained by summing-up all the potentials represented by Eqs. (17), (22) and (26) and expressed as

$$V^{*}\left(t,\boldsymbol{q}(t);\dot{\boldsymbol{q}}(t),\boldsymbol{q}(\eta)\right) = V_{c}^{*}\left(t,\boldsymbol{q}(\eta)\right)+ V_{R}^{*}\left(t,\boldsymbol{q}(t),\dot{\boldsymbol{q}}(t)\right)+V_{P}^{*}\left(t,\boldsymbol{q}(\eta)\right), \quad (28)$$

and the umbra-Lagrangian would, therefore, be

$$L^{*}\left(t,\boldsymbol{q}(t),\dot{\boldsymbol{q}}(t),\boldsymbol{q}(\eta),\dot{\boldsymbol{q}}(\eta)\right) = T_{c}^{*}\left(t,\dot{\boldsymbol{q}}(\eta)\right)-V^{*}\left(t,\boldsymbol{q}(t),\dot{\boldsymbol{q}}(t),\boldsymbol{q}(\eta)\right). \quad (29)$$

New Lagrange's equations for a general class of systems may be given as

$$\frac{d}{dt}\left\{\underset{\eta\to t}{Lim}\frac{\partial L^{*}}{\partial\dot{q}_{i}(\eta)}\right\}-\underset{\eta\to t}{Lim}\frac{\partial L^{*}}{\partial q_{i}(\eta)}=0 \text{ ,for } i=1...n. \quad (30)$$

Noether's theorem (Noether, 1918) states that, if the Lagrangian of a system is invariant under a family of single parameter groups, then each such group renders a constant of motion. The extended Noether's theorem, as discussed in paper (Mukherjee et al., 2009) may lead to a constant of motion, or trajectories, on which some dynamical quantity remains conserved.

The umbra-Lagrangian may be defined on extended manifold, which consists of real displacements and velocities as well as umbra-displacements and velocities and real time (Mukherjee et al., 2009), i.e.

$$L^{*} = L^{*}\left(t,\boldsymbol{q}(t),\dot{\boldsymbol{q}}(t),\boldsymbol{q}(\eta),\dot{\boldsymbol{q}}(\eta)\right). \quad (31)$$

Here, the super dot ($^{\bullet}$) denotes a derivative with respect to real time or umbra time, depending on the context. Unlike the classical formulation, this analysis requires single but extended manifold comprising of both umbra and real displacements and velocities and real time. The umbra-Lagrangian of a system admits several one-parameter transformation groups, and then the infinitesimal generator (Hassani, 1999; Olver, 1986) corresponding to j^{th} parameter (or group) may be decomposed as follows:

$$\boldsymbol{V}^{j} = \boldsymbol{V}_{\eta}^{j} + \boldsymbol{V}_{t}^{j} \text{, where } j=1...m. \quad (32)$$

The general forms for $\boldsymbol{V}_{\eta}^{j}$ and $\boldsymbol{V}_{t}^{j}$ would be

$$\boldsymbol{V}_{\eta}^{j}=\sum_{i=1}^{n}\alpha^{j,i}\frac{\partial}{\partial q_{i}(\eta)}+\sum_{i=1}^{n}\beta^{j,i}\frac{\partial}{\partial\dot{q}_{i}(\eta)}, \quad (33)$$

$$\text{and}\quad \boldsymbol{V}_{t}^{j}=\sum_{i=1}^{n}\gamma^{j,i}\frac{\partial}{\partial q_{i}(t)}+\sum_{i=1}^{n}\xi^{j,i}\frac{\partial}{\partial\dot{q}_{i}(t)}, \quad (34)$$

where $\alpha^{j,i}$, $\beta^{j,i}=\dfrac{d\alpha^{j,i}}{d\eta}$, $\gamma^{j,i}$ and $\xi^{j,i}=\dfrac{d\gamma^{j,i}}{dt}$ are the general functions of umbra and real displacement and real time. The fact that the given umbra-Lagrangian is invariant under the j^{th} transformation may then be expressed as

$$\left\{\boldsymbol{V}^{j}\left(L^{*}\right)\right\}=0. \quad (35)$$

Using Eq. (30) along with Eqs. (33) and (34) in the previous equation (35), the extended Noether's theorem may be obtained and written as

$$\frac{d}{dt}\left\{\underset{\eta\to t}{Lim}\sum_{i=1}^{n}\frac{\partial L^{*}}{\partial\dot{q}_{i}(\eta)}\boldsymbol{V}_{\eta}^{j}\left(q_{i}(\eta)\right)\right\}=-\underset{\eta\to t}{Lim}\left\{\boldsymbol{V}_{t}^{j}\left(L^{*}\right)\right\}, \quad (36)$$

with $j=1...m$.

In terms of the differential one-forms dq_i and dL^{*}, the above relation may be expressed as

$$\frac{d}{dt}\left\{\underset{\eta\to t}{Lim}\left(\sum_{i=1}^{n}\frac{\partial L^{*}}{\partial\dot{q}_{i}(\eta)}dq_{i}(\eta)\left(\boldsymbol{V}_{\eta}^{j}\right)\right)\right\}=-\underset{\eta\to t}{Lim}\left\{dL^{*}\left(\boldsymbol{V}_{t}^{j}\right)\right\}, \quad (37)$$

The term on the left-hand side is the classical Noether term while the term on the right-hand side is additional and termed here as modulatory convection term. This modulatory convection term is made zero to obtain the conserved quantity. So, whenever the extended Lagrangian is found invariant, there is either the general conserved quantity or a trajectory on which such quantity remains conserved. The aforementioned methodology may be explained with two simple examples provided in the next subsections.

Example 1. Simple mass-spring-damper system with time fluctuating parameters

Let us consider a simple mass spring damper system as shown in Fig. 3. Using the aforementioned procedure, the umbra-Lagrangian for this system may be expressed as

$$L^{*}=\frac{1}{2}m(t)\dot{x}^{2}(\eta)-\frac{1}{2}K(t)x^{2}(\eta)-R(t)\dot{x}(t)x(\eta)+F(t)x(\eta).$$

The equation of motion obtained from the above umbra-Lagrangian through Eq. (30) may be written as

$$\frac{d}{dt}\left(m(t)\dot{x}(t)\right)+K(t)x(t)+R(t)\dot{x}(t)=F(t).$$

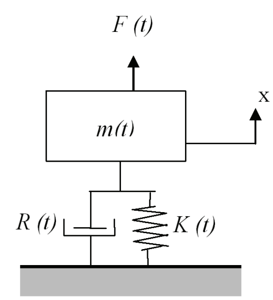

Figure 3. Mass-spring-damper system with time fluctuating parameters.

Example 2. Two oscillators with gyroscopic coupling

Let us consider an example of a system with two similar oscillators having mass m and stiffness K, and with a gyroscopic coupling of strength γ as shown in Fig. 4(a). The umbra-Lagrangian of the system may be written as

$$L^* = \frac{1}{2}m\dot{x}^2(\eta) + \frac{1}{2}m\,\dot{y}^2(\eta) - \frac{1}{2}Kx^2(\eta) - \frac{1}{2}Ky^2(\eta) - \gamma[\dot{y}(t)x(\eta) - \dot{x}(t)y(\eta)].$$

If the umbra-Lagrangian admits the one-parameter rotational group, then the infinitesimal generators of the rotational SO (2) group may be written as

$$\boldsymbol{V} = y(\eta)\frac{\partial}{\partial x(\eta)} - x(\eta)\frac{\partial}{\partial y(\eta)} + y(t)\frac{\partial}{\partial x(t)} - x(t)\frac{\partial}{\partial y(t)} + \dot{y}(\eta)\frac{\partial}{\partial \dot{x}(\eta)} - \dot{x}(\eta)\frac{\partial}{\partial \dot{y}(\eta)} + \dot{y}(t)\frac{\partial}{\partial \dot{x}(t)} - \dot{x}(t)\frac{\partial}{\partial \dot{y}(t)},$$

and the symmetry (invariance) condition for umbra-Lagrangian may be expressed as

$$\boldsymbol{V}(L^*) = \frac{1}{2}m\{\dot{y}(\eta).2\dot{x}(\eta) - \dot{x}(\eta).2\dot{y}(\eta)\} - \frac{1}{2}K\{y(\eta).2x(\eta) - x(\eta).2y(\eta)\} + \gamma\{\dot{y}(t)y(\eta) + \dot{x}(t)x(\eta) - \dot{y}(t)y(\eta) - \dot{x}(t)x(\eta)\}.$$

$$\Rightarrow \boldsymbol{V}(L^*) = 0.$$

Through Eq. (36), one obtains

$$\frac{d}{dt}\left[\lim_{\eta\to t}\left\{\frac{\partial L^*}{\partial \dot{x}(\eta)}(+y(\eta)) + \frac{\partial L^*}{\partial \dot{y}(\eta)}(-x(\eta))\right\}\right] + \lim_{\eta\to t}\left\{\begin{array}{l}\frac{\partial L^*}{\partial x(t)}(+y(t)) + \frac{\partial L^*}{\partial y(t)}(-x(t)) \\ + \frac{\partial L^*}{\partial \dot{x}(t)}(+\dot{y}(t)) + \frac{\partial L^*}{\partial \dot{y}(t)}(-\dot{x}(t))\end{array}\right\} = 0,$$

or $\frac{d}{dt}[m(\dot{x}(t)y(t) - \dot{y}(t)x(t))] + \gamma(y(t)\dot{y}(t) + x(t)\dot{x}(t)) = 0$,

or $\frac{d}{dt}\left[m(\dot{x}(t)y(t) - \dot{y}(t)x(t)) + \frac{\gamma}{2}(x^2(t) + y^2(t))\right] = 0$,

$$\Rightarrow m(\dot{x}(t)y(t) - \dot{y}(t)x(t)) + \frac{\gamma}{2}(x^2(t) + y^2(t)) = C,$$

where C is a constant of integration. The first term is the moment of momentum, and the second term is contributed by the gyroscopic coupling.

The umbra-Hamiltonian (discussed in Appendix A) of the system may be expressed as

$$H^* = \underbrace{\frac{1}{2m}(p_x^2(\eta) + p_y^2(\eta)) + \frac{1}{2}K(x^2(\eta) + y^2(\eta))}_{H_i^*} \underbrace{+ \gamma(\dot{x}(t)y(\eta) - \dot{y}(t)x(\eta))}_{H_e^*}$$

Applying the second theorem of umbra-Hamiltonian and finding $\lim_{\eta\to t}\frac{dH_e^*}{d\eta}$ of the system, one obtains

$$\lim_{\eta\to t}\frac{dH_e^*}{d\eta} = 0.$$

Again, the corollary of the second theorem of umbra-Hamiltonian gives

$$\frac{dH_i^*}{dt} = 0,$$

and on substituting the expression of H_i^* yields

$$\frac{d}{dt}\left(\frac{1}{2m}(p_x^2(t) + p_y^2(t)) + \frac{1}{2}K(x^2(t) + y^2(t))\right) = 0,$$

Hence, $H_i^* = \frac{1}{2m}(p_x^2(t) + p_y^2(t)) + \frac{1}{2}K(x^2(t) + y^2(t))$ is a constant of motion.

Both Examples illustrated in this paper provide an overview of the whole concept. It is apparent throughout the paper that the proposed extension of Lagrangian-Hamiltonian mechanics in terms of umbra philosophy gives a new dimension for analyzing the dynamical systems with non-conservative and non-potential forces.

Conclusions

The paper presented a brief review on the literature in Lagrangian-Hamiltonian Mechanics. Most of the research papers and books available in this field are incorporated, which undoubtedly enhanced and enriched the field of Mechanics. In this paper, authors have presented a brief review on extension of Lagrangian-Hamiltonian Mechanics. Various previous extensions on the subject matter were discussed with a particular regard to extension of Noether's theorem with nonconservative and non-holonomic systems for general class of systems. After review on literature on this subject matter, the following points are concluded:

(i) The procedure and methodology developed by other researchers don't have any generalization of Noether's theorem, as it has been mainly applied on the particular structure of the problems, which were rather mathematical without much physical interpretations of the real system. In this way, there is a substantial loss of generalization of the theorem, which may be applied to any engineering problems. However, in recent years, few researchers have applied the generalized Noether's theorem in few engineering applications.

(ii) In contrast to all the previous extensions, the philosophy developed by the authors has addressed the issue of nonconservative and dissipative forces by assuming a new Lagrangian, which find wider applications for engineering problems. The authors have devised a new methodology to find invariants of motions of the dynamical systems. Gauge transformations [48], bi-symmetric rotor-

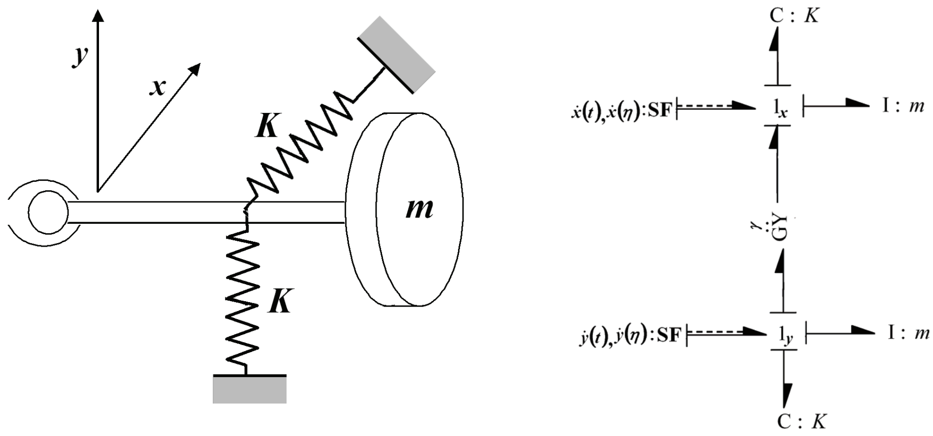

Fig. 4(a) Fig. 4(b)

Figure 4. (a) Schematic diagram of two oscillators with gyroscopic coupling; (b) Bond graph model for system represented by (a) with artificial flow sources to obtain umbra-Lagrangian.

motor system [49], dynamics of the rotor with internal damping [50] and few others are the applications already published in archival literature.

(iii) It is noteworthy to say that the alternative methods developed by the authors give more transparent physical interpretations, which enable the analyst to make further use of these first integrals in stability analysis.

(iv) In this article, the authors intended to provide critical evaluations of other extensions, which are rarely applied in the real-world problems.

Appendix A

Concept of umbra-Lagrangian and umbra-Hamiltonian

Mukherjee (1994) introduced a concise and modified form of Lagrange's equation and manifested the use of this new scheme to arrive at system models in the presence of time fluctuating parameters, general dissipation and gyroscopic couplings, etc. In this scheme, real and virtual energies (or work) are separated by introduction of an additional time like parameter, which is termed as 'umbra-time'. The prefix 'umbra' was appended to all type of energies, and corresponding Lagrangian was termed as the "umbra Lagrangian". The basic idea presented in reference (Mukherjee, 1994; 2001) leading to umbra-Lagrangian and umbra-Lagrange's equation may be briefly expressed as follows:

(a) Umbra-time is the beholder of D'Alembert's basic idea of allowing displacements, when the real time is frozen.
(b) Umbra-time may be viewed as the interior time of a system.
(c) Potential, kinetic and co-kinetic energies stored in storage elements like symmetric compliant and inertial fields can be expressed as functions in umbra-time (umbra-displacements and umbra-velocities).
(d) The effort of any external force, resistive element or field, gyroscopic element (treated as anti-symmetric resistive field), transformer or lever element, anti-symmetric compliant field and sensing element depends on displacements and velocities in real time. The potentials associated with them are obtained by evaluation of work-done through umbra-displacements.

The broad principle on which the creation of umbra-Lagrangian and other relevant energies (Mukherjee, 1994; Mukherjee, 2001; Rastogi, 2005; Mukherjee et al., 2006; Mukherjee et al., 2007; Mukherjee et al., 2009) are summarized in the section IV. However, the umbra-Hamiltonian (Mukherjee, 2001) may be represented as

$$\begin{aligned} &H^*\left[\boldsymbol{q}(\eta),\boldsymbol{p}(\eta),\boldsymbol{q}(t),\dot{\boldsymbol{q}}(t),t\right] \\ &=\dot{\boldsymbol{q}}(\eta)\,\boldsymbol{p}(\eta)-L^*\left[\boldsymbol{q}(\eta),\dot{\boldsymbol{q}}(\eta),\boldsymbol{q}(t),\dot{\boldsymbol{q}}(t),t\right], \end{aligned} \tag{A1}$$

The umbra-Hamiltonian H^* is composed of two components as H_i^* and H_e^*. H_i^* is the interior Hamiltonian, which does not depend on any function of real displacement, real velocity and real time, and H_e^* is the rest of the umbra-Hamiltonian, called the exterior Hamiltonian. Thus

$$H^*=H_i^*\{\boldsymbol{q}(\eta),\boldsymbol{p}(\eta)\}+H_e^*\{\boldsymbol{q}(\eta),\boldsymbol{p}(\eta),\boldsymbol{q}(t),\dot{\boldsymbol{q}}(t),t\}. \tag{A2}$$

The two theorems of the umbra-Hamiltonian may be given as

$$\lim_{\eta\to t}\left[\frac{dH^*}{d\eta}\right]=0\,. \qquad \text{(Theorem 1)}$$

$$\frac{dH_i^*}{dt}=-\lim_{\eta\to t}\left[\frac{dH_e^*}{d\eta}\right]. \qquad \text{(Theorem 2)}$$

Corollary of Theorem 2

If for a system $\lim_{\eta\to t}\left[\frac{dH_e^*}{d\eta}\right]=0$, then $H_i^*\left(q(t),p(t)\right)$ is a constant of motion.

Appendix B

Generation of umbra-Lagrangian through Bond graphs

Karnopp [51] proposed an algorithm to arrive at Lagrange's equations for complex systems through its bondgraph model. The steps of Karnopp algorithm may be briefed as

(1) Apply the required causality at all effort and flow sources and use the junction structure elements (only) to extend the causality as far as possible within the bondgraph. If causal conflicts arise at this stage, there is a fundamental contradiction within the model and it must be reformulated.
(2) Choose a ‘1’ junction for which the flow is not yet causally determined or insert a ‘1’ junction into any causally undetermined bond and attach an artificial flow source to ‘1’ junction.
(3) Apply the required causality to the artificial source and extend the causality as far as possible into the bondgraph using junction structure element.
(4) Return to step (2) and continue until all bonds have been causally oriented.

Now at this stage, an extension of Karnopp’s algorithm is presented with a detailed procedure and may appear more elaborate for generation of umbra-Lagrangian of the system. The models may be classified as follows:

(a) System with no modulated two-port transformers. Such bondgraph models may be called holonomic.
(b) System with modulated two-port transformers. Such bondgraph may be called as non-holonomic.

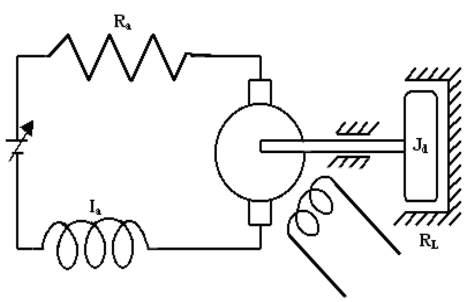

Figure 5(a). System with a DC motor and a rotating disc in viscous medium.

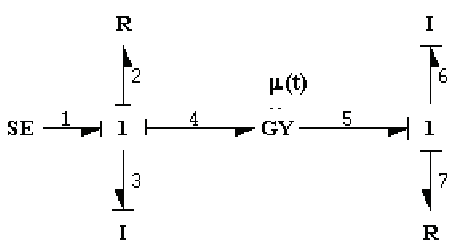

Figure 5(b). Bond graph model of system 5(a).

To explain the procedure, one may take an example as shown in Fig. 5(a) with its bondgraph model in Fig. 5(b). Now, the additional steps for generation of umbra-Lagrangian may be given as:

(i) Create two copies of each junction excluding two-port elements side by side; associate one with η-variable (the umbra-time), and other with *t*-variable (the real time). The space between these two may be designated as trans-temporal space.

(ii) Insert the artificial sources to their corresponding junctions. Those inserted in the η-component should be designated as function of η and their copies inserted in the *t*-component are designated as function of *t* (see Fig. 6(a)).

(iii) Insert the original flow sources at their respective junctions on the η and t component designating them as function of *t*; insert the effort sources in η component only.

(iv) Insert all *I*- and *C*-elements and fields at their respective junctions on η-component.

(v) *R*-elements and fields (including gyrators) observe the motion in real time *t* and apply the force on the system, the corresponding umbra-potentials associated with them is generated through work done by these forces undergoing umbra-displacements. These features may be incorporated by inserting them in trans-temporal space and adding suitable trans-temporal bonds; 0-junction and suitable activations as shown in Fig.6 (b). Such bondgraph may be termed as umbra-Lagrangian generator bond graphs.

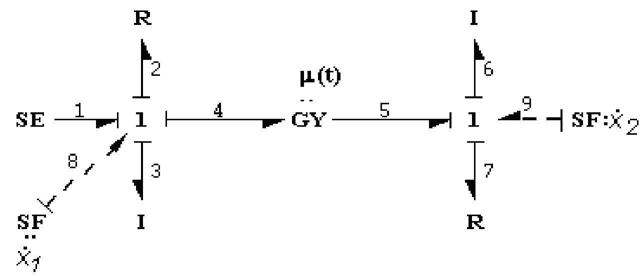

Figure 6(a). Causal bond graph model of system 5(a) with artificial flow sources.

The umbra-Lagrangian for Fig. 6(b) may be expressed as

$$\begin{aligned} L^* &= \tfrac{1}{2} m_3 \dot{x}_1^2(\eta) + \tfrac{1}{2} m_6 \dot{x}_2^2(\eta) \\ &\quad - R_2 \dot{x}_1(t) x_1(\eta) - R_7 \dot{x}_2(t) x_2(\eta) \\ &\quad - \mu(t)\{\dot{x}_2(t) x_1(\eta) - \dot{x}_1(t) x_2(\eta)\}. \end{aligned} \qquad \text{(B1)}$$

Now, it is easy to verify that umbra-Lagrangian of Eq. (B1) renders right equation of motion through Eq. (30) for the system.

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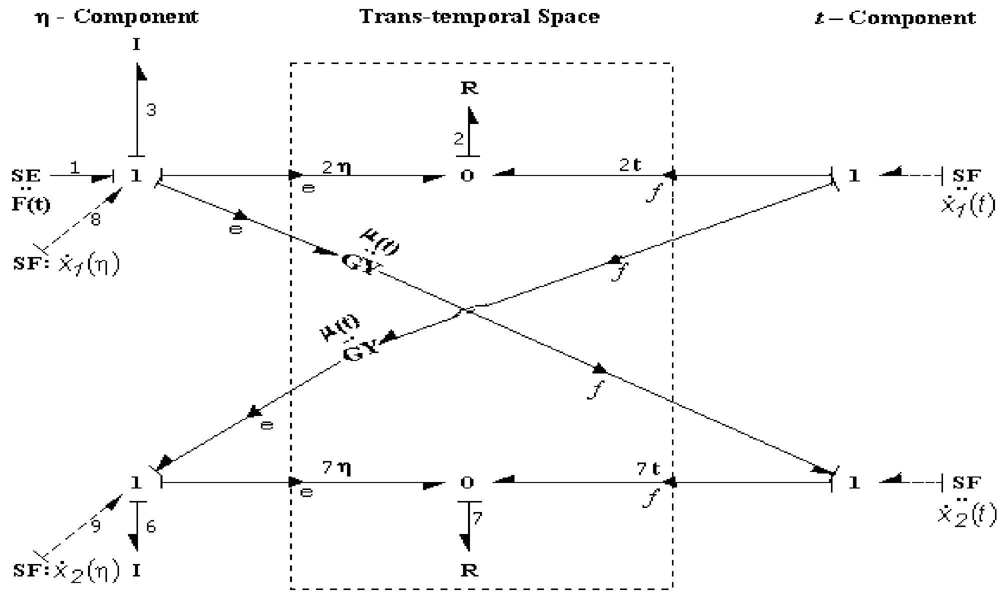

Figure 6(b). Umbra-Lagrangian generator bondgraph of system 5(a).

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