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Non-linear vibration of Euler-Bernoulli beams

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Abstract

In this paper, variational iteration (VIM) and parametrized perturbation (PPM) methods have been used to investigate non-linear vibration of Euler-Bernoulli beams subjected to the axial loads. The proposed methods do not require small parameter in the equation which is difficult to be found for nonlinear problems. Comparison of VIM and PPM with Runge-Kutta 4th leads to highly accurate solutions.

Keywords

Variational Iteration Method (VIM), Parametrized Perturbation Method (PPM), Galerkin method, non-linear vibration, Euler-Bernoulli beam.

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1 INTRODUCTION

The demand for engineering structures is continuously increasing. Aerospace vehicles, bridges, and automobiles are examples of these structures. Many aspects have to be taken into consideration in the design of these structures to improve their performance and extend their life. One aspect of the design process is the dynamic response of structures. The dynamics of distributed-parameter and continuous systems, like beams, were governed by linear and nonlinear partial-differential equations in space and time. It was difficult to find the exact or closed-form solutions for nonlinear problems. Consequently, researchers were used two classes of approximate solutions of initial boundary-value problems: numerical techniques [28, 31], and approximate analytical methods [2, 26]. For strongly non-linear partial-differential, direct techniques, such as perturbation methods, were not utilized to solve directly the non-linear partial-differential equations and associated boundary conditions. Therefore first partial-differential equations are discretized into a set of non-linear ordinary-differential equations using the Galerkin approach and the governing problems are then solved analytically in time domain.

Approximate methods for studying non-linear vibrations of beams are important for investigating and designing purposes. In recent years, some promising approximate analytical solutions have been proposed, such as Frequency Amplitude Formulation [13], Variational Iteration [5, 6, 14, 17], Homotopy-Perturbation [3, 4, 7, 24], Parametrized-Perturbation [18], Max-Min [15, 19, 29], Differential Transform Method [16], Adomian Decomposition Method [22], Energy Balance [23, 30], etc.

Kopmaz et al. [20] considered different approaches to describing the relationship between the bending moment and curvature of an Euler-Bernoulli beam undergoing a large deformation. Then, in the case of a cantilevered beam subjected to a single moment at its free end, the difference between the linear and the nonlinear theories based on both the mathematical curvature and the physical curvature was shown. Biondi and Caddemi [8] studied the problem of the integration of the static governing equations of the uniform Euler-Bernoulli beams with discontinuities, considering the flexural stiffness and slope discontinuities.

The vibration problems of uniform Euler- Bernoulli beams can be solved by analytical or approximate approaches [10, 21]. Pirbodaghi et al. [25] studied non-linear vibration behaviour of geometrically non-linear Euler-Bernoulli beams subjected to axial loads using homotopy analysis method. Also, the effect of vibration amplitude on the non-linear frequency and buckling load is discussed. Burgreen [9] investigated the free vibrations of a simply supported buckled beam using a single-mode discretization. He pointed out the natural frequencies of buckled beams depend on the amplitude of vibration. Eisley [11, 12] used a single-mode discretization to investigate the forced vibrations of buckled beams and plates. He considered both simply supported and clamped-clamped boundary conditions. For a clamped-clamped buckled beam, Eisley [11, 12] used the first buckling mode in the discretization procedure. He obtained similar forms of the governing equations for simply supported and clamped-clamped buckled beams.

The main purpose of this study is to obtain the analytical expression for geometrically non-linear vibration of clamped-clamped Euler-Bernoulli beams fixed at one end. Geometric non-linearity arises from non-linear strain-displacement relationships. This type of nonlinearity is most commonly treated in the literature. Sources of this type of nonlinearity include mid-plane stretching, large curvatures of structural elements, and large rotation of elements. First, the governing non-linear partial differential equation using Galerkin method was reduced to a single non-linear ordinary differential equation. It was then assumed that only fundamental mode was excited. The later equation was solved analytically in time domain using VIM and PPM. Ultimately, VIM and PPM methods are compared with Runge-Kutta 4th method.

2 DESCRIPTION OF THE PROBLEM

Consider a straight beam on an elastic foundation with length $\boldsymbol{L}$, a cross-section $\boldsymbol{A}$, a mass per unit length $\boldsymbol{\mu}$, moment of inertia $\boldsymbol{I}$, and modulus of elasticity $\boldsymbol{E}$ that subjected to an axial force of magnitude $\tilde{F}$ as shown in Fig. 1. It is assumed that the cross-sectional area of the beam is uniform and its material is homogenous. The beam is also modeled according to the Euler-Bernoulli beam theory. Planes of the cross sections remain planes after deformation, straight lines normal to the mid-plane of the beam remain normal, and straight lines in the transverse direction of the cross section do not change length. The first assumption ignores the in plane deformation. The second assumption ignores the transverse shear strains and consequently the rotation of the cross section is due to bending only. The last assumption, which is called the incompressibility condition, assumes no transverse normal strains. The last two assumptions are the basis of the Euler-Bernoulli beam theory [27].

The equation of motion including the effects of mid-plane stretching is given by:

$$EI\frac{\partial^4\tilde{W}}{\partial\tilde{X}^4}+\mu\frac{\partial^2\tilde{W}}{\partial\tilde{t}^2}+\tilde{F}\frac{\partial^2\tilde{W}}{\partial\tilde{X}^2}+C\frac{\partial\tilde{W}}{\partial\tilde{t}}+\tilde{K}\tilde{W}-\frac{EA}{2L}\frac{\partial^2\tilde{W}}{\partial\tilde{X}^2}\int_0^L\left(\frac{\partial\tilde{W}}{\partial\tilde{X}}\right)^2 d\tilde{X}=U(\tilde{X},\tilde{t}) \tag{1}$$

Where C is the viscous damping coefficient, $\tilde{K}$is a foundation modulus and U is a distributed load in the transverse direction.

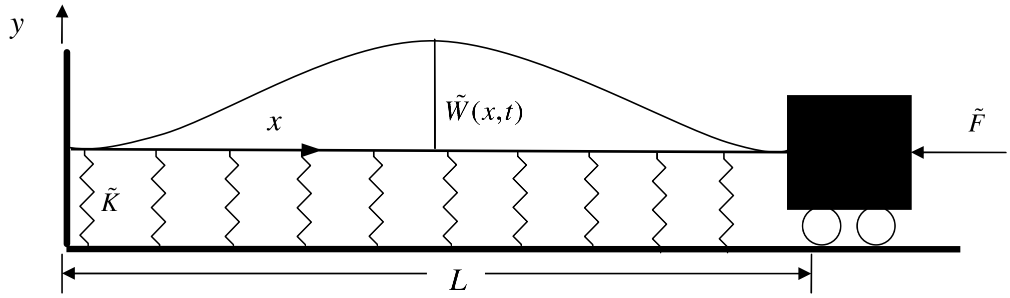

Figure 1 A schematic of an Euler-Bernoulli beam subjected to an axial load.

Assume the non-conservative forces were equal to zero. Therefore Eq. (1) can be written as follows:

$$EI\frac{\partial^4\tilde{W}}{\partial\tilde{X}^4}+\mu\frac{\partial^2\tilde{W}}{\partial\tilde{t}^2}+\tilde{F}\frac{\partial^2\tilde{W}}{\partial\tilde{X}^2}+\tilde{K}\tilde{W}-\frac{EA}{2L}\frac{\partial^2\tilde{W}}{\partial\tilde{X}^2}\int_0^L\left(\frac{\partial\tilde{W}}{\partial\tilde{X}}\right)^2 d\tilde{X}=0. \tag{2}$$

For convenience, the following non-dimensional variables are used:

$$X=\frac{\tilde{X}}{L},\qquad W=\frac{\tilde{W}}{R},\qquad t=\tilde{t}\sqrt{\frac{EI}{\mu L^4}},\qquad F=\frac{\tilde{F}L^2}{EI},\qquad K=\frac{\tilde{K}L^4}{EI}. \tag{3}$$

Where $R=(I/A)^{0.5}$ is the radius of gyration of the cross-section. As a result, Eq. (2) can be written as follows:

$$\frac{\partial^4W}{\partial X^4}+\frac{\partial^2W}{\partial t^2}+F\frac{\partial^2W}{\partial X^2}+KW-\frac{1}{2}\frac{\partial^2W}{\partial X^2}\int_0^1\left(\frac{\partial W}{\partial X}\right)^2 dX=0. \tag{4}$$

Assuming $W(X,t)=\phi(X)\psi(t)$where$\phi(X)$is the first eigenmode of the beam [32] and applying the Galerkin method, the equation of motion is obtained as follows:

$$\ddot{\psi}_{(t)}+\alpha\psi_{(t)}+\beta\psi_{(t)}^3=0 \tag{5}$$

Where $\alpha = \alpha_1 + \alpha_2 F + K$ and α_1 , α_2 and β are as follows:

$$\alpha_1 = \frac{\int_0^1 \phi^{iv}\phi dx}{\int_0^1 \phi^2 dx}, \qquad \alpha_2 = \frac{\int_0^1 \phi''\phi dx}{\int_0^1 \phi^2 dx}, \qquad \beta = \frac{-0.5\int_0^1 \left(\phi''\int_0^1 \phi'^2 dx\right)\phi dx}{\int_0^1 \phi^2 dx} \tag{6}$$

The Eq. (5) is the governing non-linear vibration of Euler-Bernoulli beams. The center of the beam subjected to the following initial conditions:

$$\psi_{(0)} = A, \qquad \dot{\psi}_{(0)} = 0 \tag{7}$$

where A denotes the non-dimensional maximum amplitude of oscillation.

3 BASIC IDEA OF VARIATIONAL ITERATION METHOD

To illustrate the basic concepts of the VIM, we consider the following differential equation:

$$Lu + Nu = g(t). \tag{8}$$

Where L is a linear operator, N a nonlinear operator and *g(t)* an inhomogeneous term. According to VIM, we can write down a correction functional as follows:

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda\left(Lu_n(\eta) + N\tilde{u}_n(\eta) - g(\eta)\right)d\eta. \tag{9}$$

Where λ is a general Lagrange multiplier which can be identified optimally via the variational theory [17]. The subscript n indicates the nth approximation and $\tilde{u}_n$ is considered as a restricted variation [17], i.e. $\delta\tilde{u}_n = 0$.

4 APPLICATION OF VARIATIONAL ITERATION METHOD

To solve Eq. (5) by means of VIM, we start with an arbitrary initial approximation:

$$\psi_0 = A\cos(\omega t). \tag{10}$$

From Eq. (5), we have:

$$\ddot{\psi} = -\alpha\psi - \beta\psi^3 \qquad \Rightarrow \qquad \ddot{\psi} = -\alpha A\cos(\omega t) - \beta A^3\cos^3(\omega t). \tag{11}$$

Integrating twice yields:

$$\psi_1 = \frac{-9\alpha A - 7\beta A^3 + 9\alpha A\cos(\omega t) + \beta A^3\cos^3(\omega t) + 6\beta A^3\cos(\omega t)}{9\omega^2}. \tag{12}$$

Equating the coefficients of $\cos(\omega t)$ in ψ_0 and ψ_1, we have:

$$\omega_{VIM} = \sqrt{\alpha + 0.75\beta A^2}, \tag{13}$$

And therefore,

$$\psi_0 = A\cos\left(\sqrt{\alpha + 0.75\beta A^2}\, t\right). \tag{14}$$

Where $\delta\tilde{u}_n = 0$ is considered as restricted variation.

$$\psi_{n+1}(t) = \psi_n(t) + \int_0^t \lambda\left(\frac{d^2\psi_n}{d\eta^2} + \alpha\psi_n + \beta\psi_n^3\right) d\eta. \tag{15}$$

Its stationary conditions can be obtained as follows:

$$1 - \lambda'|_{\eta = t} = 0, \tag{16}$$

$$\lambda|_{\eta = t} = 0, \tag{17}$$

$$\lambda'' + \omega^2\lambda = 0. \tag{18}$$

Therefore, the multiplier, can be identified as

$$\lambda = \frac{1}{\omega}\sin\omega(\eta - t). \tag{19}$$

As a result, we obtain the following iteration formula:

$$\psi_{n+1}(t) = \psi_n(t) + \int_0^t (\frac{1}{\omega}\sin\omega(\eta - t)).(\frac{d^2\psi_n}{d\eta^2} + \alpha\psi_n + \beta\psi_n^3)d\eta. \tag{20}$$

By the iteration formula (20), we can directly obtain other components as:

$$\psi_1(t) = A\cos(\omega t) - \frac{A^3\beta\cos(\omega t) - (16\omega^2 - 16\alpha - 12A^2\beta)A\omega t\sin(\omega t) - A^3\beta\cos(3\omega t)}{32\omega^2}. \tag{21}$$

Where ω is evaluated from Eq. (13).

In the same manner, the rest of the components of the iteration formula can be obtained.

5 APPLICATION OF PARAMETRIZED PERTURBATION METHOD

Equation of motion, which reads:

$$\ddot{\psi}_{(t)} + \alpha\psi_{(t)} + \beta\psi_{(t)}^3 = 0, \qquad \psi(0) = A, \qquad \dot{\psi}(0) = 0. \tag{22}$$

We let

$$\psi = \varepsilon U, \tag{23}$$

By substituting Eq. (23) in Eq. (22):

$$\ddot{U} + \alpha U + \varepsilon^2 \beta U^3 = 0, \qquad U(0) = A/\varepsilon, \qquad \dot{U}(0) = 0. \tag{24}$$

We suppose that the solution of Eq. (24) and the constant α, can be expressed in the forms:

$$U = U_0 + \varepsilon^2 U_1 + \varepsilon^4 U_2 + \varepsilon^6 U_3 + ... \tag{25}$$

$$\alpha = \omega^2 + \varepsilon^2 \omega_1 + \varepsilon^4 \omega_2 + \varepsilon^6 \omega_3 + ... \tag{26}$$

Substituting Eqs. (25) and (26) into Eq. (24) and equating coefficients of same powers of ε yields the following equations:

$$\ddot{U}_0 + \omega^2 U_0 = 0, \qquad U_0(0) = A/\varepsilon, \qquad \dot{U}_0(0) = 0. \tag{27}$$

$$\ddot{U}_1 + \omega^2 U_1 + \omega_1 U_0 + \beta U_0^3 = 0, \qquad U_1(0) = 0, \qquad \dot{U}_1(0) = 0. \tag{28}$$

Solving Eq. (27) we obtain:

$$U_0 = \frac{A}{\varepsilon} \cos(\omega t). \tag{29}$$

Therefore, Eq. (28) can be re-written as:

$$\ddot{U}_1 + \omega^2 U_1 + \left(\omega_1 + \frac{3\beta A^2}{4\varepsilon^2}\right) \frac{A}{\varepsilon} \cos(\omega t) + \frac{\beta A^3}{4\varepsilon^3} \cos(3\omega t) = 0. \tag{30}$$

Avoiding the presence of a secular terms needs:

$$\omega_1 = -\frac{3\beta A^2}{4\varepsilon^2}. \tag{31}$$

Substituting Eq. (31) into Eq. (26)

$$\omega_{PPM} = \sqrt{\alpha + 0.75\beta A^2}. \tag{32}$$

Solving Eq. (30), we obtain:

$$U_1 = -\frac{A^3 \beta}{32\omega^2 \varepsilon^3} \left(\cos(\omega t) - \cos(3\omega t)\right). \tag{33}$$

Its first-order approximation is sufficient, and then we have:

$$\psi = \varepsilon U = \varepsilon(U_0 + \varepsilon^2 U_1) = A\cos(\omega t) - \frac{A^3 \beta}{32\omega^2} [\cos(\omega t) - \cos(3\omega t)]. \tag{34}$$

Where the angular frequency can be written by Eq. (32).

6 RESULTS AND DISCUSSIONS

The behavior of $\psi(A,t)$ obtained by VIM and PPM at $\alpha = \pi$ and $\beta = 0.15$ is shown in Figs. 2 and 3. Influence of coefficients β and α on frequency and amplitude has been investigated and plotted in Figs. 4 and 5, respectively. The comparison of the dimensionless deflection versus time for results obtained from VIM, PPM and Runge-Kutta 4th order has been depicted in Fig. 6 for $\alpha = \pi$ and $\beta = 0.15$, with maximum deflection at the center of the beam equal to five ($A=5$). The solutions are also compared for $t=0.5$ in Table 1. It can be observed that there is an excellent agreement between the results obtained from VIM and PPM with those of Runge-Kutta 4th order method [1].

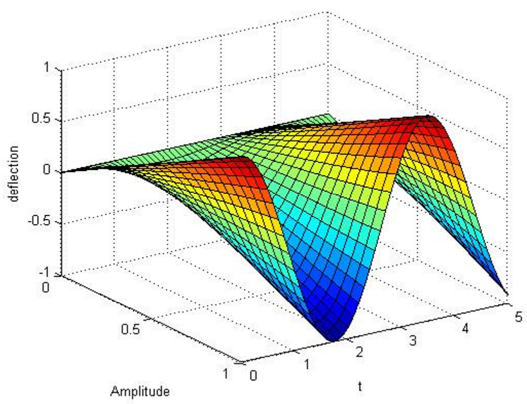

Figure 2 VIM deflection at $\alpha = \pi$ and $\beta = 0.15$.

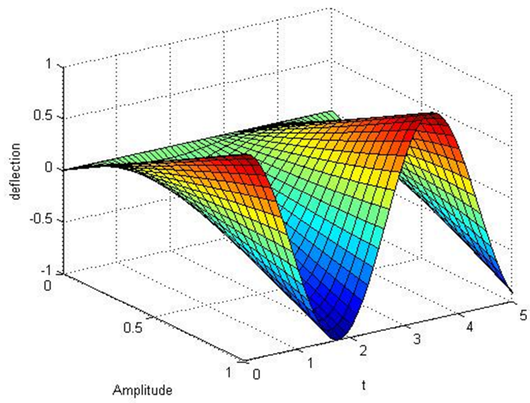

Figure 3 PPM deflection at $\alpha = \pi$ and $\beta = 0.15$.

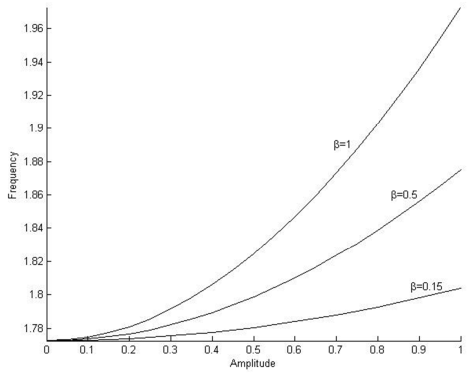

Figure 4 Results of frequency versus amplitude associated with influence of β at $\alpha = \pi$, for PPM or VIM.

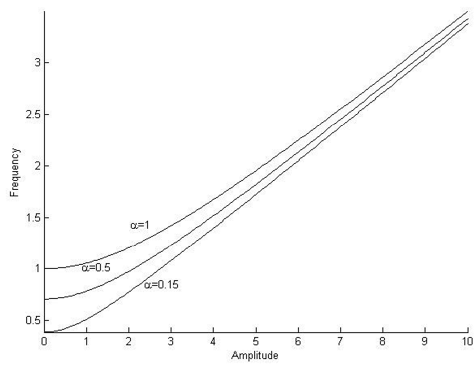

Figure 5 Results of frequency versus amplitude associated with influence of α at $\beta = 0.15$, for PPM or VIM.

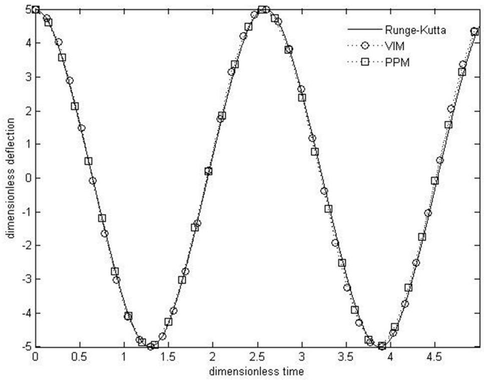

Figure 6 Results comparison between VIM and PPM deflection versus time at $\alpha = \pi$, $\beta = 0.15$ and $A = 5$ with Runge-Kutt 4th order.

Table 1 Comparison between PPM & VIM with time marching solution for the motion equation (4), when $t = 0.5(s)$, $\alpha = 1$ and $\beta = 1$.

A	PPM	VIM	Runge-Kutta	Error(PPM)	Error(VIM)
0.01	0.008775710	0.008775735	0.008775693	-0.000000017	-0.000000042
0.1	0.087643206	0.087668378	0.087643042	-0.000000168	-0.000025336
0.2	0.174597455	0.174797831	0.174597438	-0.000000017	-0.000200393
0.3	0.260180487	0.260851146	0.260182593	-0.000002106	-0.000668553
0.4	0.343723299	0.345294531	0.343733783	0.000010484	-0.001560748
0.5	0.424576467	0.427599316	0.424609486	0.000033019	-0.00298983
1	0.767843030	0.789109858	0.768790533	-0.000947503	-0.020319325
10	-2.960606970	-3.461741578	-3.700321826	-0.739714856	-0.238580248

7 CONCLUSIONS

In this paper, nonlinear responses of a clamped-clamped buckled beam are investigated. Mathematically, the beam is modeled by a partial differential equation possessing cubic non-linearity because of mid-plane stretching. Governing non-linear partial differential equation of Euler-Bernoulli's beam is reduced to a single non-linear ordinary differential equation using Galerkin method. Variational Iteration Method (VIM) and Paremetrized Perturbation Method (PPM) have been successfully used to study the non-linear vibration of beams. The frequency of both methods is exactly the same and transverse vibration of the beam center is illustrated versus amplitude and time. Also, the results and error of these methods are compared with Runge-Kutta 4th order. It is obvious that VIM and PPM are very powerful and efficient technique for finding analytical solutions. These methods do not require small parameters needed by perturbation method and are applicable for whole range of parameters. However, further research is needed to better understanding of the effect of these methods on engineering problems especially mechanical affairs.

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