WITTGENSTEIN, FORMALISM, AND SYMBOLIC MATHEMATICS*

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ABSTRACT *In a recent essay, Sören Stenlund tries to align Wittgenstein's approach to the foundations and nature of mathematics with the tradition of symbolic mathematics. The characterization of symbolic mathematics made by Stenlund, according to which mathematics is logically separated from its external applications, brings it closer to the formalist position. This raises naturally the question whether Wittgenstein holds a formalist position in philosophy of mathematics. The aim of this paper is to give a negative answer to this question, defending the view that Wittgenstein always thought that there is no logical separation between mathematics and its applications. I will focus on Wittgenstein's remarks about arithmetic during his middle period, because it is in this period that a formalist reading of his writings is most tempting. I will show how his idea of autonomy of arithmetic is not to be compared with the formalist idea of autonomy, according to which a calculus is "cut off" from its applications. The autonomy of arithmetic, according to Wittgenstein, guarantees its own applicability, thus providing its own* raison d'être.

Keywords *Wittgenstein, formalism, symbolic mathematics, applicability, autonomy of arithmetic.*

RESUMO *Em um recente artigo, Sören Stenlund procura alinhar a abordagem de Wittgenstein em relação aos fundamentos e à natureza da*

matemática com a tradição da matemática simbólica. A caracterização da matemática simbólica feita por Stenlund, de acordo com a qual a matemática é logicamente separada de suas aplicações externas, a aproxima da posição formalista. Isto naturalmente levanta a questão de se Wittgenstein defende uma posição formalista em filosofia da matemática. O objetivo deste artigo é dar uma resposta negativa a esta questão, ao defender que Wittgenstein sempre pensou não haver separação lógica entre a matemática e suas aplicações. Atenção especial será dada às observações de Wittgenstein sobre a aritmética pertencentes ao seu período intermediário, pois é neste período que uma leitura formalista de seus escritos é mais sedutora. Mostrarei como sua ideia de autonomia da aritmética não deve ser comparada à ideia formalista de autonomia, segundo a qual um cálculo é "cortado" de suas aplicações. A autonomia da aritmética, para Wittgenstein, garante ela própria sua aplicabilidade, provendo, assim, sua própria raison d'être.

Palavras-chave *Wittgenstein, formalismo, matemática simbólica, aplicabilidade, autonomia da aritmética.*

1. Introduction

In his essay "On the Origin of Symbolic Mathematics and Its Significance for Wittgenstein's Thought", Sören Stenlund tries to align Wittgenstein's approach to the foundations and nature of mathematics with the tradition of symbolic mathematics, which dates back to the works of Franciscus Vieta and René Descartes. According to Stenlund, "the symbolic view of mathematics offers us a perspective from which the unity of Wittgenstein's philosophy of mathematics becomes apparent" (2015, p. 25). This unity refers not only to a systematic coherence of Wittgenstein's remarks about mathematics, but also to a genetic continuity of his thought. Stenlund holds that Wittgenstein already defended a symbolic conception of mathematics in the *Tractatus*, and continues to defend such a view in his middle and late period as well.

The symbolic view of mathematics is contrasted in Stenlund's essay with the Greek or the so-called "ontological conception" of mathematics.[1] This contrast is developed in three interrelated levels: the level of mathematical objects (numbers, in particular), the level of mathematical systems (*e.g.*, arithmetic, algebra) and, finally, the level of the relation between mathematics

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1   The terminology is due to Klein (1968).

and ordinary verbal language. In each of these levels, the symbolic approach to mathematics is meant to be illuminated by means of a comparison with the ontological approach.

The conception of number is characterized in the ontological approach by its essential relation to a determinate multitude of things. Central to this conception is the idea that numbers do not exist *per se* as elements of a calculus, but are always numbers *of* a given plurality of things. On the other side, the symbolic view holds (according to Stenlund) that numbers are nothing but symbols within a calculus, determined by how we operate with numerals in arithmetic. Because numerals are identified as pieces inside a calculus, arithmetic is not considered to be a theory or doctrine *about* something, but as an activity, a "pure calculus". And this holds true in the symbolic view for all mathematical systems, including logic and geometry. This implies that, on one side, logic does not hold any special position among the calculi, and cannot be a theory *about* "correct reasoning" or "correct inferring". On the other side, geometry is not a theory about space (physical or phenomenological), but only a calculus side by side with others. Another consequence of this view is that the notion of truth in mathematics does not gain its significance through the relations that a mathematical system could possibly maintain with what it is about. The notion of truth is, therefore, internal to the system, and it is related to the operational practices that constitute it.

Regarding the relation of mathematics and ordinary verbal language, the ontological approach present in Greek mathematics is characterized by a continuity and proximity with language in its verbal form. Stenlund argues that this view is still present in foundational writers, which have the tendency "to give meaning and significance to basic notions in mathematics and formal logic by translation or paraphrase into verbal language" (2015, p. 37). And this tendency is, for Stenlund, responsible for the phenomenon that Wittgenstein labels "prose" (*Prosa*), the "everyday prose that accompanies the calculus" (WWK, p. 129), which is regarded by Wittgenstein as the main source of the confusions about the foundations of mathematics. Stenlund presents the symbolic conception of mathematics in sharp contrast to this tendency. According to him, the symbolic conception holds that mathematical symbols have content only because (and just to the extent that) they are part of a mathematical system, of a calculus, and not because they are related to and continuous with ordinary language. Mathematics is, therefore, autonomous and independent of everything that is external to the calculus, including its possible external applications. In Stenlund words: "an essential feature of the symbolic point of view was the *logical separation* of a symbolic system from

its application to some subject-matter outside pure mathematics" (2015, p. 46, emphasis mine).

This characterization of symbolic mathematics raises naturally the question of whether the symbolic view coincides with formalism, in the sense of Thomae (and not in the sense of what is called "Hilbert's formalism"). Stenlund argues that the problem with this identification is the superficial and pejorative sense that the word "formalism" is used by authors like Frege and Brouwer. If, however, we disregard this negative attitude towards the word "formalism" and the misconstruals of this conception[2], then the identity between symbolic mathematics and formalism is, according to Stenlund, appropriate. He goes on to say that that Thomae's concept of "formal arithmetic" is "one of the most clear and distinct examples of the use of the symbolic point of view" (2015, p. 49).

If the symbolic conception of mathematics is identical with formalism and is, at the same time, the perspective from which we can consider the unity of Wittgenstein's remarks on mathematics, the question to be asked is whether Wittgenstein holds after all a formalist position in philosophy of mathematics. The present paper addresses this question, with a particular emphasis on Wittgenstein's views in his middle period. Although the discussion of the relations between Wittgenstein's views and formalism is certainly not a novelty, I think that there is no presentation in the literature that adequately explains Wittgenstein's trains of thought in his middle period about the applicability of arithmetic (this point being the crux of the debate between Frege and the formalists). I shall not argue here with Stenlund's characterization of symbolic mathematics, but only take, as point of departure, his reading according to which "formal", as used by Thomae, is *essentially synonymous* with "symbolic", as this word has been used in the tradition of symbolic mathematics.[3] However, the way I will try to show why Wittgenstein should not be read as a formalist *could be useful* for a better appraisal of the relations between Wittgenstein's views on mathematics and symbolic mathematics. In his characterization of symbolic mathematics, Stenlund focuses too much, I think, on the *operational* aspects

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2   Frege, for instance, tries to put Thomae's views together with Heine's views, according to which arithmetic is about the signs themselves. This is clearly a misconstrual of Thomae's views, and this misconstrual is the responsible for the dichotomy Frege sustained between arithmetic conceived as referring to abstract objects and arithmetic conceived as referring to the signs used in arithmetical calculations. See, in particular, §88 and §§95-6 of Frege (1903). Shapiro (2000) distinguishes Thomae's views from Heine's by calling the first a "game formalism" and the second a "term formalism".

3   See Stenlund (2015, p. 49): "One of the most clear and distinct examples of the use of the symbolic point of view is the mathematician Johannes Thomae's concept of "formal arithmetic" [...]. As Thomae uses the word 'formal' it is essentially synonymous with 'symbolic', as this word has been used in mathematics ever since Vieta".

of the symbolism, and very little on the *ecthetic* aspects of it and its relations with the notions of scheme, paradigm and aspect. These notions will play an important role in my argument against the characterization of Wittgenstein as a formalist.

In the next section, I will start with some remarks about the approximation of Wittgenstein's philosophy of mathematics and formalism, indicating why the reading of Wittgenstein as a formalist is most inviting when we consider the writings from his middle period. Section 3 tries then to remove the temptation to regard the middle Wittgenstein as a formalist. Section 4 casts light on the reasons Wittgenstein had, in his middle period, to think that arithmetic guarantees alone its applicability, and shows that Wittgenstein's idea of the autonomy of arithmetic is not the formalist idea of autonomy according to which a calculus is logically separated from its applications. Section 5 turns to the case of algebra, which is contrasted with arithmetic in that, unlike arithmetic, it is not an autonomous mathematical system. Section 6 is devoted to some concluding remarks.

2. Wittgenstein and formalism

Stenlund was not the first to characterize Wittgenstein's attitude towards mathematics as closer to formalism. Rodych (1997) is even more explicit and suggests that Wittgenstein is to be read as advocating a formalist stance. Rodych, however, points out that there are some subtleties that need to be consider to properly characterize the development of Wittgenstein's views about mathematics. For this reason, he draws a distinction between two varieties of formalism: strong formalism and weak formalism[4]. They are defined as follows:

> *Strong Formalism (SF)*: A mathematical calculus is defined by its accepted or stipulated propositions (e.g., axioms) and rules of operation. Mathematics is syntactical, not semantical: the meaningfulness of propositions within a calculus is an entirely intrasystemic matter. A mathematical calculus may be invented as an uninterpreted formalism, or it may result from the axiomatization of a "meaningful language." If, however, a mathematical calculus has a semantic interpretation or an extrasystemic application, it is inessential, for a calculus is essentially a "sign-game" – its signs and propositions do not refer to or designate extramathematical objects or truths.
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> *Weak Formalism (WF)*: A mathematical calculus is a formal calculus in the sense of SF, but a formal calculus is a mathematical calculus only if it has been given an extrasystemic application to a real world domain. (Rodych, 1997, pp. 196-7)

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4   Rodych (1997) also defines other two varieties of formalism, namely, extreme formalism and radical formalism, but these variants of formalism are not attributed to Wittgenstein.

Rodych argues that both in the *Tractatus* and in RFM Wittgenstein adopted weak formalism, whereas in his middle period Wittgenstein espoused strong formalism. In this essay, I will focus on criticizing the view that, in his middle period, Wittgenstein was a formalist (strong or weak), because it is in this period that a formalist reading of his writings is more tempting. With regard to the late period, even if Rodych is right on characterizing Wittgenstein's position as a weak formalism,[5] from this it would follow, contrary to what Stenlund says, that there is no "logical separation" of a mathematical system from its application to some subject-matter outside pure mathematics, because it is only by being "logically connected" with (at least one of) its applications that a formal system deserves to be called mathematics. In this connection, Rodych quotes the well-known passage of RFM in which Wittgenstein states that "it is the use outside mathematics, and so the *meaning* of the signs, that makes the sign-game into mathematics" (RFM V, §2). Now, Stenlund also recognizes that, for the late Wittgenstein, "the use of mathematical signs in applications outside mathematics contributes to the meaning of mathematical signs" (2015, pp. 63-4). But then it is unclear what is the "logical separation" of a symbolic system from its application to some subject-matter outside pure mathematics. We should then conclude that Wittgenstein was not in agreement with this "essential feature of the symbolic point of view" according to Stenlund and, therefore, that this view is not a good perspective to consider the unity of Wittgenstein's thought about mathematics.

A second misgiving one might well have about calling Wittgenstein a formalist arises from his characterization of mathematical propositions as *grammatical*. In his essay, Stenlund argues as if the grammatical view of mathematical propositions reinforced Wittgenstein's "symbolic, non-ontological conception of mathematics" (2015, p. 25). But in which sense "grammar" can be said to be logically separated from its applications? As far as I can see, Stenlund's chain of reasoning throughout his essay is roughly the following: because mathematics is symbolic, and therefore non-ontological (not about something), it is cut off from its extra-mathematical applications. This reasoning in turn could be regarded as a kind of *modus tollens* of Frege's view about mathematics. For Frege, because mathematics has to guarantee all its applications (whether internal or external), mathematics has to be about something. Now, the conception of mathematical propositions as grammatical, far from leading us to accept this implication and conclude by the contrapositive that mathematics does not need to account for its applicability, allows us to

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5   But see Mühlholzer (2010, pp. 72ff) for some misgivings about this characterization.

resist it. Wittgenstein may be read, thus, as pointing out that it does not follow from the non-descriptive character of mathematics that mathematics is cut off from its applications. If we conceive of mathematical propositions as rules of grammar, it is just not the case that pure mathematics is non-descriptive and applied mathematics is descriptive. As During puts it: "it is as long as mathematics is applied that it does not deal with anything" (During, 2005, p. 203, my translation).

The view that mathematics does not need to be descriptive to guarantee its applicability (*Anwendbarkeit*) was already Wittgenstein's view in the *Tractatus*. Now, it is controversial whether the *Tractarian* view of mathematics could be called "formalist" even in the weak sense given by Rodych. For it is not that there is a formal system, namely, arithmetic, that becomes mathematics when one shows that it has an extrasystemic application. When numbers are introduced in section 6.02, they are already defined *in the context of their application*. The number zero is introduced by the definition "x = Ω^0x Def.". In this definition, Ω is a variable whose values are given by the general form of an operation, which is defined in section 6.01. Other numbers are introduced by the recursive definition "Ω' $\Omega^{v'}$x =$\Omega^{v+1'}$ x Def." and abbreviated by means of additional definitions, like "0+1+1 = 2 Def.". These definitions set the sole context in which numbers can occur: they can only occur as exponents of operations. This context is present even in proofs of mathematical propositions such as 2×2=4, as the proof Wittgenstein gives in section 6.241 shows. Therefore, numbers in the *Tractatus* are not syntactic symbols waiting for a semantic bridge to become mathematical. The applicability of numbers and of arithmetic is arguably not an *additional* feature of a pure formal calculus, but is *a priori* guaranteed by the way arithmetical symbols are presented.

In the writings from Wittgenstein's middle period, however, there are some passages that seem to support Rodych's interpretation according to which Wittgenstein defended (in that period) a strong formalism about mathematical calculi. In PR, for instance, Wittgenstein wrote:

> You could say: Why bother to limit the application of arithmetic, that takes care of itself. (I can make a knife without bothering which sorts of material it will cut: that will show soon enough.) (PR X, §109j)
> It's always a question of whether and how it's possible to represent the most general form of the application of arithmetic. And here the strange thing is that in a certain sense it doesn't seem to be needed. And if in fact it isn't needed then it's also impossible. (PR X, §110b)

It is worth noting that, in the *Tractatus*, Wittgenstein thought it was possible to represent the most general form of the application of arithmetic. In

fact, if numbers are exponents of operations, the specification of the general form of an operation is the representation of the most general form of the application of arithmetic. So, in the passage above, Wittgenstein is apparently giving up the *Tractarian* account of the applicability of arithmetic, saying that "to represent the most general form of the application of arithmetic [...] doesn't seem to be needed. And if in fact it isn't needed then it's also impossible". Therefore, this last passage seems to suggest that Wittgenstein is changing his mind about the relations between arithmetic and its applications: we do not need to define arithmetic symbols in the most general context in which they are applied, because the limits of applicability of arithmetic take care of itself. The space within which arithmetic can be applied will "show soon enough": we do not need to bother about it. Wittgenstein can be read here as saying that we cannot (and need not) guarantee *a priori* that a mathematical calculus could be applied, but at most say that it "can be applied only to what it can be applied to" (WWK, p. 104). The applicability of a formal calculus of mathematics would then be a *contingent* matter, and not something that could be guaranteed beforehand. Before the particular applications "show up", what we have is only a formal calculus defined by the rules for the manipulation of signs and nothing guarantees that it can be applied to something "outside", in the "real world". But the calculus as a calculus (and thus as a piece of mathematics) is all right as it is, because its life as a "sign-game" does not depend on its capacity to be applied to some external domain.

It is, thus, tempting to read the middle Wittgenstein as a strong formalist. In what follows, I shall argue that this temptation should be resisted, however. While it is true that, in his middle period, Wittgenstein came to realize that the *Tractarian* account of the applicability of arithmetic is not needed, I shall argue that this change in his thought does not imply that he started to regard the applicability of arithmetic as a contingent or external matter. As we shall see in the next section, Wittgenstein still sticks in his middle period to the idea that the applicability is an essential component of mathematics.

3. The applicability as a criterion for mathematics proper

Let me return for a moment to Stenlund's characterization of symbolic mathematics and its relation to formalism. One page after having raised the question whether the symbolic view of mathematics coincides with formalism, Stenlund quotes a nice passage from Couturat and qualifies this passage as a "very pertinent formulation of the symbolic view of mathematics" (2015, p. 52). The passage in question is the following:

> A mathematician never defines magnitudes [or numbers] in themselves, as a philosopher would be tempted to do; he defines their equality, their sum and their product, and these definitions determine, or rather constitute, all the mathematical properties of magnitudes. In a yet more abstract and more formal manner he lays down symbols and at the same time prescribes the rules according to which they must be combined; these rules suffice to characterize these symbols and to give them a mathematical value. Briefly, he creates mathematical entities by means of arbitrary conventions, in the same way that the several chessmen are defined by the conventions which govern their moves and the relations between them. (Quoted in Bell, 1937, p. 624)[6]

This characterization of the symbolic view of mathematics somewhat coincides with strong formalism as defined by Rodych. Under this picture, mathematical signs are, like chess pieces, elements of a calculus-game and they receive their content as symbols by means of the rules that define this game. In this sense, it is enough to determine the operations with mathematical signs to fill these signs with mathematical content and give them mathematical citizenship. In arithmetic, for example, there is no need to say what numbers are, but only how they are moved in the arithmetic-game.

Now, if the middle Wittgenstein is read as a strong formalist, then it is difficult to understand some passages in which he is apparently not satisfied with the fact that some rules for the manipulation of signs are laid down, that he thinks this is not enough to consider a symbol or an operation as *mathematical*. Commenting on intuitionistic choice sequences, for instance, he states the following:

> If Weyl believes that [a freely developing sequence] is a mathematical structure because I can derive a freely developing sequence from another by means of a general law, e.g.,
> $$m_1, m_2, m_3, \dots$$
> $$m_1, m_1 + m_2, m_1 + m_3, \dots$$
> then the following is to be said against this: No, this shows only that I can add numbers, but not that a freely developing sequence is an admissible mathematical concept. (WWK, p. 83)

The last sentence would be a very curious remark from the perspective of a formalist. Indeed, if we are given laws for manipulating choice sequences, for computing their sum, their product, and so on, what else is required from a formalist standpoint? Another example is Wittgenstein resistance to accept the so-called pseudo-irrationals (such like π', P and F[7]) and pseudo-operations

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6   The addition between brackets is from Stenlund.

7   In BT, §139, Wittgenstein defines these pseudo-irrationals as following: "π' is a rule for the formation of decimal fractions; specifically, the expansion of π' is the same as the expansion of π except where the sequence 777 occurs in the expansion of π; in that case the sequence 000 replaces the sequence 777. There is no method

(like $\times$'[8]) as *bona fide* arithmetical symbols. It is unclear why a formalist should have any qualms with these perfectly computable objects.

A further difficulty to read the middle Wittgenstein as a formalist is that, contrary to what is expected from a formalist, he seems to answer what a number *is*, i.e., he seems to succumb to the temptation warned by Couturat in the passage above. In PR we are told that "numbers are pictures (*Bild*) of the extensions of concepts" (PR X, §100a), that "a cardinal number is an internal property of a list" (PR XI, §118d) and that "a real number is: an arithmetical law which endlessly yields the places of a decimal fraction" (PR XVII, §186b). Why should a formalist bother defining what numbers are? Are not numbers just what is determined by how we manipulate numerals?

Still, if the middle Wittgenstein is not a formalist about mathematics, what does he think that is required for mathematics besides a system of rules for manipulating signs? I think that here Wittgenstein would repeat Frege almost *verbatim*: it is the applicability alone which elevates a system of rules from a game to the rank of mathematics (see Frege, 1903, §91).[9] Contrary to Frege, however, Wittgenstein does not require that mathematical sentences express thoughts in order to be applied. As I shall argue, in his middle period Wittgenstein thought that arithmetic could guarantee its applicability completely *a priori* by being its own application. In this sense, the lemma "the application of arithmetic takes care of itself" is not to be read as saying that the applicability of arithmetic is a contingent matter, but as saying that arithmetic is autonomous (*autonom*), and so it guarantees by itself its own applicability (thus providing its own *raison d'être*).

At this point, it may seem that I am being unfair to Rodych's characterization of Wittgenstein as a formalist. This is because Rodych does not deny that, for a formalist, a mathematical system could have internal or intrasystemic interpretations. He also emphasized Wittgenstein *dictum* that "arithmetic is its own application" (see Rodych, 1997, p. 199). However, Rodych interprets this *dictum* in such a way that it becomes valid for all system of rules for

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known to our calculus for discovering where we will encounter such a sequence in the expansion of π. P is a rule for the construction of binary fractions. At the nth place of the expansion there occurs a 1 or a 0, depending on whether n is prime or not. F is a rule for the construction of binary fractions. At the nth place there is a 0, except when a triple x, y, z from the first 100 cardinal numbers satisfies the equation $x^n + y^n = z^n$".

8   See fn. 10 for the definition of this pseudo-operation and Section 4 below for further discussion.

9   See Medina (2002, p. 87): "Despite the influence of formalist ideas on Wittgenstein's view, the primary concern of his discussion of rules is the issue of applicability emphasized by Frege. In this respect Wittgenstein's view of rules is more indebted to Frege's critique of formalism than to formalism itself". Medina, however, does not provide, in my opinion, a satisfactory reading of the passages where Wittgenstein says that arithmetical constructions themselves guarantee their applicability. His reading, for instance, is unable to explain Wittgenstein's qualms about the applicability of a pseudo-operation like $\times$'.

manipulating signs whatsoever. The idea of an internal or intrasystemic application is, for him, "the strong formalist idea that the meaningfulness and the truth or falsity of propositions within a calculus are determined entirely by the axioms and rules of operations of that calculus, without any necessary reference to an extrasystemic application" (1997, p. 200). In this sense, the requirement of applicability of a calculus becomes tautologous, because it is satisfied by any calculus. I shall argue, on the contrary, that the idea that arithmetic is its own application is a substantial feature of arithmetic and, moreover, that this feature is important to understand Wittgenstein's thought about mathematics in his middle period.

There is a passage in PR where the non-tautologous character of the applicability of a calculus becomes apparent, namely: "It is clear that were I able to apply ×' all doubts about its legitimacy would be dispelled. For the possibility of application is the real criterion for arithmetical reality"[10] (PR XVII, §186f). It is obvious that I can apply the rule ×' in Rodych's sense. For the formalist, the (pseudo-)operation ×' should be an operation for manipulating signs in equal footing with any other operation. If Wittgenstein raises doubt about the applicability of this rule, then his notion of application is not the same as the application of a rule inside a calculus-game. In the same passage, Wittgenstein says, I repeat, that the applicability is *the real criterion for arithmetical reality*. I would say that this criterion holds, in Wittgenstein's middle period views, not only for arithmetic, but for mathematics in general. This point will become particularly evident when we later consider the case of algebra, because algebra, unlike arithmetic, is not "its own application", depending for its "mathematical existence" on the donation of sense (applicability) for its formulae. Arithmetic, by contrast, guarantees its own existence by being its own application. Arithmetic is, in this sense, autonomous, not because it is cut off from its external applications, but because it is *causa sui*. The next section is devoted to explain this idea of autonomy of arithmetic.

4. The autonomy of arithmetic

The idea that arithmetic is autonomous is explored in PR together with the idea that arithmetic is a "kind of geometry". Both in the case of arithmetic and in the case of geometry it is possible to say, according to Wittgenstein,

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10 By "×'" Wittgenstein means the operation which is the same as ordinary multiplication except for the fact that, in the result, every occurrence of the digit '7' is replaced by '3'.

that they are their own application. This shows that they alone guarantee their own applicability:

> One always has an aversion to giving arithmetic a foundation by saying something about its application. It appears firmly enough grounded in itself. And that of course derives from the fact that arithmetic is its own application. (PR X, §109a)
> You could say arithmetic is a kind of geometry; i.e. what in geometry are constructions on paper, in arithmetic are calculations (on paper). – You could say it is a more general kind of geometry. (PR X, §109h)
> The point of the remark that arithmetic is a kind of geometry is simply that arithmetical constructions are autonomous like geometrical ones, and hence so to speak themselves guarantee their applicability. / For it must be possible to say of geometry, too, that it is its own application. (PR X, §111a)

The main difficulty of these passages is to explain in which precise sense arithmetic (and geometry) could be said to be its own application. The following passage from the conversations with the Vienna Circle recorded by Waismann, where Wittgenstein says that this idea is "tremendously important", can be helpful:

> Mathematics is its own application. This is tremendously important. A lot follows from it. When I say '3 plums + 4 plums = 7 plums', '3 men + 4 men = 7 men', etc., I do not apply numbers to different objects; it is always the same application that I have before my eyes. Numbers are not represented by proxies; numbers *are there*. Only objects are represented by proxies. / The correctness of an arithmetical proposition is never expressed by a proposition's being a tautology. In the Russellian way of expressing it, the proposition $3 + 4 = 7$ for example can be represented in the following manner:
> $$(E3x)\varphi x \ . \ (E4x)\psi x \ . \ \sim(\exists x)\varphi x.\psi x : \exists : (E7x).\varphi x \ \exists \ \psi x.$$
> Now one might think that the proof of this equation consisted in this: that the proposition written down was a tautology. But in order to be able to write down this proposition, I have to *know* that $3 + 4 = 7$. The whole tautology is an application and not a proof of arithmetic. Arithmetic is used in constructing this proposition. The fact that a tautology is the result is in itself inessential. For I can apply an arithmetical equation both to propositions with sense and to tautologies. (WWK, p. 35)

In the passage above, Wittgenstein is saying that from the fact that mathematics is its own application, two important things follow, namely:

1. all applications of arithmetical equations are, in a sense, the *same* application (this being connected with the non-surrogative character of numbers).

2. the tautology correspondent in Russell's symbolism is not the proof of the equation, but an application of the equation. As a consequence, the tautology is not needed to apply the equation, since it is already one of its applications.

It is important to stress here the direction of the reasoning. It is not because the tautology in Russellian symbolism is not needed for the application of arithmetic that arithmetic is its own application, but quite the opposite: it is because arithmetic is its own application that we do not need to bridge a supposed gap between arithmetic and its applications by means of a logical schema like the tautology above. The same direction is emphasized in the following passage of PR: "Every mathematical calculation is an application of itself and only as such does it have a sense. *That* is why it isn't necessary to speak about the general form of logical operation when giving a foundation to arithmetic" (PR X, §109e). Again, it is *because* arithmetic is its own application that the *Tractarian* explanation for the applicability of arithmetic is not needed.

When Wittgenstein says, in the passages above, that "mathematics is its own application" and that "every mathematical calculation is an application of itself", the term "mathematics" should be read, according to my interpretation, contextually as "arithmetic". This is because, according to the reading I will suggest to these passages, it is simply not true that *every mathematical calculation* is an application of itself. Algebraic calculations, for instance, are not their own application (I will return to this point later). The context in which these sentences are inserted does allow this reading: in section 109 of PR Wittgenstein is dealing with the applicability of arithmetic and in page 35 of WWK (fn. 1) the examples mentioned are arithmetical ones. The justification for such a reading, however, will only come to light when we consider further the case of algebra.

Another point worth noticing in this last quotation from PR is that the idea of mathematics being its own application is rephrased and interpreted as the self-applicability of mathematical calculations to themselves. This is a bit puzzling, since this notion of self-applicability, as far as I know, does not occur elsewhere in the writings from the middle period, *except* for a single passage of MS 107 where Wittgenstein says that "the calculation with strokes is also, at the same time, an application of the calculus. This ceases to be the case in this direct way in decimal system"[11] (MS 107, p. 68; Wi2, p. 40, my translation). Despite the scarce textual evidence, I think that this last passage provides an important hint to the direction where the answer to the question "what does it mean to say that arithmetic is its own application?" could be searched for. For it tells us that this idea of self-applicability is present in a "direct way" in

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11 Original: "Das Rechnen mit Strichen ist zugleich auch eine Anwendung der Rechnung. Das hört in dieser direkten Weise im Dezimalsystem auf".

calculations with strokes and that this immediacy is lost when we calculate in decimal system.

In a short text entitled "What is a number" written between 1929-30[12] (and intended to present Wittgenstein's views), Waismann presents this immediacy feature of the stroke-notation in connection with the idea of pictoriality (*Bildhaftigkeit*), this idea being earlier related in the text with the notion of form (*Form*):

> We are no doubt able to understand a sign of this kind: |||| plums. [...] The sign contains a *picture* but not the specification of a *property* or a *relation*. **[...]** *Numbers are forms.* The expression of a number is a picture that occurs in propositions. (WWK, p. 223)

> In speaking of 5 men I can represent the men by strokes. But those men's being 5 is not represented; it manifests itself in the fact that the number of strokes is 5. Here the number sign is *immediately* conceived as a picture. (WWK, p. 225, last emphasis mine)

This in turn helps us to understand Wittgenstein's remark, mentioned earlier, that numbers are "pictures of the extensions of concepts": they are, as Waismann says, *forms*[13] which present themselves both in the fact (in the symbol) we use to represent the extension of a concept and in this extension itself. In stroke-notation, these forms are at the same time symbolized *by means of* the notation and exemplified *in* the notation. That is, in stroke-notation these forms are *ostensively* exhibited in the notation. This feature is lost (in this direct way) when we work in decimal notation. In the continuation of his text, Waismann makes clear, however, how and in which sense this feature is present even in decimal notation, albeit indirectly:

> The usual way of representing the numbers by means of the system of digits rests on exactly the same principle. At first blush the number 387 does not seem to be a picture of the quantity it means. We must, however, take into account that in addition to the signs there are the rules of syntax too. The signs 3, 8, 7 are defined. If we reduce them to their definitions, that is, if we analyse these signs step by step, then they assume the very multiplicity they mean; e.g. 3 = 1+1+1. Second, the position of figures, too, depicts something. Our number signs contain the possibility of being transformed into other signs that are pictures in an immediate way. That is, our number signs, together with the rules of syntax, are instructions for the construction of picture-like symbols. There must always remain a clear way back to a picture-like representation of numbers leading through all arithmetical symbols, abbreviations, signs for operations, etc. The

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12 See WWK, pp. 20-1.
13 The idea that numbers are forms also occur in PR (X, 113): "The natural numbers are a form given in reality through things, as the rational numbers are through extensions etc. I mean, by actual forms. In the same way, the complex numbers are given by actual manifolds. (The symbols are actual.)".

symbolism of the representation of numbers is a system of rules for translation into something picture-like (*Bildhafte*). (WWK, pp. 225-6)

In short, numbers are *forms* and they occur in arithmetical symbolism either directly (in stroke-notation) or indirectly (in decimal notation plus the rules of syntax). These forms are not represented by proxies, but they *"are there"*. In each application of numbers, these forms are there *in the same way* that they are there in arithmetical symbolism itself. It is this sameness of form alone that makes the application possible and justify it. The symbolism itself is, thus, an application of these forms (an instantiation of them).

Now, when we calculate with strokes in stroke-notation, the manipulation of these strokes (grouping some of them together, separating others, etc.) can function as pictures of possible states of affairs. This is because the form of these possible states of affairs is represented as a structure in the symbolism itself. For instance, if I demonstrate that $2 + 2 = 4$ in stroke-notation by means of the following diagram:
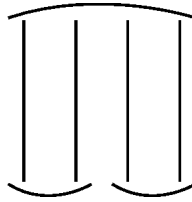
then this diagram could be used as a propositional sign for the projection of a situation in which 4 objects are grouped in 2 and 2, i.e., the diagram can *picture* such a situation. As in the *Tractatus* (cf. aphorism 2.201), the picture depicts (*das Bild bildet*) reality by representing a possibility of existence and non-existence of a determinate state of affairs, in this case, the grouping of four things in two and two. But for this to be possible both the picture and the state of affairs must share the *same* pictorial form (*Form der Abbildung*). This form is presented as an actual structure in the symbolism and as the possibility of this structure at the level of the things themselves.[14]

In Chapter X of PR, Wittgenstein insists on the fact that it is not necessary for a multitude of things to be presented as the actual extension of a concept. They could be presented by means of a list, and a list of things is not necessarily

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14 See PR (X, §100d). Note, however, that the "things" are not objects in the Tractarian sense, but ordinary things (plums, men, apples, and so on).

the extension of a material concept. This point is important for his critique of Frege's idea according to which an ascription of number (*Zahlangabe*) always refers to a concept.[15] In section 102 of PR, for instance, he says: "Only 3 of the objects a, b, c, d have the property φ. That can be expressed through a disjunction. Obviously *another case where an ascription of number doesn't refer to a concept* (although you could make it look as though it did by using an '='.)." (emphasis mine, translation slightly modified). The disjunction mentioned above is the following: "φa · φb · φc · ⊕ · φa · φb · φd · ⊕ · φa · φc · φd · ⊕ · φb · φc · φd" (where the sign "⊕" is used for exclusive disjunction). In this case, the proposition ("only 3...") does not ascribe the number 3 to a concept, but the number 3 is an internal property of the list of objects being referred in each disjunct.[16] Thus, even if this proposition is true, say, because *a*, *b*, and *c* (but not *d*) have the property φ, the number 3 is not a picture of an extension of a concept (for there is no such a concept in this case), but just an internal property of the list {a, b, c}.[17]

That is why it is more generally valid to say that the cardinal number is an internal property of a list than to say that cardinal numbers are "pictures of the extensions of concepts". When an extension is given by means of a concept, then the cardinal number is such a picture, but when we have just a list of things, the cardinal number is an internal property of this list.[18]

In stroke-notation, these internal properties are exemplified in the symbolism for numbers. If we say that a "scheme" or a "paradigm" is the "explicit symbolic exemplification of an internal property" (Narboux, 2001, p. 583) and if we say, moreover, that an "aspect" is the "internal relation exemplified within a scheme by the scheme itself" (*ibid*, p. 583), then we can say that, in stroke-notation, numbers are aspects of number-schemes or number-paradigms. That is, in stroke-notation numbers are symbolized by means of an application of them, namely, to strokes.[19] In stroke-notation, number-schemes and calculations with them display the form of their applications. If I am right, it

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15 If we remember that it was precisely by means of this idea that Frege presented the most general form of the application of number, Wittgenstein's critique of it can be understood simply as a way to further justify his point that the representation of the most general form of the application of arithmetic is not needed.

16 For Wittgenstein, a property is internal to an object just when "it is unthinkable that its object should not possess it" (*Tractatus*, 4.123).

17 Frege would of course understand the same proposition ("only 3...") as ascribing the number 3 to the concept "φ(ξ) · (ξ = a ∨ ξ = b ∨ ξ = c ∨ ξ = d)" but, as Wittgenstein remarks in other occasion, "if identity drops out, however, nothing remains" (WWK, p. 165).

18 It could be said, however, that the symbol for, say, the cardinal number 3 is a picture of every list with three elements.

19 Frascolla (1994, pp. 44-54) also points out the relation between the arithmetic of strokes and the notions of paradigm and aspect. However, Frascolla (2004) does not draw the connection between these notions and Wittgenstein's remarks about the autonomy and self-applicability of arithmetic.

is this paradigmatic feature of arithmetic (presented directly in stroke-notation or indirectly in other notations) that is being targeted by Wittgenstein when he says that "arithmetic is its own application". And this is, in turn, what makes arithmetic autonomous and guarantees *a priori* its applicability.

When we work in other notations (like the decimal system), the paradigmatic feature of arithmetical operations is not immediately present, but is present if we consider calculations and rules for manipulating symbols in these notations. These calculations, however, need to be always translatable in stroke-notation. The problem with a pseudo-operation like ×' mentioned above is that there is nothing corresponding to it in stroke-notation. In this case, the decimal system ceases to be a mere mode of presentation of what is being considered (namely, arithmetical relations) and becomes itself an object of consideration. But then the whole symbolic chain that connects this notation with stroke-notation is lost and, as a result, it becomes doubtful what is the application of such symbolic manipulations.

5. The case of algebra

At this point, let me first summarize the main conclusions reached so far:

(1) The autonomy of arithmetic comes from the fact that it is its own application. The same holds for geometry.

(2) Arithmetic is its own application because arithmetic calculations can be applied to themselves.

(3) The sense in which arithmetical calculations can be applied to themselves is that these calculations display (directly or indirectly) in the notation forms that occur in *each one* of their applications. Arithmetic is, therefore, a paradigm of *all* its applications.

(4) All applications are, in a sense, the *same* application (they all have the same form).

(5) This feature, namely, the identity of forms displayed by the symbolism and occurring in every application of arithmetic is *enough* to justify the applicability of arithmetic to these applications.

In what follows, I will try to show briefly that these conclusions do not hold for algebra, according to Wittgenstein. My strategy will be to argue that, for Wittgenstein, algebraic equations by themselves are not *enough* to justify their applicability whenever it is possible to apply them. In particular, they are not enough to justify their applicability to arithmetic, but must be supplemented with inductions. This implies, if I am right, that these equations do not display in algebraic notation the forms that occur in each one of their

applications. A *justification* is, then, needed for why they can be applied to a particular domain. Notice that it is enough to show that *one* application stands in need of justification to show that algebra is not autonomous, because in this case it would not be true that algebraic equations are paradigms of all their applications.

But why, it may well be asked, the fact that the applicability of algebra to arithmetic stands in need of justification *implies* that its applicability *simpliciter* stands in need of justification? Could not algebraic calculations display "general forms", which (i) would explain why the application of algebra to the particular forms of arithmetic stands in need of justification (because in this case the possibility of application is not explicable by the identity of forms) and (ii) would satisfy the requirement of self-applicability? Is there not an intermediate case in which a calculus is autonomous and its applications do not share the same *particular* form, but share instead a *general* form, a "super-form"?

Waismann's text is once again useful here. Soon after having emphasized that the application of arithmetic is the same everywhere and that there is no "problem of application" for arithmetic, Waismann says: "This is connected with the fact that one form cannot fall under another (super- and subordination exist only for concepts.) *The method of representing numbers is the method of picturing*. A number shows itself in a symbol" (WWK, p. 225).

As I understand him, he is pointing out that, if I could represent a form without displaying it in a symbol, two consequences would follow: (i) arithmetic would not necessarily display the forms of its possible applications and, thus, there would be a "problem of application" for arithmetic; (ii) a hierarchy of forms would be possible. But if, on the contrary, the only method of representing a form (in particular, a number) is the "method of picturing", then this explains the nonexistence of the problem of "application for arithmetic" and, at the same time, the nonexistence of super- and subordination for forms. That a form cannot fall under another is, then, just a consequence of the picture theory. I presuppose, thus, that Wittgenstein retained in PR the main features of his picture theory of language, and explored the novelty of considering numbers as forms (I take these assumptions to be the same as Waismann's in the text "What is a number").

If this is right, then either algebraic notation displays forms and then arithmetic, not having these forms, cannot be even a candidate for the application of algebra (and nothing further could make algebra applicable to arithmetic), or they do not display forms and then, by virtue of additional means, they can be applied to arithmetic. In general terms: either a calculus is autonomous and its applications share the same form (this form being displayed in the symbolism

of this calculus), or its applications do not share the same form and the calculus is not autonomous. Consequently, there is not an intermediate case in which a calculus is autonomous and its applications do not share the same form.

One particular consequence of this reasoning is that the question of the *applicability* of arithmetic and the question of the *limits* of the application of arithmetic are two sides of the same coin: the fact that arithmetic is applicable (to itself) settles the question of the domain of the applicability of arithmetic, because each other application will be of the same form, and we could say that arithmetic is applicable to anything that shares with it this same form: *this* is the domain where it can be applied. The identity of form is, in this case, enough to justify the applicability of arithmetic to them. This is not the case of algebra in relation to arithmetic, as I will now show.

Wittgenstein discussed the relation between arithmetic and algebra for the first time in his middle period in the occasion of his detailed discussion of Skolem's recursive proof of the associative law: a + (b + c) = (a + b) + c (abbreviated by Wittgenstein as "A(c)"). This discussion was undoubtedly regarded by Wittgenstein as extremely relevant to the philosophy of mathematics, and it would be later exposed in many writings from his middle period, most importantly in Chapter XIV of PR and the whole Section of BT entitled "Inductive Proofs. Periodicity". I shall here consider just some passages of PR that are important for the reading I am suggesting of the autonomy of arithmetic.

The upshot of the discussion of inductive proofs in PR is that induction is not to be conceived as the proof of an algebraic proposition, but as what shows the applicability of algebraic formulae to arithmetic[20]. In this vein, Wittgenstein warns us against confusing "the infinite possibility of its application with what is actually proved" (PR XIV, §163e). The so-called inductive or recursive "proof" only gives us the unlimited possibility of applying an algebraic formula to numbers, but it does not *prove* a general proposition about all numbers:

> A recursive proof is only a general guide to an arbitrary special proof. A signpost that shows every proposition of a particular form a particular way home. It says to the proposition 2 + (3 + 4) = (2 + 3) + 4: 'Go in this direction (run through this spiral), and you will arrive home.' (PR XIV, §164a)

> To what extent, now, can we call such a guide to proofs the proof of a general proposition? (Isn't that like wanting to ask: 'To what extent can we call a signpost a route?') Yet it surely does justify the application of A(c) to numbers. (PR XIV, §164b)

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20 On this point, see also WWK, pp. 33-4 and LWL, pp. 18-9.

> An induction doesn't prove the algebraic proposition, since only an equation can prove an equation. But it justifies the setting up of algebraic equations from the standpoint of their application to arithmetic. (PR XIV, §167d)

Notice that, in the last two passages above, Wittgenstein is saying the application of the algebraic formula A(c) to numbers is *justified* (*rechtfertigt*) by the induction. In this sense, the applicability of A(c) here does not "take care of itself". We need the induction in order to justify the applicability of A(c) to numbers. In contrast to this, we do not need anything beyond the arithmetical equations to justify the application of them to, say, plums or men.

In the sequence of the same chapter, Wittgenstein remarks that the algebraic formula only obtains its sense (its mathematical *existence*) by means of induction:

> An algebraic schema derives its sense from the way in which it is applied. So this must always be behind it. But then so must the inductive proof, since that justifies the application. (PR XIV, §167a)

> Through them alone the algebraic system becomes applicable to numbers.
> And so in a particular sense they are certainly the expression of something arithmetical, but as it were the expression of an arithmetical existence. (PR XIV, §167f)

If we remember that Wittgenstein equated mathematical existence (mathematical reality) with applicability, then it is clear from this last passage that Wittgenstein is saying that algebraic formulae, unlike arithmetical and geometrical, do not themselves guarantee their applicability in this case, but need to be supplemented with inductions.

In the same paragraph 167 of PR, Wittgenstein uses the pair of notions "sense / truth" to characterize the relation of induction to algebraic axioms. On one side, algebraic equations are not proved by induction and therefore do not acquire their truth through it. On the other side, induction gives *sense* to algebraic equations: it makes possible the application of algebra to arithmetic. Another pair of notions used by Wittgenstein to explain this relation is the pair "name / proposition". As stipulations, algebraic axioms like A(c) are more like *names*: they are not bipolar like propositions (another reason to say that that the induction does not prove it). But a name is only a name when it "denotes" something, that is, it receives its content only by pointing to its reference. Analogously, A(c) only receives its content by pointing to an induction.

I conclude that, for the middle Wittgenstein, algebra is not "its own application" in the sense arithmetic and geometry are and, thus, algebra is not autonomous (its existence does not follow from its essence). That is why, in

the case of algebra, we need to bridge the gap between its formulae and its applications. According to Wittgenstein, it is induction that bridges this gap between algebra and arithmetic.

6. Concluding remarks

In the preceding sections, I have intended to show why Wittgenstein should not be read as advocating a formalist view of mathematics. While he did use the formalist metaphor of a game with signs to dispel some misconceptions about the nature of mathematics, he still maintained (even in his middle period) Frege's idea that the applicability is something essential to mathematics proper. If I am right, Rodych's interpretation of the middle Wittgenstein as a (strong) formalist is misconceived. Arithmetic is autonomous not in the sense that it is essentially a sign-game with contingent applications, but in the sense that it displays the common form shared by each one of its applications in its own symbolism.

To return to Stenlund's characterization of Wittgenstein's conception of mathematics as "symbolic", it must be emphasized that it does not follow from the fact that mathematics is not "ontological" (not a theory about an independent reality) that it is logically separated from its applications. The assumption behind this implication is that the axioms of a mathematical systems become true (descriptive) propositions when applied to a "real world domain", that we are able to find many applications of a mathematical system by finding true relations in some domain that "match" those axioms. And then, if mathematics is not about something, it has to be cut off from its applications. But it seems to me that it is precisely this picture of the notion of application that Wittgenstein is trying to oppose:

> If we say "it must be essential to mathematics that it can be applied" we mean that its applica*bility* isn't the kind of thing I mean of a piece of wood when I say "I will be able to find many applications for it". (PG, p. 319)

As a final remark, I would like to suggest that the approximation made by Stenlund between symbolic mathematics and formalism was the result of an overemphasis on the operational aspect of mathematical symbols to the detriment of their pictorial character as symbols. While it is completely adequate to say that "a symbol is determined by how we operate with the sign for it" (Stenlund, 2015, p. 70) and that "it is the *operational* aspect of a symbol, its function in the calculus, its role in the manipulation and transformation of expressions, which *constitutes it as a symbol*" (*ibid*, p. 23), we must bear in

mind that the manipulation of symbols in arithmetic is not an end in itself, neither a means to "win" in the arithmetical game, but a means to call our attention to certain *aspects* of the symbolism. The function of an arithmetical proposition, as Wittgenstein conceived it in his middle period, is to "draw our attention to a particular aspect of the matter" (PR X, §114b). And the function of an arithmetical proof is to display this aspect in the symbolism itself.

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