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Flexural-Torsional Vibration Analysis of Axially Loaded Thin-Walled Beam

The present paper considers the flexure-torsion coupled vibrations of axially loaded thin-walled beams with arbitrary open cross section, by means of an exact solution. The effects of axial force, warping stiffness and rotary inertia are included in the present formulations. In the case of simply supported thin-walled beam, a closed-form solution for the coupled natural frequencies of free harmonic vibrations was derived by using a general solution of the governing differential equations of motion based on Vlasov theory. The method is illustrated by its application to two test examples, to demonstrate the effects of bending-torsion coupling and axial force on the dynamic behavior of thin-walled beams. Compared with those available in the relevant literature, numerical results demonstrate the accuracy and effectiveness of the proposed method.
Keywords: *thin-walled beams, flexural-torsional vibration, axially loaded, free vibration*

Introduction

Thin-walled beam members are widely used as basic structural elements within the fields of mechanical, civil, aeronautical engineering etc., offering a high performance in terms of minimum weight for a given strength. Because of their practical significance in engineering applications, it is essential for design engineers to evaluate the dynamic characteristics of the thin-walled beam structures accurately and so ensure that their design is reliable and safe. The effects due to axial force on the dynamic response of the thin-walled beams are especially of interest. Helicopter, turbine or propeller blades, plane and space frames and also girders of cable-stayed bridges, all could be qualified as axially loaded structures.

Due to their practical importance mentioned above, the vibration analysis of thin-walled beams have been studied by different authors and numerous approaches for calculating the free vibration frequencies have been proposed (Friberg, 1983; Dokumaci, 1987; Banerjee, 1989 and 1999; Kim et al., 1994; Tanaka and Bercin, 1999; Kollár, 2001; Arpaci and Bozdag, 2002 and 2003; Prokić, 2005 and 2006).

Relatively fewer works are available in literature toward the study of the coupled bending-torsion vibrations of axially loaded thin-walled beams. Banerjee and Williams (1992 and 1994) derived the analytical expressions for the coupled flexural-torsional dynamic stiffness matrix of an axially loaded Timoshenko beam element. But the warping stiffness was not included in their theory. The method is referred as an exact method since it is based on exact shape functions obtained from the exact solution of differential equations. As Moon-Young et al. (2003) pointed out, this analytical method, however, is sometimes inefficient because analytical operations in solving a system of simultaneous ordinary differential equations with many variables may be too complex. Hashemi and Richard (2000) presented a new dynamic finite element for the coupled bending-torsional vibration of axially loaded beams based on the closed-form solutions of the Bernoulli-Euler and St. Venant beam theories. But the warping of the cross-sections, shear deformation and rotary inertia were not included in the formulation. Taking the warping effect into account, Li et al. (2004a) carried out the free vibration analysis of an axially loaded beam with nonsymmetrical open cross section by means of dynamic transfer matrix method. Subsequently, Li et al. (2004b) extended this work to include the effects of rotary inertia and shear deflection, but limited the flexural motion to a single plane. The dynamic transfer matrix method implies mathematical procedures which are sometimes difficult to deal with determining the frequencies values of the complex transcendental characteristic equation.

The investigation in the present paper is partly motivated by the fact that natural frequencies of axially loaded thin-walled beams are often required in the design of many structures. Also, some studies have shown that the effect of axial force on natural frequencies is more pronounced than those of the shear deformation and/or rotary inertia. The proposed method is simple, rapid and accurate enough to be used in preliminary design stage and also for verifying numerical results of complex and time-consuming computer procedures. It is expected that undertaken investigation will be useful for better understanding of dynamic characteristics of thin-walled elements.

Nomenclature

a,b,c,d	= *coefficients in the frequency equation*
C	= *origin of coordinate system*
D	= *shear center*
e, s	= *curvilinear coordinates*
F	= *area of cross section*
G	= *shear modulus*
h_P	= *distance from tangent at arbitrary point on contour to pole*
$I_{xx}, I_{yy}, I_{\omega\omega}, I_D$	= *geometrical properties of cross section*
K	= *Saint Venant torsion constant*
L	= *length of element*
M_x, M_y	= *bending moments*
M_ω	= *bimoment*
m_x, m_y, m_D, m_ω	= *external bending moments, torsional moment and bimoment per unit length of the beam*
N	= *axial force*
O	= *starting point* (*point from which s is measured*)
P	= *external force*
p	= *frequency*
p_x, p_y, p_z	= *externally loads per unit length of the beam*
$\overline{p}_x, \overline{p}_y, \overline{p}_z$	= *externally applied loads per unit area of midplane of the beam*
T_D	= *total torsional moment*
T_s	= *Saint Venant torsional moment*
t	= *thickness of wall*
U, V, Φ	= *amplitudes of the transverse displacements and torsional rotation*
$\overline{U}$	= *work of actual stresses*
u	= *vector of displacements*
u, v, w	= *displacements of shear center*
u_*, v_*, w_*	= *displacements of an arbitrary point of cross-section*
V_x, V_y	= *shear forces*
$\overline{W}$	= *work of external load and inertia forces*

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w = *axial displacement of cross section considered as rigid*
x, y, z = *descartes coordinates*

Greek Symbols

γ_s = *shear strain*
δ = *symbol of variation*
$\boldsymbol{\varepsilon}$ = *strain tensor*
ε_z = *longitudinal strain*
λ_n = $n\pi/L$
ρ = *density*
$\boldsymbol{\sigma}$ = *stress vector*
σ_z = *normal stress*
τ = *time*
τ_w = *shear stress uniformly distributed over thickness*
τ_{zs} = *total shear stress*
τ_s = *Saint Venant shear stress*
φ = *rotation of the cross section around its shear centre*
ω = *warping function*

Subscripts

D = *relative to shear centre*

Basic Equations

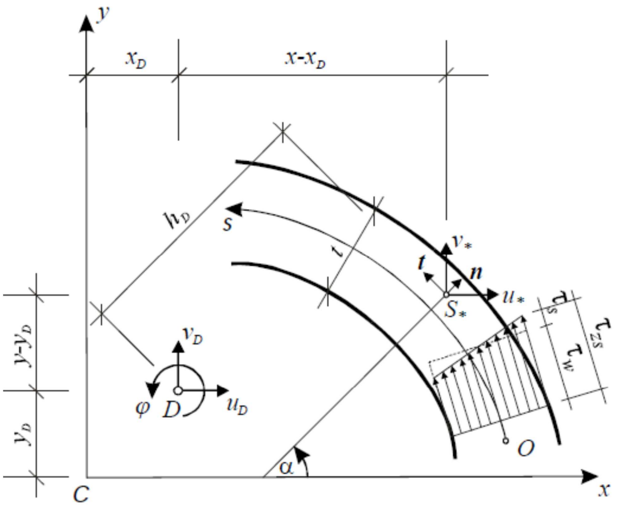

Figure 1. Section geometry.

A straight uniform thin-walled beam of length L with nonsymmetrical open cross-section is shown in Fig. 1. The beam consists of a linear elastic material with mass density ρ. The beam is referenced to a right-handed rectangular coordinates system x, y, z, where the z-axis is the initial elastic axis of the beam while x and y are the principal axes of the cross-section. The origin of these axes is located at the centroid C. The shear centre with coordinates x_D and y_D in Cxy is denoted by D. Furthermore, it is assumed that the beam is loaded by a given transverse forces per unit length p_x, p_y and p_z distributed along the centroidal axis, externally applied moment per unit length m_x, m_y and m_D and external distributed bimoment of intensity m_ω. A constant axial force P is assumed to act through the centroid of the cross-section of the thin-walled beam.

Based on the usual assumptions of Vlasov theory

- the cross-section is perfectly rigid in its own plane,
- the shear strains in the middle surface of the wall are negligible.

The displacements u_*, v_* and w_* of an arbitrary point S_* of the cross section can be described by only four components, three translations u, v and w of pole D and the cross section rotation φ about the same pole:

$$\begin{aligned} u_* &= u - \varphi\left(y - y_D\right) \\ v_* &= v + \varphi\left(x - x_D\right) \\ w_* &= w - u'x - v'y - \varphi'\omega \end{aligned} \tag{1}$$

where ω is warping function with respect to pole D.

Component deformations different from zero are given by

$$\begin{aligned} \varepsilon_z &= w' - u''x - v''y - \varphi''\omega \\ \gamma_s &= 2\varphi' e \end{aligned} \tag{2}$$

where e is the distance of the observed point from the middle surface measured along the normal $\mathbf{n}$.

Reducing the normal stresses at the center of gravity and shear stresses at the pole D, for stress resultants the following expressions are obtained

$$\begin{aligned} N &= \iint_F \sigma_z dF \\ M_x &= \iint_F \sigma_z y dF \\ M_y &= -\iint_F \sigma_z x dF \\ V_x &= -\iint_F \tau_{zs} \sin\alpha\, dF \\ V_y &= \iint_F \tau_{zs} \cos\alpha\, dF \\ T_D &= \iint_F \tau_{zs} h_D dF \\ T_s &= 2\iint_F \tau_s e dF \\ M_\omega &= \iint_F \sigma_z \omega dF \end{aligned} \tag{3}$$

In Eqs. (3), N represents the axial force, M_x and M_y the bending moments with respect to the x and y axis, V_x and V_y the shear forces in the x and y direction, T_D the torsion moment, T_s the Saint Venant torque, M_ω the bimoment and F the area of the cross section.

The equations of motion of thin-walled beam can be obtained using the principle of virtual displacements. All vector and matrix quantities are defined with respect to the right-handed rectangular coordinate system (x, y, z). The z-axis is parallel with the longitudinal centroidal axis of the beam, while x and y are arbitrarily taken.

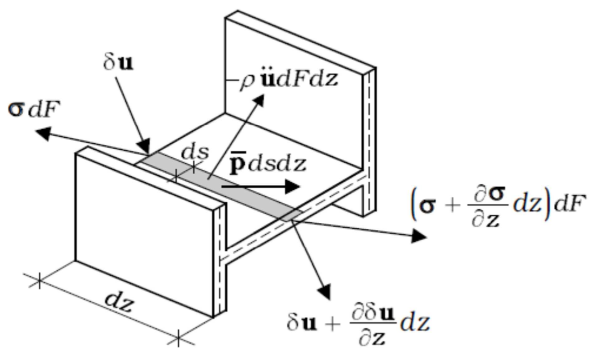

Figure 2. Differential element of beam.

A small element between cross sections $z_1 = z$ and $z_2 = z + dz$ (Fig. 2) subjected to external loads $\overline{\mathbf{p}}\left(\overline{p}_x,\, \overline{p}_y,\, \overline{p}_z\right)$ per unit area of midplane is considered.

At any point on the cross section z_1 acts as a stress vector

$$\boldsymbol{\sigma} = \tau_{zs}\mathbf{t} + \sigma_z\mathbf{i}_z = -\tau_{zs}\sin\alpha\,\mathbf{i}_x + \tau_{zs}\cos\alpha\,\mathbf{i}_y + \sigma_z\mathbf{i}_z \tag{4}$$

The vector of virtual displacements $\delta\mathbf{u}$, which satisfies the necessary continuity and displacement boundary conditions, may be adopted in the same form as a vector of real displacements

$$\begin{aligned}\delta\mathbf{u} &= \delta u_*\mathbf{i}_x + \delta v_*\mathbf{i}_y + \delta w_*\mathbf{i}_z = \\ &= \left[\delta u - \delta\varphi\left(y - y_D\right)\right]\mathbf{i}_x + \left[\delta v + \delta\varphi\left(x - x_D\right)\right]\mathbf{i}_y \\ &\quad + \left(\delta w - \delta u' x - \delta v' y - \delta\varphi'\omega\right)\mathbf{i}_z\end{aligned} \tag{5}$$

Virtual displacement parameters, which for distinction from real displacements are marked with prefix δ, are arbitrary functions of coordinates and do not depend upon external loads.

The virtual work expression is

$$\delta\overline{W} + \delta\overline{U} = 0 \tag{6}$$

where $\delta\overline{W}$ = virtual work of external load and inertia forces through virtual displacements $\delta\mathbf{u}$ and $\delta\overline{U}$ = virtual work of actual stresses $\boldsymbol{\sigma}$ realized through virtual strains $\delta\boldsymbol{\varepsilon} = \left[\delta\varepsilon_z \quad \delta\gamma_T\right]$.

The virtual work of the external load and inertia forces, including the second order effects of the constant axial stress σ_z^o, per unit length of the element is

$$\begin{aligned}\delta\overline{W} &= \iint_F \left(\boldsymbol{\sigma}_{,z}\delta\mathbf{u} + \boldsymbol{\sigma}\delta\mathbf{u}_{,z}\right)dF + \iint_F \sigma_z^o\left(u_*''\delta u_* + v_*''\delta v_*\right)dF + \\ &\quad + \int_s \overline{\mathbf{p}}\,\delta\mathbf{u}\,ds - \rho\iint_F \ddot{\mathbf{u}}\,\delta\mathbf{u}\,dF\end{aligned} \tag{7}$$

where ρ is the density (mass per unit volume), and $\ddot{\mathbf{u}}$ is the acceleration vector given by

$$\begin{aligned}&\ddot{\mathbf{u}} = \ddot{u}_*\mathbf{i}_x + \ddot{v}_*\mathbf{i}_y + \ddot{w}_*\mathbf{i}_z = \\ &\left[\ddot{u} - \ddot{\varphi}\left(y - y_D\right)\right]\mathbf{i}_x + \left[\ddot{v} + \ddot{\varphi}\left(x - x_D\right)\right]\mathbf{i}_y + \left(\ddot{w} - \ddot{u}'x - \ddot{v}'y - \ddot{\varphi}'\omega\right)\mathbf{i}_z\end{aligned} \tag{8}$$

A dot denotes differentiation with respect to time.

Substituting (4), (5) and (8) into (7), the following expression for δW is obtained

$$\begin{aligned}\delta\overline{W} = \iint_F \Big\{ &-\tau_{zs}'\sin\alpha\left[\delta u - \delta\varphi\left(y - y_D\right)\right] + \tau_{zs}'\cos\alpha\left[\delta v + \delta\varphi\left(x - x_D\right)\right] + \\ &+ \sigma_z'\left(\delta w - \delta v' y - \delta u' x - \delta\varphi'\omega\right) - \\ &- \tau_{zs}\sin\alpha\left[\delta u' - \delta\varphi'\left(y - y_D\right)\right] + \tau_{zs}\cos\alpha\left[\delta v' + \delta\varphi'\left(x - x_D\right)\right] + \\ &+ \sigma_z\left(\delta w' - \delta v'' y - \delta u'' x - \delta\varphi''\omega\right)\Big\}dF + \\ + \iint_F &\sigma_z^o\left(u_*''\delta u_* + v_*''\delta v_*\right)dF + \\ + \int_s &\Big\{\overline{p}_x\left[\delta u - \delta\varphi\left(y - y_D\right)\right] + \overline{p}_y\left[\delta v + \delta\varphi\left(x - x_D\right)\right] + \\ &+ \overline{p}_z\left(\delta w - \delta u' x - \delta v' y - \delta\varphi'\omega\right)\Big\}ds - \\ - \rho\iint_F &\left(\delta u_*\ddot{u}_* + \delta v_*\ddot{v}_* + \delta w_*\ddot{w}_*\right)dF\end{aligned} \tag{9}$$

The virtual work of the internal load due to the corresponding variation of deformation, per unit length of the element, is

$$\delta\overline{U} = -\iint_F\left(\sigma_z\delta\varepsilon_z + \tau_s\delta\gamma_s\right)dF. \tag{10}$$

Using expressions (2) for virtual strains, where real displacement should be replaced by virtual displacement, one gets for $\delta\overline{U}$:

$$\delta\overline{U} = -\left\{\iint_F\left[\sigma_z\left(\delta w' - \delta u'' x - \delta v'' y - \delta\varphi''\omega\right) + \tau_s 2\delta\varphi' e\right]dF\right\} \tag{11}$$

By suitable rearrangement of (9) and (11) in accordance with virtual displacement parameters, the principle of virtual work may be expressed as

$$\begin{aligned}&\delta w\left\{\iint_F \sigma_z' dF - \rho\iint_F \ddot{w}_* dF + \int_s \overline{p}_z ds\right\} + \\ &+ \delta u\left\{-\iint_F \tau_{zs}'\sin\alpha\, dF + \iint_F \sigma_z^o u_*'' dF - \rho\iint_F \ddot{u}_* dF + \int_s \overline{p}_x ds\right\} + \\ &+ \delta v\left\{\iint_F \tau_{zs}'\cos\alpha\, dF + \iint_F \sigma_z^o v_*'' dF - \rho\iint_F \ddot{v}_* dF + \int_s \overline{p}_y ds\right\} + \\ &+ \delta\varphi\left\{\iint_F \tau_{zs}' h_D dF - \iint_F \sigma_z^o\left[\left(y - y_D\right)u_*'' - \left(x - x_D\right)v_*''\right]dF + \right. \\ &\quad \left. + \rho\iint_F\left[\left(y - y_D\right)\ddot{u}_* - \left(x - x_D\right)\ddot{v}_*\right]dF + \int_s\left[\overline{p}_y\left(x - x_D\right) - \overline{p}_x\left(y - y_D\right)\right]ds\right\} - \\ &- \delta u'\left\{\iint_F\left(\sigma_z' x + \tau_{zs}\sin\alpha\right)dF - \rho\iint_F x\ddot{w}_* dF + \int_s \overline{p}_z x\, ds\right\} - \\ &- \delta v'\left\{\iint_F\left(\sigma_z' y - \tau_{zs}\cos\alpha\right)dF - \rho\iint_F y\ddot{w}_* dF + \int_s \overline{p}_z y\, ds\right\} - \\ &- \delta\varphi'\left\{\iint_F\left(\sigma_z'\omega - \tau_{zs}h_D + 2\tau_s e\right)dF - \rho\iint_F \omega\ddot{w}_* dF + \int_s \overline{p}_z\omega ds\right\} = 0\end{aligned} \tag{12}$$

To satisfy these equations identically for any virtual displacement parameter $\delta w_o, \delta u_P,\ \delta v_P, ,...$ it is necessary the expressions in the great brackets to vanish. Now, using the expressions for stress resultants (3), one obtains

$$\begin{aligned}&N' - \rho\iint_F \ddot{w}_* dF + p_z = 0 \\ &V_x' + \iint_F \sigma_z^o u_*'' dF - \rho\iint_F \ddot{u}_* dF + p_x = 0 \\ &V_y' + \iint_F \sigma_z^o v_*'' dF - \rho\iint_F \ddot{v}_* dF + p_y = 0 \\ &T_D' - \iint_F \sigma_z^o\left[\left(y - y_D\right)u_*'' - \left(x - x_D\right)v_*''\right]dF + \\ &\quad + \rho\iint_F\left[\left(y - y_D\right)\ddot{u}_* - \left(x - x_D\right)\ddot{v}_*\right]dF + m_D = 0 \\ &M_y' + V_x + \rho\iint_F x\ddot{w}_* dF + m_y = 0 \\ &M_x' - V_y - \rho\iint_F y\ddot{w}_* dF + m_x = 0 \\ &M_\omega' - T_D + T_s - \rho\iint_F \omega\ddot{w}_* dF + m_\omega = 0\end{aligned} \tag{13}$$

The forces V_x, V_y and T_D can be eliminated from (13) in order to obtain four equations:

$$
\begin{aligned}
& N' - \rho \iint_F \ddot{w}_* \, dF + p_z = 0 , \\
& M_y'' - \iint_F \sigma_z^o u_*'' dF + \rho \iint_F x \ddot{w}_*' \, dF + \rho \iint_F \ddot{u}_* \, dF - p_x + m_y' = 0 , \\
& M_x'' + \iint_F \sigma_z^o v_*'' dF - \rho \iint_F y \ddot{w}_*' \, dF - \rho \iint_F \ddot{v}_* \, dF + p_y + m_x' = 0 , \\
& M_\omega'' + T_s' - \iint_F \sigma_z^o \left[(y - y_D) u_*'' - (x - x_D) v_*'' \right] dF - \\
& - \rho \iint_F \omega \ddot{w}_*' \, dF + \rho \iint_F \left[(y - y_D) \ddot{u}_* - (x - x_p) \ddot{v}_* \right] dF + m_D + m_\omega' = 0
\end{aligned}
\tag{14}
$$

The stress resultants can be expressed directly in terms of the displacements (Prokić, 2005 and 2006). The equations are written in matrix form:

$$
\begin{bmatrix} N \\ M_y \\ -M_x \\ -M_\omega \\ T_s \end{bmatrix} = E \begin{bmatrix} F & 0 & 0 & 0 & 0 \\ 0 & I_{xx} & 0 & 0 & 0 \\ 0 & 0 & I_{yy} & 0 & 0 \\ 0 & 0 & 0 & I_{\omega\omega} & 0 \\ 0 & 0 & 0 & 0 & \dfrac{GK}{E} \end{bmatrix} \begin{bmatrix} w' \\ u'' \\ v'' \\ \varphi'' \\ \varphi' \end{bmatrix}
\tag{15}
$$

The Equations of motion can be obtained by substituting the stress resultants from (15) into (14)

$$
\begin{aligned}
& E \left[\begin{array}{c|c|c|c} F & 0 & 0 & 0 \\ \hline 0 & I_{xx} & 0 & 0 \\ \hline 0 & 0 & I_{yy} & 0 \\ \hline 0 & 0 & 0 & I_{\omega\omega} \end{array} \right] \begin{bmatrix} w''' \\ u'''' \\ v'''' \\ \varphi'''' \end{bmatrix} - \\
& - \left(GK \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + P \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & y_D \\ 0 & 0 & 1 & -x_D \\ 0 & y_D & -x_D & I_D/F \end{bmatrix} \right) \begin{bmatrix} w' \\ u'' \\ v'' \\ \varphi'' \end{bmatrix} - \\
& - \rho \left[\begin{array}{c|c|c|c} F & 0 & 0 & 0 \\ \hline 0 & I_{xx} & 0 & 0 \\ \hline 0 & 0 & I_{yy} & 0 \\ \hline 0 & 0 & 0 & I_{\omega\omega} \end{array} \right] \begin{bmatrix} \ddot{w}' \\ \ddot{u}'' \\ \ddot{v}'' \\ \ddot{\varphi}'' \end{bmatrix} + \rho F \left[\begin{array}{c|c|c|c} 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & y_D \\ \hline 0 & 0 & 1 & -x_D \\ \hline 0 & y_D & -x_D & \dfrac{I_D}{F} \end{array} \right] \begin{bmatrix} \ddot{w} \\ \ddot{u}_D \\ \ddot{v}_D \\ \ddot{\varphi} \end{bmatrix} \\
& = \begin{bmatrix} -p_z' \\ p_x - m_y' \\ p_y + m_x' \\ m_D + m_\omega' \end{bmatrix}
\end{aligned}
\tag{16}
$$

To achieve the compact form the order of Eq. (14-1) is artificially raised by one. The first equation in (16), describing axial vibration, is uncoupled from the rest of the system and may be analysed independently.

The free harmonic transverse and torsional vibrations are defined by the coupled homogeneous Eqs. (16-2,3,4). The solution may be expressed in the form

$$
\begin{bmatrix} u(z,t) \\ v(z,t) \\ \varphi(z,t) \end{bmatrix} = \begin{bmatrix} U(z) \\ V(z) \\ \Phi(z) \end{bmatrix} \sin p\tau
\tag{17}
$$

where p is the radian frequency and U, V and Φ are amplitudes of the transverse displacements and torsional rotation. Substituting (17) into homogeneous Eqs. (16) yields

$$
\begin{aligned}
& E \begin{bmatrix} I_{xx} & & \\ & I_{yy} & \\ & & I_{\omega\omega} \end{bmatrix} \begin{bmatrix} U'''' \\ V'''' \\ \Phi'''' \end{bmatrix} - \\
& - \left(GK \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + P \begin{bmatrix} 1 & 0 & y_D \\ 0 & 1 & -x_D \\ y_D & -x_D & I_D/F \end{bmatrix} \right) \begin{bmatrix} U'' \\ V'' \\ \Phi'' \end{bmatrix} + \\
& + \rho p^2 \begin{bmatrix} I_{xx} & & \\ & I_{yy} & \\ & & I_{\omega\omega} \end{bmatrix} \begin{bmatrix} U'' \\ V'' \\ \Phi'' \end{bmatrix} - \rho F p^2 \begin{bmatrix} 1 & 0 & y_D \\ 0 & 1 & -x_D \\ y_D & -x_D & \dfrac{I_D}{F} \end{bmatrix} \begin{bmatrix} U \\ V \\ \Phi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
\tag{18}
$$

In the case of a beam with simply supported ends (fork supports at each end which prevent rotation and can warp freely) the end conditions are

$$
\begin{bmatrix} U \\ V \\ \Phi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} U'' \\ V'' \\ \Phi'' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\tag{19}
$$

These requirements are satisfied by taking

$$
\begin{bmatrix} U(z) \\ V(z) \\ \Phi(z) \end{bmatrix} = \begin{bmatrix} C_U \\ C_V \\ C_\Phi \end{bmatrix} \sin \lambda_n z
\tag{20}
$$

where C_u, C_v and C_Φ are constants and $\lambda_n = n\pi/L \quad n = 1, 2, ...$

Substituting (20) into (18) results in

$$
\begin{aligned}
& \left(\lambda_n^4 E \begin{bmatrix} I_{xx} & & \\ & I_{yy} & \\ & & I_{\omega\omega} \end{bmatrix} + \lambda_n^2 GK \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \right. \\
& + \lambda_n^2 P \begin{bmatrix} 1 & 0 & y_D \\ 0 & 1 & -x_D \\ y_D & -x_D & I_D/F \end{bmatrix} - \lambda_n^2 \rho p^2 \begin{bmatrix} I_{xx} & & \\ & I_{yy} & \\ & & I_{\omega\omega} \end{bmatrix} - \\
& \left. - \rho F p^2 \begin{bmatrix} 1 & 0 & y_D \\ 0 & 1 & -x_D \\ y_D & -x_D & I_D/F \end{bmatrix} \right) \begin{bmatrix} C_U \\ C_V \\ C_\Phi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}
\tag{21}
$$

Setting the determinant of the above system equal to zero:

$$\begin{vmatrix} \lambda_n^4 I_{xx}+\lambda_n^2\dfrac{P}{E}-\\ -\left(\lambda_n^2 I_{xx}+F\right)p_* & 0 & \lambda_n^2\dfrac{P}{E}y_D-y_D F p_* \\ 0 & \lambda_n^4 I_{yy}+\lambda_n^2\dfrac{P}{E}-\\ & -\left(\lambda_n^2 I_{yy}+F\right)p_* & x_D F p_*-\lambda_n^2\dfrac{P}{E}x_D \\ \lambda_n^2\dfrac{P}{E}y_D-y_D F p_* & x_D F p_*-\lambda_n^2\dfrac{P}{E}x_D & \lambda_n^4 I_{\omega\omega}+\lambda_n^2\left(\dfrac{GK}{E}+P\dfrac{I_D}{EF}\right)-\\ & & -\left(\lambda_n^2 I_{\omega\omega}+I_D\right)p_* \end{vmatrix}=0 \qquad (22)$$

where $p_* = \dfrac{\rho}{E}p^2$ yields the following algebraic frequency equation of the third order

$$ap_*^3+bp_*^2+cp_*+d=0 \qquad (23)$$

with the coefficients

$$\begin{aligned}
a=&-\left(\lambda_n^2 I_{xx}+F\right)\left(\lambda_n^2 I_{yy}+F\right)\left(\lambda_n^2 I_{\omega\omega}+I_D\right)+\\
&+F^2\left[x_D^2\left(\lambda_n^2 I_{xx}+F\right)+y_D^2\left(\lambda_n^2 I_{yy}+F\right)\right]\\
b=&\lambda_n^2\left(\lambda_n^2 I_{xx}+F\right)\left(\lambda_n^2 I_{yy}+F\right)\left(\lambda_n^2 I_{\omega\omega}+\frac{GK}{E}+P\frac{I_D}{EF}\right)+\\
&+\lambda_n^2\left[\left(\lambda_n^2 I_{xx}+\frac{P}{E}\right)\left(\lambda_n^2 I_{yy}+F\right)+\left(\lambda_n^2 I_{yy}+\frac{P}{E}\right)\left(\lambda_n^2 I_{xx}+F\right)\right]\left(\lambda_n^2 I_{\omega\omega}+I_D\right)-\\
&-2\lambda_n^2 F\frac{P}{E}\left[y_D^2\left(\lambda_n^2 I_{yy}+F\right)+x_D^2\left(\lambda_n^2 I_{xx}+F\right)\right]-\\
&-\lambda_n^2 F^2\left[x_D^2\left(\lambda_n^2 I_{xx}+\frac{P}{E}\right)+y_D^2\left(\lambda_n^2 I_{yy}+\frac{P}{E}\right)\right]\\
c=&-\lambda_n^4\left(\lambda_n^2 I_{xx}+\frac{P}{E}\right)\left(\lambda_n^2 I_{yy}+\frac{P}{E}\right)\left(\lambda_n^2 I_{\omega\omega}+I_D\right)-\\
&-\lambda_n^4\left[\left(\lambda_n^2 I_{xx}+\frac{P}{E}\right)\left(\lambda_n^2 I_{yy}+F\right)+\left(\lambda_n^2 I_{yy}+\frac{P}{E}\right)\left(\lambda_n^2 I_{xx}+F\right)\right]\times\\
&\times\left(\lambda_n^2 I_{\omega\omega}+\frac{GK}{E}+P\frac{I_D}{EF}\right)+\\
&+2\lambda_n^4 F\frac{P}{E}\left[y_D^2\left(\lambda_n^2 I_{yy}+\frac{P}{E}\right)+x_D^2\left(\lambda_n^2 I_{xx}+\frac{P}{E}\right)\right]+\\
&+\lambda_n^4\frac{P^2}{E^2}\left[y_D^2\left(\lambda_n^2 I_{yy}+F\right)+x_D^2\left(\lambda_n^2 I_{xx}+F\right)\right]\\
d=&\lambda_n^6\left(\lambda_n^2 I_{xx}+\frac{P}{E}\right)\left(\lambda_n^2 I_{yy}+\frac{P}{E}\right)\left(\lambda_n^2 I_{\omega\omega}+\frac{GK}{E}+P\frac{I_D}{EF}\right)-\\
&-\lambda_n^6\frac{P^2}{E^2}\left[y_D^2\left(\lambda_n^2 I_{yy}+\frac{P}{E}\right)+x_D^2\left(\lambda_n^2 I_{xx}+\frac{P}{E}\right)\right]
\end{aligned} \qquad (24)$$

Numerical Examples

The method presented in the previous section is used to calculate the natural frequencies of the thin-walled beams. For every mode the three numerical values which characterize three different types of natural frequencies: predominantly torsional, predominantly flexural in *x* direction and predominantly flexural in *y* direction are given. The particular examples are chosen to illustrate the effects of axial force on coupled bending-torsion natural vibrations of simply supported thin-walled beams and also to confirm the predictability and accuracy of the theory.

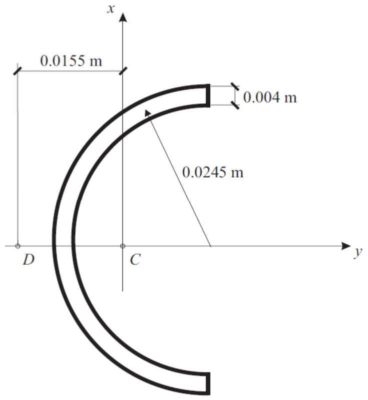

Figure 3. Cross section layout for Example 1.

The first example considers a thin-walled beam with semi-circular cross-section, Fig. 3. This example was selected because comparative results are available in the literature, Li et al. (2004a). The geometrical and material properties of thin walled beam are given below.

$L = 0.82$ m
$F = 3.08\times10^{-4}\ \mathrm{m}^2$
$E = 68.9\times10^6\ \mathrm{kN/m}^2$
$G = 26.5\times10^6\ \mathrm{kN/m}^2$
$\rho = 2.711\ \mathrm{kNs}^2/\mathrm{m}^4$
$I_{xx} = 9.26\times10^{-8}\ \mathrm{m}^4$
$I_{yy} = 1.77\times10^{-8}\ \mathrm{m}^4$
$I_{\omega\omega} = 1.52\times10^{-12}\ \mathrm{m}^6$
$K = 1.64\times10^{-9}\ \mathrm{m}^4$
$I_D = 1.843\times10^{-7}\ \mathrm{m}^4$
$x_D = 0.0$
$y_D = -0.0155$ m

First, the axial force P is simply assumed to be zero and then the effects of a constant compressive axial force (P = 1.79 kN) on the natural frequencies are considered. The natural frequencies obtained by the present method and those found by Li et al. (2004a) are given in Table 1.

Table 1. Natural frequencies (Hz) of beam studied as Example 1.

Mode	P = 0		P = 1.79 kN	
	Present paper	Li et al. (2004a)	Present paper	Li et al. (2004a)
n=1	89.24	89.27	84.65	84.69
	150.45	150.44	147.77	147.77
	319.84	320.32	318.60	319.07
n=2	356.51	357.11	352.03	352.62
	366.09	365.81	361.71	361.42
	1091.88	1106.59	1090.46	1105.14
n=3	604.52	604.13	598.56	598.16
	800.48	803.50	796.02	799.02
	2355.30	N/A	2353.87	N/A

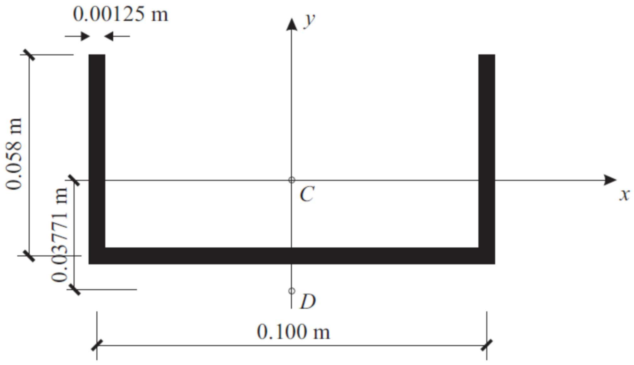

Figure 4. Cross section layout for Example 2.

The second example examines a thin walled beam with monosymmetical channel cross section as shown in Fig. 4, with geometrical and physical properties listed below.

$$L = 1.28 \text{ m}$$
$$F = 2.684\times10^{-4} \text{ m}^2$$
$$E = 2.164\times10^{8} \text{ kN / m}^2$$
$$G = 0.801\times10^{8} \text{ kN / m}^2$$
$$\rho = 7.8055 \text{ kNs}^2 / \text{m}^4$$
$$I_{xx} = 0.450\times10^{-6} \text{ m}^4$$
$$I_{yy} = 0.940\times10^{-7} \text{ m}^4$$
$$I_{\omega\omega} = 0.1636\times10^{-9} \text{ m}^6$$
$$K = 0.140\times10^{-9} \text{ m}^4$$
$$I_D = 9.256\times10^{-7} \text{ m}^4$$
$$x_D = 0.0$$
$$y_D = -0.03771 \text{ m}$$

The computational results, both including and excluding the effect of compressive axial force (P = 2.56 kN), are given in Table 2. A very good agreement was found with published results, Li et al. (2004b), obtained by the dynamic transfer matrix method.

Table 2. Natural frequencies (Hz) of beam studied as Example 2.

Mode	P = 0		P = 2.56 kN	
	Present paper	Li et al. (2004b)	Present paper	Li et al. (2004a)
n=1	67.19	67.12	65.79	65.72
	94.36	N/A	93.35	N/A
	273.68	275.75	273.35	275.42
n=2	263.55	263.67	262.14	262.25
	376.22	N/A	375.24	N/A
	1065.86	1049.80	1065.53	1048.38
n=3	589.47	591.23	588.05	589.81
	842.11	N/A	841.13	N/A
	2303.14	N/A	2302.82	N/A

Conclusion

Using the principle of virtual displacements the basic governing equations of motion of an axially loaded thin walled beam which exhibits bending-torsion coupling have been derived. The effects of warping and rotatory inertia are also included. By solving the governing differential equations of motion of the beam, the analytical expressions for the coupled bending-torsional vibration of an axially loaded beam are derived in an exact sense. When the results obtained from the present theory are compared with the published results, very good agreement is observed. The method is useful particularly when better accuracy of results or higher frequencies are required.

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Appendix

The values that determine geometrical properties of cross section are given by

$$I_{xx} = \iint_F x^2 dF$$
$$I_{yy} = \iint_F y^2 dF$$
$$I_{\omega\omega} = \iint_F \omega^2 dF$$
$$I_D = \iint_F \left[\left(x\text{-}x_D \right)^2 + \left(y - y_D \right)^2 \right] dF$$
$$K = \frac{1}{3} \int_s t^3 ds$$

Externally applied loads and moments per unit length of a beam are as follows

$$p_x = \int_s \overline{p}_x ds \quad p_y = \int_s \overline{p}_y ds \quad p_z = \int_s \overline{p}_z ds$$
$$m_x = \int_s \overline{p}_z y ds \quad m_y = -\int_s \overline{p}_z x ds \quad m_D = \int_s \left[\overline{p}_y \left(x - x_D \right) - \overline{p}_x \left(y - y_D \right) \right] ds$$
$$m_\omega = \int_s \overline{p}_z \omega ds$$

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