NECESSARY TRUTH AND PROOF[*]

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ABSTRACT *What makes necessary truths true? I argue that all truth supervenes on how things are, and that necessary truths are no exception. What makes them true are proofs. But if so, the notion of proof needs to be generalized to include verification-transcendent proofs, proofs whose correctness exceeds our ability to verify it. It is incumbent on me, therefore, to show that arguments, such as Dummett's, that verification-truth is not compatible with the theory of meaning, are mistaken. The answer is that what we can conceive and construct far outstrips our actual abilities. I conclude by proposing a proof-theoretic account of modality, rejecting a claim of Armstrong's that modality can reside in non-modal truthmakers.*

Keywords *Truthmakers, Modality, Truth, Proof*

RESUMO *O que faz verdadeiras as verdades necessárias? Defendo que qualquer verdade sobrevém das coisas como elas são, e que as verdades necessárias não são exceções. O que as faz verdadeiras são provas. Mas, se assim for, a noção de prova precisa ser generalizada para incluir provas de verificação-transcendente, provas cuja correção extrapola a nossa própria habilidade de verificação. Além disso, tenho a incumbência de mostrar que argumentos, como o de Dummett, segundo o qual a verdade em termos de verificação não é compatível com a teoria do significado, não procedem. A resposta consiste no fato de que aquilo que podemos conceber e construir ultrapassa nossas habilidades efetivas. Concluo propondo um tratamento*

* Presented at the conference '(Anti-)Realisms, Logic and Metaphysics': Nancy, 28 June-1 July 2006

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das modalidades em termos de teoria da prova, rejeitando a afirmação de Armstrong de que modalidades podem residir em fazedores de verdade não modais.

Palavras-chave *Fazedores de verdade, Modalidades, Verdade, Prova*

1 Truthmaker Realism

The realist believes that truth is in no way dependent on our ability to detect it, so there can be verification-transcendent truths, truths which may forever elude our attempts to discover their truth. In what then, does their truth consist, if it is so divorced from our investigative powers? What the realist has resorted to over many years is the language of truth-making, now formalised into a theory of truth, Truthmaker Realism. At its heart lies the principle Truthmaker, which appears in various, mostly inequivalent, forms. The one I find plausible is the principle of the **Supervenience of Truth on So-Being**:

> **ST** Truth supervenes on how things are: there can be no difference in truth without a difference in how things are.

This is distinct from two stronger principles, that of the **Supervenience of Truth on Being**, that differences in truth require differences in what exists, and the yet stronger principle, **Truthmaker**, that whatever is true, something makes it true.

The main objection to this last principle, **Truthmaker**, concerns negative existentials, e.g., 'There are no unicorns' or 'Vulcan does not exist'. It is not that there is something which makes them true, rather, they are true because something (unicorns, Vulcan) does not exist. The objection to the **Supervenience of Truth** on Being is similar. Take, e.g., the fact that I am sitting down. This is true not because I exist, or that a standing me does not exist, but because of how I am, that I am sitting and not standing. Truth supervenes on how things are, not on what there is. Truth supervenes on so-being rather than being, using the terminology of Meinong's Principle of Independence. Nonetheless, in some cases, truth will supervene on what there is, will even be made true by what there is, as **Truthmaker** says. Take essence, for example. If a being's nature is essential to it, then what supervenes on its essence, or nature, will supervene on it too, since it could not exist without its essence, that is, without being as it is.

I want to defend the two-fold thesis that necessary truths require truth-makers, and that what make necessary truths true, are proofs. In insisting that every truth, including necessities, needs a ground for its truth, I am following Leibniz, who wrote in his *Monadology* (§§31-2):

> Our reasonings are based on two great principles, the *principle of contradiction* ... and the *principle of sufficient reason*, by virtue of which we consider that we can find no existent fact, no true assertion, without there being a sufficient reason why it is thus and not otherwise.

Truth supervenes on so-being, and even necessary truths require explanation why they are true. Indeed, they require more. They require explanation why they are necessarily true. That is supplied by the proof, which if it exists, exists of necessity, and entails immediately that its conclusion is true and necessary.

Why do necessary truths require truthmakers? It is often claimed that only contingent truths need truthmakers. Necessary truths cannot fail to be true, and so there is no explanatory need for anything to make them true. But this is circular: if they must be true, then they will indeed be true, but why must they be true? Simply calling them necessary truths does not make them true. Two thoughts are often adduced:

1. Necessary truths are entailed by all truths, so whatever makes any truth true makes all necessary truths true too. Hence necessary truths are made true, not by their own special truthmakers, but by all truth-makers, and indeed, the fact that they are made true by all truthmakers, not just some, is what makes them necessarily true.

This argument depends on two contentious premises:

NAQ Necessary truths are entailed by any proposition whatever, and

ET Truthmaking is closed under entailment, that is, any proposition is made true by the truthmakers of any proposition which entails it.

I reject **NAQ** for logical reasons, along with its flipside, **EFQ** (that an impossible proposition entails any proposition whatever); and I believe **ET** needs qualification, as below.

2. Necessary truths are empty of content, a doctrine left over in positivism from Wittgenstein's *Tractatus* even when its rationale there had been rejected. Hence there are no facts corresponding to necessary truths, which are purely analytic and so true solely in virtue of the meaning of the constituent words. Thus they require no truthmaker.

One reason to reject this line of thought goes along with the rejection of **NAQ**: not all necessary truths are equivalent (which they would be, given **NAQ**) and so they cannot all be empty of content.

Given the rejection of 2, and of **NAQ**, one needs to reconsider **ET**. For **ET** also suggests that entailment itself has no content and no truthmaker. But if 'α entails β' is made true by, say, s and α is made true by t, then β must be true (since α and 'α entails β' are true). But what makes β true? It cannot just be t (as decreed by **ET**) but some combination of s and t. Some notation: I write 'α entails β' as $\alpha \Rightarrow \beta$, and 's makes α true' as $s \vDash \alpha$ (i.e., s *forces* α). Then we can endorse a revised entailment principle:

RET If $s \vDash \alpha$ and $\alpha \Rightarrow \beta$ then $f(s) \vDash \beta$,

where $f(s)$ is some function of s. Of course, until something is said about the function f, this really says no more than that if α is true and α entails β, then β is true. What will give it substantive content is to relate f to the proof that makes $\alpha \Rightarrow \beta$ true.

An obvious objection to the claim that all necessary truths are established by proof appeals to the arguments of Kripke and Putnam. Their arguments appear to show that all true identities, including many *a posteriori* identities, are necessarily true. But *a posteriori* truths do not admit of proof, or they would be *a priori*.

There are two possible responses. First, suppose Kripke and Putnam are right. Nonetheless, the role of proof here is not to establish the truth of true identities but their necessity. What Kripke, following Barcan Marcus and others did, was to show that if $a = b$ is true then it is necessarily true. There is a general proof-scheme into which one slots '$a = b$' and whose conclusion is that $a = b$ is necessarily true. But this conclusion inherits from its empirical premise its own *a posteriori* status.

It's not clear if this solution is coherent. A better approach for the defender of "proof-maker realism" may be to reject Kripke and Putnam's arguments. They depend on too simplistic a semantics for modality and necessity (and too gullible a use of thought-experiments). The point is a general one about the utility of logic. Presented with a philosophical puzzle one may turn to logic to model it. The logic may then deliver an unpalatable result. Rather than accept the result, one may query the logic (in this case, the modal semantics), until a reflective equilibrium is reached. In the present case, that may only come once a semantics for modality has been formulated which admits contingent identity.[1]

In proposing proofs as truthmakers, I am not defending a form of conventionalism. But where traditional conventionalism falls to two objections, the involvement of proofs in establishing their truth forestalls

1 For such a semantics for contingent identity, see, e.g., Priest [2008, ch. 17].

them. The objections are first, Quine's observation (Quine [1936]) that the conventionalist account is circular, appearing to explain necessity as what necessarily followed from certain conventions, and secondly, that it fails to recognise the obvious difference between trivial necessities (e.g., 'Bachelors are unmarried') and deep necessities (e.g., Fermat's Last Theorem). The distinction between trivial proofs and deep proofs immediately speaks to the second of these objections. The first objection is dealt with by the autonomy of logic and mathematics. Logical and mathematical concepts are autonomous in that their meaning is given by the rules for their use. They are self-justifying. Permitting certain inferences regarding a logical or mathematical term, inferences which a convention or decision that the term should have a certain logical or mathematical meaning encapsulate, has ineluctable consequences for what follows from assertions containing the term. These consequences conform to the principle that one can legitimately extract from an assertion as much, but no more than, one must have correctly to make it.[2]

There are three further objections to the idea that proofs could act as truthmakers, however, First, a famous result of Kurt Gödel's may be invoked, where he appeared to show that truth outstrips proof, in that he showed that in any mathematical system of sufficient strength, there are true assertions which are not provable if the system is consistent. Secondly, it may be objected that the role of proofs is not ontological, to make certain truths true (and necessary), but epistemological, to demonstrate to ourselves that they are true. Finally, it may be objected that the project is circular, in that a proof only demonstrates that its conclusion must be accepted if its axioms are, and so the idea of proofs as truthmakers is regressive, explaining the truth of the conclusion at the expense of requiring demonstration of the truth of the axioms.

The third objection falls to the same inferentialist response as did Quine's objection above, in explaining how the meaning of logical and mathematical terms is encapsulated in the valid inferences in which they appear. The inference rules are self-justifying or autonomous. The first and second objections are dealt with *seriatim*, in §§ 2 and 3.

2 Incompleteness

Hilbert's programme was a response to the nineteenth century crisis in the foundations of mathematics. The nineteenth century opened with the

2 The inferentialist answer is developed further in Read [2000] and Read [2008].

seminal contributions of Cauchy and others towards getting clear about the foundations of analysis, as it had been developed from Newton's and Leibniz' beginnings. In particular, their elucidations needed to deal with the blatant inconsistency involved in, for example, dividing by a number which in the limit was zero. The end of the century saw the creation of modern set theory by Cantor and others, as a general language and formal theory which would, it was believed, at long last provide the firm and consistent framework for nineteenth century analysis. But the century ended in worse chaos than that in which it started, the vague suspicions of inconsistency at the beginning being replaced by formally proved contradictions at the close, such as Cantor's and Burali-Forti's paradoxes. The new century opened with the discovery of the most famous of the formal contradictions, that of Zermelo and Russell, of the set of all sets which are not members of themselves.

Hilbert's reaction, like others such as Russell's, was a retrenchment, an attempt to salvage as much as possible of classical analysis. But some careful restriction of mathematical procedures was clearly needed, and Hilbert's diagnosis, like Brouwer's, was that the problem lay in unfettered use of the infinite. What was needed was some finitistic restriction on mathematics together with a proof that when finitistic constraints were (in careful ways) lifted, the result would remain consistent. It was taken for granted–or supported by a traditional inductive proof–that finitistic methods could not induce contradiction; the lesson from the preceding centuries was that the problem arose only when finite methods were extended to deal with infinite classes. Behind this methodological approach lay an ontological interpretation. Talk of the finite was of the real; talk of the infinite should be treated "as if" it were true and referred to actual entities, but no reliance should be placed on this, and no derivation or use of the infinite was complete until all statements involved were again open to a finitistic interpretation. This ontological conception backed up the methodology. For orthodox philosophy, embodied for Hilbert in the critical philosophy of Kant, believes that the real cannot be contradictory. It is the ideal elements, Hilbert's "as if" or Kant's "phenomena" (appearances), which not only can be but are contradictory, as in the paralogisms. Hence arose the need in Hilbert's programme for proofs of the consistency–and finitistic proof, at that, of course–of the use of the ideal. Proof of conservative extension then shows that any inconsistency is restricted to the ideal elements.

Arithmetic was a seemingly straightforward case in point. Peano's postulates have no finite models. Every number has a successor, if the successors are equal the numbers are equal, and 0 is not a successor. So there can be no greatest number, and no circles. Hence all models are infinite. Thus there was an urgent need to show that arithmetic, formulated clearly and exactly in some such way as Peano's and Dedekind's, was consistent, and the expectation was that a proof would be readily forthcoming. As is well known, Gödel dashed these hopes, and with them, any likelihood that Hilbert's programme could trace a way out of the crisis. For Gödel showed first, that any sufficiently strong system of arithmetic such as Peano's, if consistent, is incomplete, in leaving some arithmetical truth unproven; and more appositely, as a corollary, that if consistent, its consistency cannot be demonstrated within arithmetic, and so *a fortiori* cannot be demonstrated in any finitistic subsystem.

The relevance of Gödel's result for us is that it appears to show that no system of proof is adequate to the role of truthmakers, for there are arithmetical statements which though true are, relative to any particular system of proof, unprovable in that system. Of course, any particular truth can be proved in some system, e.g., one in which it is an axiom. But no single system can establish the truth of all arithmetical truths, and so it cannot be their proofs (in that system) which make them true. The mistake in this reasoning is to accept Gödel's account of what constitutes a proof as relevant to our concerns. It is not. Gödel's concept of proof, which has become the norm, is of an array of formulae whose conformity to a set of rules is decidable, that is, one which can be checked in a finite time as being a proof or not. This accords with the epistemological needs of Hilbert's programme, and probably with other epistemological reservations. But we are not constrained by epistemology. It is completely plausible that some necessary truths, including arithmetical truths, are true in virtue of proofs which it is, for whatever reason, impossible for us to formulate. If this seems surprising, it is surprisingly easy to give an example.

There are, in Peano arithmetic formulated on the basis of first-order predicate logic, two ways to prove a universal proposition. Suppose the logical basis is given in natural deduction. Then besides the ∀I-rule:

$$\frac{A(u)}{\forall x A(x)}\,\forall\mathrm{I}$$

(where '*u*' is not free in any assumption on which the premise depends), whose conclusion has the form of a universal proposition, there is the rule of Mathematical Induction (Ind):

$$\frac{A(0) \quad A(u) \to A(u+1)}{\forall x A(x)} Ind$$

(where again, 'u' is not free in the assumptions on which the second premise depends), whose conclusion has the same form. However, in both cases, establishing the conclusion requires following a uniform method for every number, in the first to show A(u), where u is arbitrary, in the second to show A(u) → A(u + 1), again where u is arbitrary. One might conjecture, in the face of Gödel's result, that the incompleteness of arithmetic lies in the requirement of uniformity, required for the decidability of the Gödelian notion of proof. For suppose we introduce a rule with infinitely many premises, often termed the rule of infinite induction or ω-rule:[3]

$$\frac{A(0) \quad A(1) \quad A(2)\ldots\ldots}{\forall x A(x)} \omega - rule$$

where there is a premise of the form A(n) for every natural number n. Two things should be immediately obvious: first, we cannot now check in a finite time whether a proposed array, containing at least one instance of the ω-rule, conforms to the rules, for there are infinitely many sub-proofs to be checked. Hence, provability is no longer decidable. Thus, arithmetic with the ω-rule is not subject to Gödel's theorem. The proof predicate is not primitive recursive. Secondly, it is fairly obviously complete, since the side-condition on the ω-rule exactly matches the truth-condition for 'all', so $\forall x$A(x) is true (and provable) iff each instance A(n) is true (and provable).

The unprovable formula in Gödel's theorem is just such a universal formula, saying that it has no proof: it has the form $\forall x(T(x) \to \neg\exists y \mathsf{Prov}(y, x))$, where '$T$' holds uniquely of this formula, and $\mathsf{Prov}(y, x)$ says that y is (the Gödel number of) a proof of (the wff with Gödel number) x.[4] So too, is Goldbach's Conjecture, that every even number (greater than 2) is the sum of two primes. (Hence Gödel [1990, p. 305] referred to his undecidable sentences as of "Goldbach form".) Goldbach's Conjecture has resisted proof for a long time, and it may be that its proof cannot proceed in the usual way by a uniform method, but requires demonstration in a different way for each

3 Carnap [1937, § 14], Tarski [1956, pp. 258-61].

4 Clearly, in the presence of infinite proofs, the notion of Gödel number will need to be generalized.

natural number (or at least, for an infinite partition of the numbers) that it conforms to the Conjecture. Any such particular proof is obviously beyond us in practice, but not beyond our comprehension–we could not survey the indefinitely many different methods (if we could, there would be some way to treat them uniformly and finitely, contrary to hypothesis). But the general method of proof, via an infinitary rule of proof, the ω-rule, is easily seen to be both complete and, in itself, consistent (i.e., consistent relative to the system without it).

Although first-order arithmetic with the ω-rule is complete, second-order arithmetic with the same rule (i.e., based on second-order logic) is not, for there are second-order arithmetical truths (whose existence is no counterexample to the completeness of first-order ω-arithmetic) which cannot be proven in second-order ω-arithmetic. The reason is that there are uncountably many subsets, that is, properties, of natural numbers. But the problem here is not with proofs of the first-order quantified formula, $\forall x\mathrm{A}(x)$, but the second-order formula $\forall F\mathrm{A}(F)$. For this to be true, it is not sufficient that it be true of some ω-sequence of properties F, e.g., the recursive predicates (or sets), or the definable properties. For these constitute a small island of countability in a sea of uncountability. But now language seems to constrain us, for, as normally conceived, there are only countably many predicate expressions. It is not enough that $\mathrm{A}(F)$ be true for all predicates F. We need it true for all properties F. But uncountably many of these cannot be expressed in a countable language.[5]

The moral is not to be defeated, or intimidated by infinity. Indeed, this was Cantor's lesson, when he tamed the transfinite. Even with our confinement to finite modes of expression, we can gain an understanding and comprehension of indefinite orders of infinity. Consider, e.g., Paris and Harrington's proofs of arithmetical truths not provable in Peano arithmetic. Goodstein's theorem, for example, which claims that the Goodstein function eventually takes the value 0 for a sufficiently large, but finite argument, cannot be proved in first-order Peano arithmetic (of the kind subject to Gödel's theorem), but its proof using the assignment of transfinite indices (i.e., by transfinite induction) is concise and transparent. Note once again, that our comprehension of these proofs is not the point. The essential point to realise is that there are these proofs,

5 The assumption that languages are countable can be challenged. For example, infinite decimals are expressions (expression using base-10 notation), though no (non-recurring) infinite decimal can ever be written down. Nonetheless, there are uncountably many such decimals, as the usual diagonalization argument shows.

many as yet undiscovered, and perhaps incapable in the end of discovery (not least, since there infinitely many of them), whose existence is licensed by the meaning of the terms involved.

3 Anti-realism

The idea of verification-transcendent truth has been challenged many times over the past one hundred years, however. Can there be propositions whose truth we may be incapable of establishing? In particular, the challenge has been refined into a dilemma: of any proposition, either we are capable in principle of recognising its truth or falsity or if not, then we cannot have conferred on the relevant expression any clear meaning, that is, it does not express any proposition. If an expression has a clear meaning, then there are circumstances whose obtaining is necessary and sufficient for the truth of the proposition expressed and which we can recognise to obtain when they do, and not when they do not.

The main thesis which Dummett uses to support these claims is the manifestation challenge, that every aspect of meaning so conceived must be capable of manifestation. The argument for this conclusion relies on the consideration that meaning can and must be learned. Language is a social activity which is transmitted from generation to generation. Somehow, the child or language-learner comes to understand the meaning of the terms of the language, and indeed, those terms have no meaning other than that given to them by the practice of linguistic communication. Hence there can be no element of meaning which is not exhibited in some linguistic behaviour and which the language-learner can come to appreciate. All aspects of meaning must be capable of being manifested and acquired from participation in the practice.

The meaning of a proposition determines when it has been correctly uttered and when not. In particular, its meaning is such that, uttered in certain circumstances it is true, uttered in others it is false. In order to be capable of manifestation, those truth-conditions must be ones we can recognise to obtain or not to obtain. For if we could not do so, we could not make manifest how its truth depended on those circumstances and so could not articulate its meaning in terms of those truth-conditions. Hence the conditions of truth of any proposition with a determinate meaning must be capable, in principle, of being recognised as verifying or falsifying it. There can be no verification-transcendent propositions.

To the extent to which this argument is compelling we are here confronted by a paradox. For it seems that there are clear cases of propositions we understand yet which we also realise we could never verify. Dummett gives an example himself: ‘A city will never be built here’.[6] It is straightforward enough to recognise that the proposition has been falsified: finding a city there. But to verify it, we would need to complete an infinite task of checking throughout eternity that no city had been built, and however late we left our check, that would verify only up to that time that no city had been built, leaving countless ages of subsequent history when one might appear. The example is reminiscent of the situation we noted might obtain with Goldbach’s Conjecture, which might require checking separately and independently of each even number that it was the sum of two primes. Indeed, the situation seems even more likely in the empirical case. It seems implausible that there be a general reason, valid for all times, why a city will not be built–though, of course, there could be such a reason, such as the ineradicable presence of toxic heavy metals, or the absence of an adequate water supply. Even so, to be assured that a city will never be built, means ensuring these obstacles will never be overcome, and that seems to require an open-ended check which would remain forever uncompleted.

How should this paradox be resolved? The claim is not that we cannot understand such a proposition as ‘A city will never be built here’. The claim is that the requirement that we be capable of manifesting a grasp of its meaning entails that we should draw back from a realist avowal of bivalence for it–that it must be either true or false–and so that we should manifest our understanding not in terms of a grasp of when it is true or false (for we have no justification for a belief that it must be one or the other), but in terms of when we are justified in asserting it and when are justified in denying it. Our human limitations necessitate that we could not envisage a circumstance obtaining whose recognition we would take as showing the proposition true, for that would require infinite knowledge of the absence of a city at all times. Hence our understanding and its manifestation must relate to what grounds we would accept as justifying its assertion or denial. ‘Never’ gets its meaning from quantification over finite and surveyable domains. Extrapolating it to the supposed indefinite future yields a proposition we could never be in a position to assert–a verification-transcendent one–and so one whose truth-conditions we cannot conceive. Consequently, we are not entitled to claim it is either true or false, and must reject a realist interpretation of there being a state of affairs whose existence makes it true or false.

6 Dummett [1978, p. 16].

The fault with all such anti-realist arguments is that they systematically underestimate our conceptual powers. Dummett [1978, p. 17] concedes that although we cannot ourselves check on the possible truth of 'A city will never be built here', we can conceive of a being (he calls it 'God') who could verify it, by an infinite check. Our conception of the truth of the proposition is what such a being would have verified by verifying each successive instance–'A city has not been built here yet', 'A city has still not been built here', and so on into the indefinite future. If we can conceive of such a being, then we can comprehend what such a being would have verified and of which it had such infinite knowledge. To be sure, there are anxieties, precisely those which prompted Hilbert's programme and the intuitionistic philosophy of mathematics, about the coherence of such an extrapolation of our finite powers. Yet there had always been a scepticism about the coherence of the notion of the infinite. Cantor's bold step was to propose a careful and systematic treatment of the infinite on a par with the finite.

When the notorious antinomies such as Cantor's and Russell's were discovered, a natural reaction was to continue such scepticism about the notion of the infinite, and blame them on Cantor's inception. But a careful diagnosis of the antinomies reveals that they offer "old wine in new bottles". Russell's paradox of the set of all sets which are not members of themselves has an immediate analogy which Russell recognised in the paradox of heterologicality, of the adjective which applies to all adjectives which do not apply to themselves, hence to the property of not applying to itself, that is, of not being true-of-itself, and so relates back to the infamous Liar paradox, of the proposition which says of itself that it is not true. That Russell (and Zermelo) discovered the paradox by reflection on the proof of Cantor's theorem does not show that something must be inherently wrong with Cantor's theorem itself. Rather, Cantor's world of the transfinite offered new possibilities for old paradoxes to arise again. What is needed is a satisfactory diagnosis of those paradoxes, not a hasty and universal ban on all talk of the infinite, any more than an analogous ban on self-reference in the face of the semantic paradoxes.

What lies at the heart of the realist's confidence in his position is, we noted, a belief that reality cannot be incoherent or inconsistent. Only our descriptions, beliefs and theories can. Clearly, something was wrong with the foundations proposed for late nineteenth century mathematics. But the anti-realist reaction, Hilbertian, Brouwerian or whatever, which jettisons belief in an underpinning reality in favour of revised forms of description, cannot be the right one. There cannot be anything wrong with infinite collections in themselves. The error must lie in our descriptions of them. The descriptions created by Cantor and others needed revision; but the collections themselves are as real as ever, and so are either as described or not.

Consider, once again, 'A city will never be built here'. According to (**ST**), either there is something whose being built makes it false or whose absence makes it true. Clearly, that thing is a city. If a city is built here, the proposition will be false; so if it is true, whatever would have falsified it must not have existed or be going to exist. This is no argument for bivalence and realism, of course. Rather, it is an affirmation, and explanation, of the realist's belief in there being a truth-condition for the proposition.

Dummett's other famous (purported) counterexample to bivalence is also unconvincing, but for a different reason. Consider Jones, he says, who was never placed in circumstances which might have established whether or not he was brave, and is now, sadly, dead. Was Jones brave? Dummett's proposal is that the sense of 'brave' is dispositional, such that 'Jones was brave' means 'If Jones had encountered danger, he would have acted bravely'. That may be, but he claims that 'Jones was not brave' means 'If Jones had encountered danger, he would not have acted bravely'.[7] Although Stalnaker's semantics for counterfactuals validates Conditional Excluded Middle, this is not acceptable, as Lewis [1973, pp. 79-82] observed. Jones' bravery is as good a counterexample as others: if Jones was not brave, he might nonetheless have behaved by chance as if he were. The correct analysis of 'Jones was not brave' is 'If Jones had encountered danger, he might not have acted bravely'.[8] So analysed, Jones is definitely either brave or not, in that either he would have acted thus or he might not have. Dummett's example only fails to satisfy Excluded Middle because he analyses it wrongly. If we can show that Jones might not have acted appropriately in the relevant circumstances, we have shown that it is false that he would have acted so, and hence false that he was brave. Thus, whatever the limitations on our now showing it, either Jones was brave or he was not.

But we cannot leave the example there. Perhaps Lewis can: the behaviour of Jones' counterparts acts for Lewis as a truthmaker of the modal proposition about Jones. I eschew belief in the reality of other worlds and the existence of their denizens. Fortunately, I am not presently being tortured. So there is no state of affairs of my being tortured, not even a non-actual such state of affairs.

7 Dummett [1978, p. 15]; cf. Dummett [1991, pp. 342, 347].

8 In symbols, Dummett expresses 'Jones was brave or Jones was not brave' as $(\alpha \;\Box\!\!\rightarrow \beta) \vee (\alpha \;\Box\!\!\rightarrow \neg\beta)$, where '$\Box\!\!\rightarrow$' is the subjunctive conditional, 'If it were α it would be β'. The correct analysis is $(\alpha \;\Box\!\!\rightarrow \beta) \vee (\alpha \;\Diamond\!\!\rightarrow \neg\beta)$, where $\alpha \;\Diamond\!\!\rightarrow \beta$ is equivalent to $\neg(\alpha \;\Box\!\!\rightarrow \neg\beta)$, that is, it is not the case that if it were α it would not be β, i.e., if it were α it might be β.

Without being actual, there can be no unity to constitute a state of affairs.[9] Nor is there any counterpart of me who is being (non-actually) tortured. (What an awful thought, that I escape torture only at his expense.) So such counterparts cannot act as truthmakers. What, then, does make 'Jones was brave' true or false? It is the fact that Jones was, or was not brave, his actual mental qualities. It is Jones' make-up which determines his behaviour, and so determines in the fiction, the behaviour of his counterparts–how he would act in such and such circumstances, not vice versa. It was disbelief in the bare truth of dispositional analyses which provided the initial spark for talk of truthmakers. C.B. Martin could not accept that phenomenalists or logical behaviourists had offered a satisfactory analysis of perception or belief or whatever if it ended in a conditional concerning possible sense-data or actions. What could make such conditionals concerning unperceived sense-data or unperformed actions true if all there are, are actual sense-data and actions? Unless these theories came up with elements of a kind they sought to avoid (such as Russell's sensibilia or categorical materialist bases), they were not only incomplete but incompletable. All truths must have a truthmaker, whether that truthmaker is a categorical basis (for Armstrong) or a power (for Martin) or a thing *qua* truthmaker (for Lewis). Or, by **ST**, at the very least, there is some property which Jones has or lacks which grounds his disposition to behave.

Dummett's comments show that, if he is right, we can accept the demand for truthmakers and still resist realism. But his grounds for resistance are mistaken. It is not our faith in the reality of the infinite which creates the problem. It merely allows existing problems to reappear in dramatic and virulent form. There are truths whose meaning and truth-conditions we can understand and yet whose truth (or falsity) we may be forever incapable of recognising. In particular, our understanding of mathematical notions allows us to conceive of there being proofs whose existence may lie beyond our ability to grasp or survey them. Nonetheless, if they do exist, then they establish the truth of their conclusions of necessity.

4 Logical Pluralism

Although intuitionism began as a philosophy of mathematics, a revisionary programme aiming for a fresh approach to the foundational crisis in late nineteenth century mathematics, intuitionistic logic has of late had a new

9 For an elaboration of this argument, see Read [2005].

lease of life in theoretical computer science. It is one of the prime examples of a new breed of logical relativism, what has come recently to be called "logical pluralism". The idea is that there are many logics, some suited for one purpose, others for another. A leading example is so-called linear logic. Its inventor, Girard, frequently remarked that "linear logic is not just another exotic logic". What he meant was that linear logic was an all-embracing approach to computational issues, in which the actual logic was but a tool in the analysis of computation. This logic had in fact been considered, at least in a fragmentary way, by Church [1951] in his "weak theory of implication". Once again, Church did not advocate his weak theory as the ultimate truth about implication. Rather, it was a useful tool with which to explore the differential effects of certain implicational assumptions. The weak theory encapsulated everything which (Church thought) was uncontentious about implication (primarily prefixing, suffixing and permutation), and any other assumptions (above all, contraction and weakening) could then be separately considered. So too with linear logic. Basic assumptions are built in (though in subsequent variants, even these have been revised) and the particular effects of non-linearity, namely, contraction and weakening (i.e., multiple and zero or vacuous uses of assumptions) can be separately studied through the modal ("exponential") operators.

Linear logic started as a methodology, not a philosophy of logic. It proves its usefulness as a tool for the study of computation, but some now claim linear proofs give an insight into the notion of valid inference.[10] The reverse was the case with intuitionism. Originally conceived as a philosophical response to the foundational questions concerning mathematics, its recent popularity has been methodological, once again, as a tool for the study of computation. The interest is in what can be achieved–that is, computed–with the finite, but theoretically unlimited, resources of mechanical procedures. Rarely is this coupled to any claim that non-computable functions are suspect or non-constructive results not to be trusted. Rather, the attraction of intuitionistic, or more generally, constructive methods is their usefulness in studying what can, in principle, be implemented on a computer.

At the heart of this methodology lies the so-called Curry-Howard correspondence, or Curry-Howard isomorphism. By this correspondence, the formulae of propositional intuitionistic logic are seen to match the types of function terms in a λ-calculus, such that '&' matches Cartesian product, '$\rightarrow$'

10 See the radical anti-realism of Jacques Dubucs: e.g., Dubucs [2002, p. 214].

matches functional application, '∨' matches disjoint union (or direct sum) and '⊥' (absurdity) matches the empty type. A function term is well-formed just when its type is (intuitionistically) provable, and the proof articulates how the value of the term is computed.

Such a constructivist interpretation of intuitionistic logic does not constitute logical pluralism, however. For intuitionistic provability is taken to show computability, not validity. Logical pluralism is better illustrated by the claim, for example, that classical logic is valid in finite domains, intuitionistic logic valid in infinite domains, and perhaps, relevance, or some other paraconsistent logic, valid in inconsistent domains (e.g., databases). This doctrine is refuted by an argument of Langford's and Quine's.[11] They mistakenly took the argument as a defence of classical logic. It cannot be that, at least, not without some supplementary consideration in favour of distinctively classical principles. What it does show is that logic is not relative to other purposes. It does not make sense to say that an argument is valid for purposes X but not valid for purposes Y. The reason is that truth is not relative, as Plato showed. For either truth is relative, in which case relativism is false for me (or Plato), or it is not relative. Either way, the doctrine of the relativity of truth is false. So too for any suggestion that validity is relative. For to say that an argument is valid is to say that the truth of the premises guarantees the truth of the conclusion, that is, that the argument preserves truth. So if the argument were valid for purposes X, it would preserve truth. Hence it preserves truth for purposes Y, since truth is independent of what purposes one has.

Suppose, for example, that it is claimed that classical logic is valid in finite domains, but only intuitionistic logic is valid in infinite domains. Then there will be counterexamples to classical principles in infinite domains. But if there are counterexamples to a principle, it is not valid. So classical logic is not valid, regardless of what domain is in question. Or suppose it is said that classical logic is valid in consistent situations, but only a paraconsistent logic is valid in inconsistent situations. Then there are counterexamples to classical principles, and so those principles are not valid *tout court*.

Constructivism is acceptable, therefore, as a methodology for the study of finitary procedures, but it must be rejected as an account of validity and truth. For there are verification-transcendent truths, as we noted in § 3. Moreover, what should be resisted is the all-pervasive finitary interpretation of terms such

11 Langford [1928, p. 582], Quine [1970, ch. 6].

as ‘proof’, ‘construction’ and ‘procedure’. It is nowadays almost universal to find these terms given a finitary meaning. We noted this sense of ‘proof’ in § 2. The very title ‘constructivism’ exhibits the phenomenon, for ‘constructive procedures’ is now taken to mean ‘procedures determining an outcome in a finite time’. But traditionally, many constructions were non-terminating, or as one would now say, “non-constructive”. The terminology fortunately lives on in many mathematical textbooks. Consider the famous “construction” of the reals from sets or cuts of rationals. Stewart and Tall, in their classic work on *The Foundations of Mathematics*, write:

> we must solve the problem of constructing a complete ordered field $\mathbb{R}$ starting from $\mathbb{N}_0$ . . . First the integers $\mathbb{Z}$ are constructed from $\mathbb{N}_0$, and the rationals $\mathbb{Q}$ from $\mathbb{Z}$. To construct $\mathbb{R}$ from $\mathbb{Q}$ is a more taxing operation . . . . (Stewart and Tall [1977, p. 173])

Ebbinghaus, in his book on Numbers, writes:

> Quite apart from its use in the definition of real numbers, the Cantor construction with fundamental sequences has turned out to be the most fruitful. (Ebbinghaus [1991, p. 40])

Such constructions are infinitary, and entirely “non-constructive”. The Dedekind cut construction, for example, identifies the irrational number $\sqrt{2}$ with the infinite (completed) totality of all rationals whose square is less than 2.

What I have done elsewhere [Read, 2000] is to use some of the insights of constructive (i.e., finitary) proof theory in giving a realist account of meaning in terms of proofs. The proofs in question are arrays of formulae according with carefully specified rules, but with no constraint that it be possible to check, in a finite time, whether those rules have been obeyed. That is, whether a particular array is a proof may itself be verification-transcendent. Nonetheless, I claim, the concept of proof articulated there is perfectly comprehensible and coherent. But I do not propose to embark on that articulation here. I want to close by considering two further aspects of modality, namely, possibility and contingency.

5 Contingency

My main topic is necessity, and what makes necessary truths true. But there is another side to that coin, and it would be a mistake to suppose that one can develop a theory of the one independently of the other. In particular, recall that I rejected the suggestion that what makes it true that, for example, I might undergo torture is that some counterpart of me in another possible world does do so. Such possible worlds are a mere *façon de parler* and cannot serve as truthmakers. But the question then arises, what is the truthmaker of this proposition? What does make possibilities possible?

Some possibilities are possible because they are actual, of course. If I am in agony, then it is clearly possible for me to be in agony. For $\ulcorner p \urcorner$ entails that $\ulcorner p \urcorner$ is possible, for all p. So by (RET), if $s \vDash p$ then $\alpha(s) \vDash \Diamond p$, where α is the one-step inference of $\Diamond$I.[12] But suppose $\ulcorner p \urcorner$ is false, but nonetheless, contingently so. Then $\ulcorner p \urcorner$ is still possible. What makes it true that $\ulcorner p \urcorner$ is possible, and indeed, that it is contingent? It cannot be the truthmaker for $\ulcorner p \urcorner$, since $\ulcorner p \urcorner$ has no truthmaker–*ex hypothesi* it is false.

David Armstrong claims that what makes $\ulcorner \Diamond p \urcorner$ true in this case is whatever makes $\ulcorner \neg p \urcorner$ true. His argument is this:[13] suppose $\ulcorner p \urcorner$ is false and contingent. Then $\ulcorner \neg p \urcorner$ is true and contingent also. But given $\ulcorner \neg p \urcorner$ and given that it is contingent, the truth of $\ulcorner \Diamond p \urcorner$ is entailed. He draws back from endorsing (ET) in full generality, but he claims that it seems to hold in a wide variety of cases, such as here. Armstrong [2002] spells out the argument in more detail in his contribution to the *Festschrift* for Hugh Mellor. Suppose $T \vDash p$, that is, T makes p true. Let us represent 'it is contingent that p' by 'Cp'. Then $T \vDash Cp$. For, since $\ulcorner p \urcorner$ is contingent, so too is the existence of T. "Could the contingency of T lie outside T?", he asks. "It does not seem possible. It cannot be a relation that T has to something beyond itself. So T is the truthmaker for the proposition 'p is contingent'."[14] Armstrong's reasoning can be formalized like this: suppose $T \vDash p$ and $\ulcorner p \urcorner$ is contingent. Then $T \vDash Cp$. He continues (*loc.cit.*):

Suppose	(1)	$T \vDash p$	
Then	(2)	$T \vDash Cp$	by the above reasoning
So	(3)	$T \vDash p \mathbin{\&} Cp$	
But	(4)	$p \mathbin{\&} Cp \Rightarrow \Diamond\neg p$	
So	(5)	$T \vDash \Diamond\neg p$	by his (restricted) Entailment principle

12 See, e.g., Read [2008]. Note that a one-step inference is a function, from one proof to another.
13 See Armstrong [2000, pp. 154-5]; Armstrong [2002, p. 15]; Armstrong [2004, §7.2].
14 Armstrong [2002, p. 15].

The final steps of this argument are trivial. ⌜Cp⌝ means ⌜$\Diamond p$ & $\Diamond\neg p$⌝, so clearly if $T \vDash Cp$ then $T \vDash \Diamond\neg p$, or at least, something very close to T does so, if we prefer (RET) to (ET) or some such version. The real puzzle is the step from line (1) to line (2), from $T \vDash p$ to $T \vDash Cp$, provided ⌜p⌝ is contingent. Armstrong's stated concern is to avoid, if he can, postulating a special categorical property of contingency *in re*, which he hopes the above argument will allow him to do. But what is the justification for line (2), rather than its motivation? Armstrong [2004, p. 84] appeals to **ET**. He claims that

(*) if p is contingent, then p entails $\Diamond\neg p$.

So, since $T \vDash p$, $T \vDash \Diamond\neg p$ by **ET**. But one cannot appeal to **ET** here, for ⌜p⌝ does not entail ⌜Cp⌝, even when ⌜p⌝ is contingent. Indeed, no wff of the form $p \Rightarrow Cp$ is valid in S5, for atomic p. In fact, note that ⌜Cp⌝ is itself never contingent, for $Cp \Rightarrow \Box Cp$ (given S5-principles). If ⌜p⌝ is contingent, then it is necessary that ⌜p⌝ is contingent. But it is a basic tenet of relevant logic that the necessary does not follow from the contingent, and indeed by a result of Coffa's, contingencies do not relevantly entail necessitives (that is, wffs of the form $\Box\alpha$) unless they already contain them as a part.[15] Whatever makes ⌜Cp⌝ true must make it necessarily true. So suppose ⌜p⌝ is contingent, and that the proposition asserting the existence of what makes it true does not contain any necessitives essentially. Then it cannnot make ⌜Cp⌝ true, for by Coffa's result, the proposition asserting its existence cannot entail ⌜Cp⌝, since ⌜Cp⌝ is a necessitive. Whatever makes (contingent) ⌜p⌝ true must exist only contingently, but what makes ⌜Cp⌝ true must exist of necessity. So far from being a reason for it, (1) is inconsistent with (2). What makes ⌜p⌝ true, where ⌜p⌝ is contingent, cannot make ⌜Cp⌝ true.

This ironically shows that Armstrong's subsequent defence of (*) collapses. Armstrong writes:

> Given the attractive S5 modal logic, if p is contingent, it is a necessary truth that it is contingent. This may help to quell any doubts we may have about step [(*)] in the argument. (Armstrong [2004, pp. 84-5])

Quite the contrary. It is the fact that (*) has a necessarily true conclusion which shows that it must fail.

15 See Anderson and Belnap [1975, §§ 22.2.1-2]. Armstrong [2004, p. 10] concedes that the entailment in **ET** must be relevant.

Reverting to our original case, we are left with the original puzzle: if ⌜p⌝ is false but contingent, what makes it possible that p? The answer lies in a remark of Pierre Bayle's, cited in Leibniz' *Theodicy* (§173): "the possible is whatever does not contain a contradiction." ⌜$\Diamond p$⌝ is equivalent to ⌜$\neg\Box\neg p$⌝, and that is true, by (**ST**), if there is no proof of ⌜$\neg p$⌝. According to (**ST**), ⌜$\neg p$⌝ can be true simply by default, that is, in the absence of a truthmaker for ⌜p⌝. But that is not enough for the truth of ⌜$\Box\neg p$⌝, that is, to make it necessarily false that p. For that we need something to ensure that ⌜p⌝ is false, and as we have seen, necessary truths require proofs to make them true. But what would a proof of ⌜$\neg p$⌝ be? We can learn here, as so often, from the constructivists. Intuitionistic logic is often developed on the basis of the connectives '&', '$\vee$', '$\rightarrow$' and '$\perp$', and '$\neg p$' is defined as '$p \rightarrow \perp$'. Indeed, this is sometimes referred to as an intuitionistic or constructivist definition of negation. But its constructive character is given by imbuing '$\rightarrow$' with that character. Realist negation can also be taken to identify '$\neg p$' with '$p \rightarrow \perp$', provided the theory of '$\rightarrow$' is suitably strong and realist, and for a suitable choice of '$\perp$'.[16]

Hence, what makes it possible that I might be in insufferable agony is that nothing rules it out, that there is no proof that shows that the assumption that I am in agony is absurd. Fortunately, I am not. But there is nothing absurd in the suggestion that I might be. Hence it is possible.[17]

In general, necessary truths require proofs to make them true and non-necessary truths require their absence. Some of those proofs will be simple and straightforward; others can be more complex than the human imagination can conceive. Nonetheless, what the existence of such proofs depends on are the basic inferences which encapsulate the meanings of the terms and operations involved.

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16 See Read [2000].

17 Note that nothing I have said above commits me to the S5 principle $\Diamond p \rightarrow \Box\Diamond p$. If one were committed to it, one would need a proof that there is no proof of $\neg p$. More plausibly, a theory of necessity as proof will reject $\Diamond p \rightarrow \Box\Diamond p$ and endorse an S4 theory of modality.

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