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# Single variable new first-order shear deformation theory for isotropic plates

**Abstract**

This paper presents a single variable new first-order shear deformation plate theory with only one fourth-order partial governing differential equation. It may be noted that, first-order shear deformation plate theory of Mindlin has three coupled partial governing differential equations involving three unknown functions. Even a recently developed new first-order shear deformation plate theory has two uncoupled partial governing differential equations involving two unknown functions for static problems. The displacement functions of the proposed theory give rise to constant transverse shear strains through thickness of the plate and, as is the case of Mindlin plate theory, the proposed theory also requires a shear correction factor. The governing differential equation, expressions for moments and shear forces of the proposed theory have a striking resemblance to the corresponding expressions of classical plate theory. The proposed theory is the only first-order shear deformation plate theory with two different types of physically meaningful clamped boundary conditions. To obtain solutions for the flexure of the plate, efforts required using the proposed theory are comparable to those involved in the case of classical plate theory. The effectiveness of the proposed theory is demonstrated through illustrative examples and by comparing results obtained with other plate theories.

**Keywords**

Single variable plate theory, shear deformation plate theory.

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## 1 INTRODUCTION

The simplest plate theory is classical plate theory (*CPT*) which was developed in the late 19th century. It has been widely used as a first level check for analysis of structures that can be approximated as plates. However, *CPT* does not take into account the effects of transverse shear deformations present through the plate thickness. These effects of shear in plate deformations are significant especially in case of thick plates. Hence, *CPT* can provide reasonably accurate results only for thin plates. Use of *CPT* provides underestimated deflections and overestimated frequencies and buckling loads for thick plates (Ghugal and Shimpi (2002)).

To address these drawbacks of *CPT*, Reissner (1945) and Mindlin (1951) introduced first-order shear deformation plate theories. Displacement based Mindlin plate theory assumes constant transverse shear strains through the plate thickness and requires a shear correction factor. Even though the use of Mindlin plate theory provides satisfactory results for analysis of thick plates, it involves three coupled partial governing differential equations involing three unknown functions. It also requires specification of three boundary conditions per edge as opposed to *CPT* which requires specification of only two boundary conditions per edge.

Recently, Shimpi et al. (2007) developed new first-order shear deformation plate theories (*NFSDT*). *NFSDT* involves two partial governing differential equations involving two unknown functions. These two equations are inertially and elastically uncoupled in case of static problems. Whereas, these two equations are only inertially coupled and elastically uncoupled in case of dynamics problems. As is the case of Mindlin plate theory, *NFSDT* also requires specification of three boundary conditions per edge.

The objective of this paper is to present a single variable new first-order shear deformation theory for isotropic plates with only one partial governing differential equation. In this regards, Senjanović et al. (2013) have also derived fourth-order partial governing differential equation for moderately thick plate vibrations. It must be noted that the proposed theory is based on *NFSDT* by Shimpi et al. (2007) and refined plate theory (*RPT*) by Shimpi (2002). It is evident from equations (9), (10) and (20) of Senjanović et al. (2013) that they have also used the key concepts of *NFSDT*. They have developed the plate theory for thick plate vibrations only. The proposed theory is a displacement based theory for the plate flexture. Unlike any other first-order shear deformation plate theory, the proposed theory describes two different types of physically meaningful clamped boundary conditions which are analogous to those discussed by Timoshenko and Goodier (1951) in the context of two-dimensional theory of elasticity approach for beam analysis. As opposed to Mindlin plate theory, the proposed theory requires specification of only two boundary conditions per edge.

## 2 NOTATIONS USED

The notations used in this paper for displacements ($u$, $v$, $w$); direct strains ($\epsilon_x$, $\epsilon_y$, $\epsilon_z$); shear strains ($\gamma_{xy}$, $\gamma_{yz}$, $\gamma_{zx}$); direct stresses ($\sigma_x$, $\sigma_y$, $\sigma_z$) and shear stresses ($\tau_{xy}$, $\tau_{yz}$, $\tau_{zx}$) of the plate are the same as the one used by Timoshenko and Goodier (1951) on page no. xvii and xviii.

## 3 PLATE UNDER CONSIDERATION

The following are the features of the plate under consideration:

1. The plate considered is as shown in Figure 1 and it has uniform thickness $h$.
2. The plate is made of linearly elastic, homogeneous, isotropic material. Modulus of elasticity $E$, modulus of rigidity $G$ and Poisson's ratio $\mu$ of the plate material are related by $G = E / [2(1+\mu)]$.
3. Area $\Omega(x, y)$ is the mid-surface of the undeformed plate which is enclosed by a boundary curve $\Psi(x, y)$; as shown in Figure 1.
4. The right handed Cartesian co-ordinate system $0 - x - y - z$ would be utilized throughout this paper.
   a. The $xy-$ plane of this co-ordinate system is assumed to coincide with mid-surface of the undeformed plate.
   b. The origin "$0$" of this co-ordinate system can be selected at a convenient location on the mid-surface of the undeformed plate.
5. The plate is loaded on its surface $z = -h / 2$ by a transverse load of intensity $q(x, y)$. The loading is considered as positive when it acts in the positive direction of $z-$ axis.
6. Local directions $n$, $s$ and $z'$ at a typical point '$P$' on the edge are as shown in Figure 1. $n$ and $s$ are normal and tangent respectively to a boundary curve at that point. Direction $z'$ is parallel to the $z-$ axis of co-ordinate system. Physically meaningful boundary conditions can be prescribed at the boundary of the plate in terms of such local co-ordinate systems.

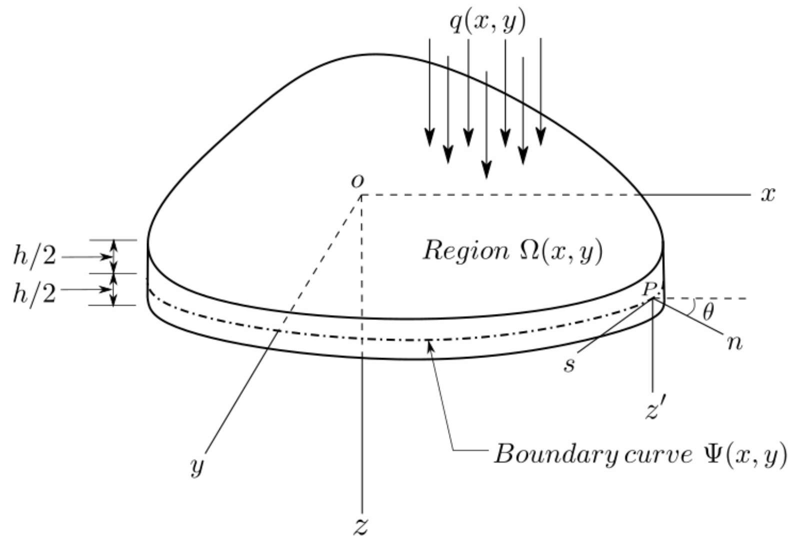


***Figure 1:*** *Geometry of the plate.*

## 4 ASSUMPTIONS MADE IN THE PRESENT THEORY

Assumptions of the present theory are built on those of Shimpi et al. (2007).

1. Displacements involved are small in comparison to the plate thickness. Hence, strains produced in the plate are infinitesimal.
2. In general, in-plane normal stresses $\sigma_x$ and $\sigma_y$ developed in the plate are very high as compared to transverse normal stress $\sigma_z$. Hence, transverse normal stress $\sigma_z$ can be neglected.

3. Transverse normal perpendicular to the mid-surface of the plate before deformation, remains straight but may or may not remain normal to the mid-surface of the plate after deformation.
4. Following points should be noted regarding displacement functions of the present theory:
    a. The transverse displacement $w$ along the $z-$ direction consists of two components:
    i. Bending component $w_b$
    ii. Shear component $w_s$
    These two components are functions of $x$ and $y$ co-ordinates only.
    b. In-plane displacement $u$ along the $x-$ direction and in-plane displacement $v$ along the $y-$ direction are analogous to those of *CPT*.
    c. In-plane displacements $u$ and $v$ in conjunction with bending component $w_b$ (of transverse displacement $w$ ) do not contribute towards transverse shear strains. Shear component $w_s$ (of transverse displacement $w$ ) alone contributes towards transverse shear strains. These transverse shear strains remain constant across the plate thickness.
5. As is the case with Mindlin plate theory (Mindlin (1951)), the present theory assumes transverse shear strains $\gamma_{yz}$ and $\gamma_{zx}$ to remain constant through the plate thickness. It is well known fact that, in reality, these transverse shear strains vary, more or less, in parabolic manner through the plate thickness. Hence, a shear correction factor associated with Mindlin plate theory (Mindlin (1951)) will be utilized in the present theory.

## 5 EXPRESSIONS FOR DISPLACEMENTS OF THE PRESENT THEORY

As a result of assumption 1, strain-displacement relations of linear theory of elasticity will hold good.

$$\epsilon_x = \frac{\partial u}{\partial x}; \qquad \epsilon_y = \frac{\partial v}{\partial y}; \qquad \epsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}; \qquad \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}; \qquad \gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \tag{1}$$

As a result of assumption 2, constitutive relations between stresses and strains can be used to relate direct stresses $\sigma_x$ and $\sigma_y$ to linear strains $\epsilon_x$ and $\epsilon_y$ as follows:

$$\sigma_x = \frac{E}{1-\mu^2}\left(\epsilon_x + \mu\,\epsilon_y\right) \tag{2}$$

$$\sigma_y = \frac{E}{1-\mu^2}\left(\epsilon_y + \mu\,\epsilon_x\right) \tag{3}$$

As a result of assumption 3 and assumption 4, expressions for in-plane displacements $u$ and $v$ of the present theory can be written as follows:

$$u(x,y,z) = -z\frac{\partial w_b}{\partial x} \tag{4}$$

$$v(x,y,z) = -z\frac{\partial w_b}{\partial y} \tag{5}$$

The transverse displacement $w$ has a bending component $w_b$ and a shear component $w_s$. Hence,

$$w(x,y) = w_b(x,y) + w_s(x,y) \tag{6}$$

With some efforts, shear component $w_s$ can be expressed in terms of bending component $w_b$ (as shown in Appendix A) as:

$$w_s = -\frac{h^2}{6\,k\,(1-\mu)}\left(\frac{\partial^2 w_b}{\partial x^2} + \frac{\partial^2 w_b}{\partial y^2}\right) \tag{7}$$

Here, '$k$' is a shear correction factor which is analogous to shear correction factor proposed by Mindlin (1951).

Hence, transverse displacement $w$ can be expressed in terms of bending component $w_b$ as follows:

$$w = w_b - \frac{h^2}{6\,k\,(1-\mu)}\left(\frac{\partial^2 w_b}{\partial x^2} + \frac{\partial^2 w_b}{\partial y^2}\right) \tag{8}$$

It can be observed from expressions (4), (5) and (8) that expressions for in-plane displacements $u$ and $v$, and transverse displacement $w$ contain only one unknown variable, *i.e.* $w_b$.

Now, using displacement functions of the present theory in strain-displacement relations given by equation (1), expressions for strains can be written as follows:

$$\epsilon_x = -z\,\frac{\partial^2 w_b}{\partial x^2} \tag{9}$$

$$\epsilon_y = -z\,\frac{\partial^2 w_b}{\partial y^2} \tag{10}$$

$$\gamma_{xy} = -2z\,\frac{\partial^2 w_b}{\partial x \partial y} \tag{11}$$

$$\gamma_{yz} = -\frac{h^2}{6\,k\,(1-\mu)}\frac{\partial}{\partial y}\left(\frac{\partial^2 w_b}{\partial x^2} + \frac{\partial^2 w_b}{\partial y^2}\right) \tag{12}$$

$$\gamma_{zx} = -\frac{h^2}{6\,k\,(1-\mu)}\frac{\partial}{\partial x}\left(\frac{\partial^2 w_b}{\partial x^2} + \frac{\partial^2 w_b}{\partial y^2}\right) \tag{13}$$

From expressions (12) and (13), it can be observed that, as is the case with Mindlin plate theory (Mindlin (1951)), transverse shear strains $\gamma_{yz}$ and $\gamma_{zx}$ remain constant through the plate thickness in the present theory. Hence, transverse shear stresses $\tau_{yz}$ and $\tau_{zx}$ also remain constant through the plate thickness.

It is well known fact that, in reality, transverse shear stresses $\tau_{yz}$ and $\tau_{zx}$ vary, more or less, in parabolic manner through the plate thickness. In general, these transverse shear stresses are zero on surfaces ($z = \pm h/2$) of the plate. Hence, a shear correction factor associated with Mindlin plate theory (Mindlin (1951)) will be utilized in the present theory.

## 6 MODIFIED CONSTITUTIVE RELATIONS OF THE PRESENT THEORY

The constitutive relations of theory of elasticity in respect of shear strains and shear stresses are $\tau_{xy} = G\gamma_{xy}$, $\tau_{yz} = G\gamma_{yz}$ and $\tau_{zx} = G\gamma_{zx}$. However as a result of assumption 3 and assumption 5, the relations between transverse shear strains and transverse shear stresses get modified by incorporating a shear correction factor '$k$' as: $\tau_{yz} = kG\gamma_{yz}$ and $\tau_{zx} = kG\gamma_{zx}$. Hence, constitutive relations between stresses and strains can be written as follows:

$$\sigma_x = \frac{E}{1-\mu^2}\left(\epsilon_x + \mu\,\epsilon_y\right) \tag{14}$$

$$\sigma_y = \frac{E}{1-\mu^2}\left(\epsilon_y + \mu\,\epsilon_x\right) \tag{15}$$

$$\tau_{xy} = \frac{E}{2(1+\mu)}\gamma_{xy} \tag{16}$$

$$\tau_{yz} = \frac{E\,k}{2(1+\mu)}\gamma_{yz} \tag{17}$$

$$\tau_{zx} = \frac{E\,k}{2(1+\mu)}\gamma_{zx} \tag{18}$$

## 7 EXPRESSIONS FOR STRESSES

Now, using equations (9) through (13) in constitutive relations given by equations (14) through (18), expressions for stresses can be written as follows:

$$\sigma_x = -\frac{E\,z}{1-\mu^2}\left(\frac{\partial^2 w_b}{\partial x^2} + \mu\,\frac{\partial^2 w_b}{\partial y^2}\right) \tag{19}$$

$$\sigma_y = -\frac{E\,z}{1-\mu^2}\left(\frac{\partial^2 w_b}{\partial y^2} + \mu\,\frac{\partial^2 w_b}{\partial x^2}\right) \tag{20}$$

$$\tau_{xy} = -\frac{E\,z}{1-\mu^2}(1-\mu)\frac{\partial^2 w_b}{\partial x \partial y} \tag{21}$$

$$\tau_{yz} = -\frac{E\,h^2}{12(1-\mu^2)}\frac{\partial}{\partial y}\left(\frac{\partial^2 w_b}{\partial x^2} + \frac{\partial^2 w_b}{\partial y^2}\right) \tag{22}$$

$$\tau_{zx} = -\frac{E\,h^2}{12(1-\mu^2)}\frac{\partial}{\partial x}\left(\frac{\partial^2 w_b}{\partial x^2} + \frac{\partial^2 w_b}{\partial y^2}\right) \tag{23}$$

## 8 EXPRESSIONS FOR MOMENTS AND SHEAR FORCES

Moments $M_x$, $M_y$ and $M_{xy}$; shear forces $Q_x$ and $Q_y$ can now be defined as follows:

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \\ Q_x \\ Q_y \end{Bmatrix} = \int_{z=-h/2}^{z=h/2} \begin{Bmatrix} \sigma_x\, z \\ \sigma_y\, z \\ \tau_{xy}\, z \\ \tau_{zx} \\ \tau_{yz} \end{Bmatrix} dz \tag{24}$$

Now, using equations (19) through (23) in equation (24), expressions for bending moments $M_x$ and $M_y$; twisting moment $M_{xy}$; shear forces $Q_x$ and $Q_y$ can be written as follows:

$$M_x = -D\left(\frac{\partial^2 w_b}{\partial x^2} + \mu\frac{\partial^2 w_b}{\partial y^2}\right) \tag{25}$$

$$M_y = -D\left(\frac{\partial^2 w_b}{\partial y^2} + \mu\frac{\partial^2 w_b}{\partial x^2}\right) \tag{26}$$

$$M_{xy} = -D\left(1-\mu\right)\frac{\partial^2 w_b}{\partial x \partial y} \tag{27}$$

$$Q_x = -D\frac{\partial}{\partial x}\left(\frac{\partial^2 w_b}{\partial x^2} + \frac{\partial^2 w_b}{\partial y^2}\right) \tag{28}$$

$$Q_y = -D\frac{\partial}{\partial y}\left(\frac{\partial^2 w_b}{\partial x^2} + \frac{\partial^2 w_b}{\partial y^2}\right) \tag{29}$$

Here, $D$ represents the rigidity of the plate which is given by

$$D = \frac{Eh^3}{12\left(1-\mu^2\right)} \tag{30}$$

It can be observed that expressions for moments given by equations (25), (26), (27) and shear forces given by equations (28), (29) have striking resemblance to those of *CPT*.

## 9 OBTAINING GOVERNING DIFFERENTIAL EQUATION FOR THE FLEXURAL ANALYSIS OF THE PLATE

The equilibrium equations as per three-dimensional linear theory of elasticity can be written as follows:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = 0 \tag{31}$$

$$\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} = 0 \tag{32}$$

$$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \sigma_z}{\partial z} = 0 \tag{33}$$

Here in equations (31) through (33), body forces (such as self weight due to gravity) are not mentioned separately as they can be merged with externally applied loads without causing much loss of accuracy. It is important to note that, theory of elasticity equations can be satisfied only in case of few problems. Whereas, gross equilibrium equations can be satisfied comparatively easily. Gross equilibrium equations can be obtained using theory of elasticity equilibrium equations (31) through (33) as obtained by Shimpi (2002).

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0 \tag{34}$$

$$\frac{\partial M_{yx}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = 0 \tag{35}$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = 0 \tag{36}$$

Equations (34), (35) and (36) are gross equilibrium equations. Expressions for moments given by equations (25), (26), (27) and shear forces given by equations (28), (29) satisfy gross equilibrium equations (34) and (35).

It must be noted that, shear forces $Q_x$ and $Q_y$ as given by expressions (28) and (29) respectively are obtained using expression for $w_s$ as given by equation (7); displacement functions (4), (5) and (6); strain-displacement relation (1); stress-strain relation (17) and (18); expressions for $Q_x$ and $Q_y$ as given in equation (24). Expressions for $Q_x$ and $Q_y$ which are identical to expressions (28) and (29) can also be obtained using expressions for moments $M_x$, $M_y$ and $M_{xy}$ as given by equations (25), (26) and (27) respectively in gross equilibrium equations (34) and (35) (as shown in Appendix A).

Now, using expressions for shear forces $Q_x$ and $Q_y$ as given by expressions (28) and (29) in gross equilibrium equation (36), governing differential equation for the flexural analysis of the plate can be obtained as follows:

$$\frac{\partial^4 w_b}{\partial x^4} + 2\frac{\partial^4 w_b}{\partial x^2 \partial y^2} + \frac{\partial^4 w_b}{\partial y^4} = \frac{q}{D} \tag{37}$$

Here, it can be observed that, governing differential equation (37) of the present theory has striking resemblance to the governing differential equation of *CPT*. The only difference is that, in equation (37), $w_b$ is appearing, whereas in case of *CPT*, in its place transverse displacement $w$ appears.

Using governing differential equation (37) and appropriate boundary conditions, $w_b$ can be obtained. Transverse displacement $w$ can be obtained using equation (8).

The appropriate physically meaningful boundary conditions would now be discussed.

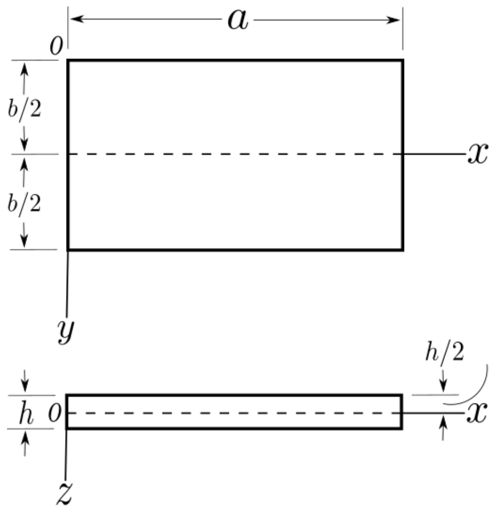

***Figure 2:*** *Plate co-ordinate system.*

## 10 BOUNDARY CONDITIONS

In the present theory; $M_x$, $M_y$, $M_{xy}$, $Q_x$ and $Q_y$ as given by expressions (25), (26), (27), (28) and (29) respectively, have striking resemblence to the corresponding expressions of *CPT*. Only difference is that in the present theory, $w_b$ is appearing in these expressions, whereas in case of *CPT*, in its place transverse displacement $w$ appears. Boundary conditions in the present theory can be prescribed as guided by the experience of *CPT* and theory of elasticity.

In this section, few commonly used physically meaningful boundary conditions would be discussed for the rectangular plate (Figure 2) at the edge $y = \frac{b}{2}$, for the sake of illustration. To prescribe the boundary conditions at other edges, *i.e.* $y = -\frac{b}{2}$, $x = 0$ and $x = a$, one could follow the similar logic as that of the edge $y = \frac{b}{2}$. In case of the plate with arbitrary shape, boundary conditions can be specified in similar manner by considering local directions $n$, $s$ and $z'$ at a typical point '$P$' on the edge as shown in Figure 1.

### 1. Plate edge $y = \frac{b}{2}$ is simply-supported

At simply-supported edge $y = \frac{b}{2}$, transverse displacement $w$ is zero and bending moment $M_y$ is zero.

$$\left[ w \right]_{y=b/2} = \left[ w_b - \frac{h^2}{6\,k\,(1-\mu)} \left( \frac{\partial^2 w_b}{\partial x^2} + \frac{\partial^2 w_b}{\partial y^2} \right) \right]_{y=b/2} = 0 \tag{38}$$

$$\left[ M_y \right]_{y=b/2} = -D \left[ \frac{\partial^2 w_b}{\partial y^2} + \mu \frac{\partial^2 w_b}{\partial x^2} \right]_{y=b/2} = 0 \tag{39}$$

Expressions (38) and (39) lead to following boundary conditions at simply-supported edge $y = \frac{b}{2}$:

$$\left[ w_b \right]_{y=b/2} = 0 \tag{40}$$

$$\left[ \frac{\partial^2 w_b}{\partial y^2} \right]_{y=b/2} = 0 \tag{41}$$

### 2. Plate edge $y = \frac{b}{2}$ is free:

At free edge $y = \frac{b}{2}$; using same reasoning as that in the case of *CPT* for free edge as discussed by Timoshenko and Woinowsky-Krieger (1959) on page no. 83 through 85, boundary conditions can be specified as follows:

$$\left[ M_y \right]_{y=b/2} = -D \left[ \frac{\partial^2 w_b}{\partial y^2} + \mu \frac{\partial^2 w_b}{\partial x^2} \right]_{y=b/2} = 0 \tag{42}$$

$$\left[ Q_y + \frac{\partial M_{xy}}{\partial x} \right]_{y=b/2} = -D \left[ \frac{\partial^3 w_b}{\partial y^3} + (2-\mu) \frac{\partial^3 w_b}{\partial x^2 \partial y} \right]_{y=b/2} = 0 \tag{43}$$

### 3. Plate edge $y = \frac{b}{2}$ is clamped:

At clamped edge $y = \frac{b}{2}$; it is feasible to represent two types of boundary conditions. In this paper, these two types of clamped boundary conditions are denoted as "clamped edge: type 1" and "clamped edge: type 2". These boundary conditions are analogous to those discussed by Timoshenko and Goodier (1951) on page no. 35 through 39 in the context of two-dimensional theory of elasticity approach for beam analysis.

a. Plate edge $y = \frac{b}{2}$ is clamped with "clamped edge: type 1"

For "clamped edge: type 1" boundary conditions on the edge $y=\frac{b}{2}$, $w$ is taken as zero and slope $\left[\frac{\partial v}{\partial z}\right]$ is taken as zero at the clamped edge $y=\frac{b}{2}$. This slope, can be expressed in terms of $w_b$ by using equation (5). Hence, following conditions are to be used:

$$\left[w\right]_{y=b/2} = \left[w_b - \frac{h^2}{6\,k\,(1-\mu)}\left(\frac{\partial^2 w_b}{\partial x^2}+\frac{\partial^2 w_b}{\partial y^2}\right)\right]_{y=b/2} = 0 \tag{44}$$

$$\left[\frac{\partial w_b}{\partial y}\right]_{y=b/2} = 0 \tag{45}$$

b. Plate edge $y=\frac{b}{2}$ is clamped with "clamped edge: type 2"

For "clamped edge: type 2" boundary conditions on the edge $y=\frac{b}{2}$, the displacement boundary condition remains the same as that of earlier case, but now slope $\left[\frac{\partial w}{\partial y}\right]$ is taken as zero at the clamped edge $y=\frac{b}{2}$, Hence, following conditions are to be used:

$$\left[w\right]_{y=b/2} = \left[w_b - \frac{h^2}{6\,k\,(1-\mu)}\left(\frac{\partial^2 w_b}{\partial x^2}+\frac{\partial^2 w_b}{\partial y^2}\right)\right]_{y=b/2} = 0 \tag{46}$$

$$\left[\frac{\partial w}{\partial y}\right]_{y=b/2} = \left[\frac{\partial w_b}{\partial y} - \frac{h^2}{6\,k\,(1-\mu)}\frac{\partial}{\partial y}\left(\frac{\partial^2 w_b}{\partial x^2}+\frac{\partial^2 w_b}{\partial y^2}\right)\right]_{y=b/2} = 0 \tag{47}$$

It can be seen from the boundary condition given by equation (45) that at the clamped edge $y=\frac{b}{2}$, slope of only bending component $w_b$ (of transverse displacement $w$) along $y-$ axis is equal to zero. Hence in case of "clamped edge: type 1" boundary condition, effects of shear deformation on transverse displacement are significant.

Whereas, it can be seen from the boundary condition given by equation (47) that at the clamped edge $y=\frac{b}{2}$, slope of transverse displacement $w$ along $y-$ axis is equal to zero. Hence in case of "clamped edge: type 2" boundary condition, effects of shear deformation on transverse displacement are less significant as compared to those of "clamped edge: type 1" boundary condition.

It is pointed out by Groh and Weaver (2015) that inconsistencies with regards to the shear force arise in the formulation of flexural behaviour of plates with clamped boundary conditions using a certian class of axiomatic shear deformation theories. Discussion provided by Groh and Weaver (2015) points out that shear forces erroneously vanish at a clamped edge when the constitutive relations and boundary conditions of the particular theory are utilized. Hence, the shear forces obtained by using the equilibrium considerations of the forces would not match with those obtained using constitutive relations and boundary conditions of the particular theory. Hence, this would result in the inconsistency of the shear forces.

However, it is worth mentioning that such a discrepancy of vanishing shear forces, when the constitutive relations and boundary conditions of the theory are utilized, will not take place in the present theory. In the present theory, clamped edge boundary conditions specified are analogous to those discussed by Timoshenko and Goodier (1951) in the context of two-dimensional theory of elasticity approach for beam analysis. It would be feasible to obtain appropriate shear forces at the clamped edge of the plate by using the boundary conditions prescribed either by equations (44) and (45) or by equations (46) and (47), as the case may be.

## 11 COMMENTS ON THE PRESENT THEORY

The following are the noteworthy features of the present theory:

**1.** The proposed theory is a displacement based theory. As is case of Mindlin plate theory (Mindlin (1951)), the present theory assumes constant transverse shear strains across the plate thickness and it requires a shear correction factor to be specified. The governing differential equation of the present theory is obtained by utilizing gross equilibrium equations of the plate. These gross equilibrium equations are in terms of moments, shear forces and the applied loading. Based on physical understanding, the boundary conditions have been specified.

**2.** In the present theory, shear strains are obtained using assumed displacement functions along with strain-displacement relations. These shear strains are then used to obtain shear stresses using modified stress-strain relations. These shear stresses are finally used to obtain shear forces. Whereas, in case of *CPT*, shear forces are obtained using gross equilibrium equations.

**3.** Unlike Mindlin plate theory (Mindlin (1951)) which contains three coupled partial governing differential equations and three unknown functions, the present theory has only one fourth-order partial governing differential equation (equation (37)). The present theory involves only one unknown function ( $w_b$ ).

**4.** For the plate flexure problems, the expressions for moments, shear forces and the governing differential equation of the present theory have striking resemblance to those of *CPT*. The only difference is that, in the present theory, $w_b$ appears in these expressions, whereas in case of *CPT*, in its place transverse displacement $w$ appears.

**5.** Unlike any other first-order shear deformation theory, the present theory provides two different types of physically meaningful clamped boundary conditions. These clamped boundary conditions of the present theory are analogous to those discussed by Timoshenko and Goodier (1951) in the context of two-dimensional theory of elasticity approach for beam analysis.

**6.** Following points should be noted with regards to the present theory, the work presented by Shimpi et al. (2017) and the conceptual differences between them:

**a.** The difference between the present theory and the work presented by Shimpi et al. (2017) in their eqs. (46) - (48) is on similar lines as the difference between *NFSDT* by Shimpi et al. (2007) and *RPT* by Shimpi (2002). In other words, the present theory belongs to the category of first-order shear deformation plate theories. Whereas the work presented by Shimpi et al. (2017) belongs to the category of higher-order shear deformation plate theories.

**b.** It should be noted that in case of both Shimpi et al. (2017) in their eq. (48) and the present theory, transverse displacement $w$ consists of a bending component $w_b$ and a shear component $w_s$. But on similar lines of *RPT*, the assumed in-plane displacement field of Shimpi et al. (2017) in their eqs. (46) - (47) has linear as well as cubic variations in terms of the plate thickness coordinate. Whereas on the similar lines of *NFSDT*, the assumed in-plane displacement field of the present theory has only linear variation in terms of the plate thickness coordinate.

## 12 ILLUSTRATIVE EXAMPLE

In this section, illustrative examples for the flexure of an isotropic rectangular plates will be presented to demonstrate the effectiveness of the present theory.

### Strategy for solutions

All the illustrative examples involve rectangular plates as shown in Figure 2. The plate is simply-supported on the edges $x = 0$ and $x = a$. For individual problems, the boundary conditions at the edges $y = \frac{b}{2}$ and $y = -\frac{b}{2}$ would be specified. The plate is acted upon by uniformly distributed load of intensity $q_o$ per unit area over the entire surface $z = -\frac{b}{2}$ of the plate and the load $q_o$ acts in the positive $z-$ direction.

For such a plate, it is possible to obtain expression for $w_b$ using Lévy method of analysis as discussed by Timoshenko and Woinowsky-Krieger (1959) on page no. 113 through 115. This expression for $w_b$ satisfies simply-supported boundary conditions on the edges $x = 0$ and $x = a$ as well as the governing differential equation (37). Expression for $w_b$ is as follows:

$$w_b = \sum_{m=1,3,5,..}^{\infty} \left(\frac{q_o a^4}{D}\right) \left[\begin{array}{c} \left(\frac{4}{\pi^5 m^5}\right) + A_m \cosh\left(\frac{m\pi y}{a}\right) + B_m \left(\frac{m\pi y}{a}\right) \sinh\left(\frac{m\pi y}{a}\right) \\ + C_m \sinh\left(\frac{m\pi y}{a}\right) + D_m \left(\frac{m\pi y}{a}\right) \cosh\left(\frac{m\pi y}{a}\right) \end{array}\right] \sin\left(\frac{m\pi x}{a}\right) \tag{48}$$

Now, by using equations (48) and (8), expression for transverse displacement $w$ can be written as follows:

$$w=\sum_{m=1,3,5,..}^{\infty}\left(\frac{q_o a^4}{D}\right)\left\{\begin{bmatrix}\left[\left(\frac{4}{\pi^5 m^5}\right)+A_m\cosh\left(\frac{m\pi y}{a}\right)+B_m\left(\frac{m\pi y}{a}\right)\sinh\left(\frac{m\pi y}{a}\right)\right.\\ \left.+C_m\sinh\left(\frac{m\pi y}{a}\right)+D_m\left(\frac{m\pi y}{a}\right)\cosh\left(\frac{m\pi y}{a}\right)\right]\\ +\left(\frac{h}{a}\right)^2\left[\frac{m^2\pi^2}{3\,k\,(1-\mu)}\right]\left[\begin{matrix}\left(\frac{2}{\pi^5 m^5}\right)-B_m\cosh\left(\frac{m\pi y}{a}\right)\\ -D_m\sinh\left(\frac{m\pi y}{a}\right)\end{matrix}\right]\end{bmatrix}\right\}\sin\left(\frac{m\pi x}{a}\right) \quad (49)$$

Now, arbitrary constants $A_m$, $B_m$, $C_m$ and $D_m$ are obtained by substituting expressions (48) and (49) into appropriate boundary conditions of remaining two edges *i.e.* $y=\frac{b}{2}$ and $y=-\frac{b}{2}$ and by solving obtained set of linear algebraic equations.

1. **Example 1**: Plate with edges $y=\frac{b}{2}$, $y=-\frac{b}{2}$, $x=0$ and $x=a$ are all simply-supported (*SSSS*).
2. **Example 2**: Plate with edges $y=\frac{b}{2}$, $y=-\frac{b}{2}$ are clamped and edges $x=0$, $x=a$ are simply-supported (*SCSC*).
3. **Example 3**: Plate with edges $y=\frac{b}{2}$, $y=-\frac{b}{2}$ are free and edges $x=0$, $x=a$ are simply-supported (*SFSF*).
4. **Example 4**: Plate with edge $y=\frac{b}{2}$ is clamped, edge $y=-\frac{b}{2}$ is simply-supported and edges $x=0$, $x=a$ are simply-supported (*SCSS*).
5. **Example 5**: Plate with edge $y=\frac{b}{2}$ is clamped, edge $y=-\frac{b}{2}$ is free and edges $x=0$, $x=a$ are simply-supported (*SCSF*).
6. **Example 6**: Plate with edge $y=\frac{b}{2}$ is simply-supported, edge $y=-\frac{b}{2}$ is free and edges $x=0$, $x=a$ are simply-supported (*SSSF*).

As pointed out by Lee et al. (2002), in the open literature, the only analytical Mindlin plate results on Lévy plates have been reported by Cooke and Levinson (1983). Most of the results present in the literature for the plate flexure problems pertain to *SSSS*, *SCSC* and *SFSF* plates. Hence for examples 4, 5 and 6, authors have compared the results for *SCSS*, *SCSF* and *SSSF* plates obtained using the present theory with corresponding results given by Zenkour (2003), Thai and Choi (2013) and Thai et al. (2013). Zenkour (2003) has used mixed plate theory, Thai and Choi (2013) have used refined plate theory and Thai et al. (2013) have used simple refined shear deformation theory for the plate flexure analysis.

In example 1; $m=1,3,5,....,49$ is taken, whereas for examples 2, 3, 4, 5 and 6; $m=1,3,5,7$ is taken for series expansion of $w_b$ and $w$.

## 13 NUMERICAL RESULTS AND DISCUSSIONS

Using expressions (48) and (49) for $w_b$ and $w$ respectively, numerical results will now be presented for flexural analysis of *SSSS*, *SCSC*, *SFSF*, *SCSS*, *SCSF* and *SSSF* plates. Numerical results in terms of non-dimensional transverse displacement ($\overline{w}$), non-dimensional bending moment ($\overline{M}_x$) and non-dimensional shear force ($\overline{Q}_x$) obtained using the present theory and corresponding results available in the literature are tabulated in Tables 1 through 6.

Convergence analysis in terms of effect of number of terms in series expansion of bending component $w_b$ of transverse displacement $w$ on various non-dimensional parameters of the plate with *SSSS* boundary conditions (Example 1) for various values of plate thickness ratio is included in Appendix B.

Poisson's ratio ( $\mu$ ) for the plate material is assumed to be 0.3. The value of shear correction factor ( $k$ ) is taken as 5/6.

The non-dimensional parameters used in the tabulation are defined as follows:

$\overline{w} = \frac{wD}{q_o a^4}$ : Non-dimensional transverse displacement (in the context of *SSSS*, *SCSC*, *SFSF*, *SCSS*, *SCSF* and *SSSF* plates carrying uniformly distributed loads).

$\overline{M}_x = \frac{M_x}{q_o a^2}$ : Non-dimensional bending moment (in the context of *SSSS* plates carrying uniformly distributed loads).

$\overline{Q}_x = \frac{Q_x}{q_o a}$ : Non-dimensional shear force (in the context of *SSSS* plates carrying uniformly distributed loads).

***Table 1:*** *Comparison of various non-dimensional parameters of the plate with SSSS boundary conditions (Example 1) and carrying a uniformly distributed load of intensity* $q_o$ *and Poisson's ratio (*$\mu$ *= 0.3).*

| Theory | Non-dimensional displacement $\overline{w}\left(=\frac{wD}{q_o a^4}\right)$ at $x = a/2,\ y = 0,\ b/a = 1.0$ | | | |
|---|---|---|---|---|
| | $h/a = 0.01$ | $h/a = 0.05$ | $h/a = 0.10$ | $h/a = 0.20$ |
| *CPT* | 0.00406 | 0.00406 | 0.00406 | 0.00406 |
| Levinson (Reddy et al. (2001)) | - | - | 0.00427 | 0.00490 |
| Mindlin (Lee et al. (2002)) | 0.00406 | 0.00411 | 0.00427 | 0.00490 |
| Reissner (Salerno and Goldberg (1960)) | 0.00406 | 0.00411 | 0.00424 | 0.00478 |
| Present | 0.00407 | 0.00412 | 0.00427 | 0.00490 |
| **Theory** | **Non-dimensional bending moment $\overline{M_x}\left(=\frac{M_x}{q_o a^2}\right)$ at $x = a/2,\ y = 0,\ b/a = 1.0$** | | | |
| | $h/a = 0.01$ | $h/a = 0.05$ | $h/a = 0.10$ | $h/a = 0.20$ |
| *CPT* | 0.0479 | 0.0479 | 0.0479 | 0.0479 |
| Mindlin (Lee et al. (2002)) | 0.0479 | 0.0479 | 0.0479 | 0.0479 |
| Reissner (Salerno and Goldberg (1960)) | 0.0479 | 0.0479 | 0.0481 | - |
| Present | 0.0479 | 0.0479 | 0.0479 | 0.0479 |
| **Theory** | **Non-dimensional shear force $\overline{Q_x}\left(=\frac{Q_x}{q_o a}\right)$ at $x = 0,\ y = 0,\ b/a = 1.0$** | | | |
| | $h/a = 0.01$ | $h/a = 0.05$ | $h/a = 0.10$ | $h/a = 0.20$ |
| *CPT* | 0.338 | 0.338 | 0.338 | 0.338 |
| Mindlin (Lee et al. (2002)) | 0.333 | 0.333 | 0.333 | 0.333 |
| Refined HSDT (Kant (1982)) | 0.337 | 0.337 | 0.337 | 0.337 |
| Present | 0.334 | 0.334 | 0.334 | 0.334 |

***Table 2:*** *Comparison of non-dimensional displacements of the plate with SCSC boundary conditions (Example 2) and carrying a uniformly distributed load of intensity $q_o$ and Poisson's ratio ($\mu$ = 0.3).*

| Theory | Non-dimensional displacement $\overline{w}\left(=\frac{wD}{q_o a^4}\right)$ at $\mathrm{x = a/2,\ y = 0,\ b/a = 1.0}$ | | | |
|---|---|---|---|---|
| | $\mathrm{h/a = 0.01}$ | $\mathrm{h/a = 0.05}$ | $\mathrm{h/a = 0.10}$ | $\mathrm{h/a = 0.20}$ |
| *CPT* | 0.00192 | 0.00192 | 0.00192 | 0.00192 |
| Levinson (Reddy et al.(2001)) | - | - | 0.00227 | 0.00322 |
| Mindlin (Lee et al. (2002)) | 0.00192 | 0.00199 | 0.00221 | 0.00302 |
| Reissner (Salerno and Goldberg (1960)) | 0.00192 | 0.00199 | 0.00220 | 0.00298 |
| Present clamp type 1 | 0.00192 | 0.00199 | 0.00222 | 0.00308 |
| Present clamp type 2 | 0.00192 | 0.00193 | 0.00196 | 0.00210 |
| Theory | Non-dimensional displacement $\overline{w}\left(=\frac{wD}{q_o a^4}\right)$ at $\mathrm{x = a/2,\ y = 0,\ b/a = 2.0}$ | | | |
| | $\mathrm{h/a = 0.01}$ | $\mathrm{h/a = 0.05}$ | $\mathrm{h/a = 0.10}$ | $\mathrm{h/a = 0.20}$ |
| *CPT* | 0.00845 | 0.00845 | 0.00845 | 0.00845 |
| Levinson (Reddy et al.(2001)) | - | - | 0.00889 | 0.01013 |
| Mindlin (Lee et al. (2002)) | 0.00845 | 0.00855 | 0.00885 | 0.01000 |
| Reissner (Salerno and Goldberg (1960)) | 0.00845 | 0.00854 | 0.00882 | 0.00985 |
| Present clamp type 1 | 0.00845 | 0.00855 | 0.00886 | 0.01005 |
| Present clamp type 2 | 0.00845 | 0.00850 | 0.00867 | 0.00936 |

***Table 3:*** *Comparison of non-dimensional displacements of the plate with SFSF boundary conditions (Example 3) and carrying a uniformly distributed load of intensity $q_o$ and Poisson's ratio ($\mu$ = 0.3).*

| Theory | Non-dimensional displacement $\overline{w}\left(=\frac{wD}{q_o a^4}\right)$ at $\mathrm{x = a/2,\ y = 0,\ b/a = 1.0}$ | | | |
|---|---|---|---|---|
| | $\mathrm{h/a = 0.01}$ | $\mathrm{h/a = 0.05}$ | $\mathrm{h/a = 0.10}$ | $\mathrm{h/a = 0.20}$ |
| *CPT* | 0.01309 | 0.01309 | 0.01309 | 0.01309 |
| Levinson (Reddy et al.(2001)) | 0.01310 | 0.01319 | 0.01346 | 0.01454 |
| Mindlin (Lee et al. (2002)) | 0.01310 | 0.01319 | 0.01346 | 0.01454 |
| Reissner (Salerno and Goldberg (1960)) | - | - | 0.01341 | 0.01433 |
| Present | 0.01310 | 0.01318 | 0.01342 | 0.01441 |
| Theory | Non-dimensional displacement $\overline{w}\left(=\frac{wD}{q_o a^4}\right)$ at $\mathrm{x = a/2,\ y = b/2,\ b/a = 1.0}$ | | | |
| | $\mathrm{h/a = 0.01}$ | $\mathrm{h/a = 0.05}$ | $\mathrm{h/a = 0.10}$ | $\mathrm{h/a = 0.20}$ |
| *CPT* | 0.01501 | 0.01501 | 0.01501 | 0.01501 |
| Levinson (Reddy et al.(2001)) | 0.01504 | 0.01522 | 0.01560 | 0.01690 |
| Mindlin (Lee et al. (2002)) | 0.01504 | 0.01522 | 0.01560 | 0.01690 |
| Reissner (Salerno and Goldberg (1960)) | - | - | 0.01557 | 0.01678 |
| Present | 0.01501 | 0.01508 | 0.01530 | 0.01616 |

***Table 4:** Comparison of non-dimensional displacements of the plate with SCSS boundary conditions (Example 4) and carrying a uniformly distributed load of intensity $q_o$ and Poisson's ratio ($\mu$ = 0.3).*

| Theory | Non-dimensional displacement $\bar{w}\left(=\frac{wD}{q_o a^4}\right)$ at $x = a/2,\ y = 0,\ b/a = 2.0$ | | | |
|---|---|---|---|---|
| | $h/a = 0.001$ | $h/a = 0.04$ | $h/a = 0.10$ | $h/a = 0.20$ |
| *CPT* | 0.00927 | 0.00927 | 0.00927 | 0.00927 |
| MPT(Zenkour (2003)) | 0.00927 | 0.00933 | 0.00964 | 0.01070 |
| RPT(Thai and Choi (2013)) | 0.00927 | 0.00932 | 0.00960 | 0.01057 |
| SRSDT (Thai et al. (2013)) | 0.00927 | 0.00932 | 0.00960 | 0.01057 |
| Present clamp type 1 | 0.00927 | 0.00933 | 0.00964 | 0.01073 |
| Present clamp type 2 | 0.00927 | 0.00931 | 0.00955 | 0.01037 |

MPT: Mixed plate theory, RPT: Refined plate theory, SRSDT: Simple refined shear deformation theory.

***Table 5:** Comparison of non-dimensional displacements of the plate with SCSF boundary conditions (Example 5) and carrying a uniformly distributed load of intensity $q_o$ and Poisson's ratio ($\mu$ = 0.3).*

| Theory | Non-dimensional displacement $\bar{w}\left(=\frac{wD}{q_o a^4}\right)$ at $x = a/2,\ y = 0,\ b/a = 2.0$ | | | |
|---|---|---|---|---|
| | $h/a = 0.001$ | $h/a = 0.04$ | $h/a = 0.10$ | $h/a = 0.20$ |
| *CPT* | 0.01061 | 0.01061 | 0.01061 | 0.01061 |
| MPT(Zenkour (2003)) | 0.01061 | 0.01066 | 0.01098 | 0.01209 |
| RPT(Thai and Choi (2013)) | 0.01061 | 0.01066 | 0.01095 | 0.01197 |
| SRSDT (Thai et al. (2013)) | 0.01061 | 0.01066 | 0.01095 | 0.01197 |
| Present clamp type 1 | 0.01061 | 0.01067 | 0.01099 | 0.01212 |
| Present clamp type 2 | 0.01061 | 0.01065 | 0.01089 | 0.01176 |

MPT: Mixed plate theory, RPT: Refined plate theory, SRSDT: Simple refined shear deformation theory.

***Table 6:** Comparison of non-dimensional displacements of the plate with SSSF boundary conditions (Example 6) and carrying a uniformly distributed load of intensity $q_o$ and Poisson's ratio ($\mu$ = 0.3).*

| Theory | Non-dimensional displacement $\bar{w}\left(=\frac{wD}{q_o a^4}\right)$ at $x = a/2,\ y = 0,\ b/a = 2.0$ | | | |
|---|---|---|---|---|
| | $h/a = 0.001$ | $h/a = 0.04$ | $h/a = 0.10$ | $h/a = 0.20$ |
| *CPT* | 0.01149 | 0.01149 | 0.01149 | 0.01149 |
| MPT(Zenkour (2003)) | 0.01150 | 0.01155 | 0.01183 | 0.01284 |
| RPT(Thai and Choi (2013)) | 0.01150 | 0.01155 | 0.01184 | 0.01286 |
| SRSDT (Thai et al. (2013)) | 0.01150 | 0.01155 | 0.01184 | 0.01286 |
| Present | 0.01150 | 0.01155 | 0.01183 | 0.01285 |

MPT: Mixed plate theory, RPT: Refined plate theory, SRSDT: Simple refined shear deformation theory.

Non-dimensional transverse displacements for *CPT* reported in Tables 1 through 6 have been calculated by the present authors.

Following points must be noted regarding clamped boundary conditions:

1. At the clamped edges for the plate with *SCSC* boundary conditions, Reddy et al. (2001) assume transverse deflection of the mid-surface of the plate and rotations of normals to the mid-surface of the plate about $y$ and $x-$ axes to be zero. However, it has been pointed out by Groh and Weaver (2015) that, irrespective to the choice of shape function, restraining such rotations perpendicular to clamped edges leads to a static inconsistency at the clamped edge.

2. At the clamped edges for the plate with *SCSC* boundary conditions, Lee et al. (2002) assume transverse deflection of the mid-surface of the plate and rotations of normals to the mid-surface of the plate about $y$ and $x-$ axes to be zero. Lee et al. (2002) specify only one type of clamped boundary condition.

Similarly, at the clamped edges for plates with *SCSS* and *SCSF* boundary conditions, Zenkour (2003) assumes transverse deflection of the mid-surface of the plate and rotation angles of a line normal to the mid-surface of the plate before deformation about $y$ and $x-$ axes to be zero. Zenkour (2003) specifies only one type of clamped boundary condition.

Senjanović et al. (2013) have used key concepts of Shimpi et al. (2007) by reducing number of unknown functions to one for the vibrations of thick plates. Although the work reported by Senjanović et al. (2013) is similar to the present theory, it also specifies only one type of clamped boundary condition.

3. At the clamped edges for plates with *SCSS* and *SCSF* boundary conditions, Thai and Choi (2013) and Thai at al. (2013) assume bending component of transverse displacement, shear component of transverse displacement and their slopes to be zero. This is incorrect way of specifying boundary conditions as it amounts to shear forces to be zero at the clamped edge.

As discussed earlier, to the best of authors' knowledge, the present theory is the only first-order shear deformation theory that provides two different types of physically meaningful clamped boundary conditions. Both of these clamped boundary conditions are analogous to those discussed by Timoshenko and Goodier (1951) in the context of two-dimensional theory of elasticity approach for beam analysis. The results obtained for plates with clamped edges using clamped edge: type 1 boundary condition of the present theory almost match with corresponding results available in the literature.

## 14 COMMENTS ON THE PLATE FLEXURE RESULTS

Following observations are in connection with the present theory and the numerical results presented in Tables 1 through 6:

1. It must be noted that, the present theory and Mindlin plate theory are first-order shear deformation plate theories. Even though the present plate theory can predict the non-dimensional transverse displacement ( $\overline{w}$ ) to the same accuracy level as that of Mindlin plate theory (refer to Tables 1 through 6); it can be noted that, the present theory involves only one partial governing differential equation and one unknown function. On the other hand, Mindlin plate theory involves three coupled partial governing differential equations and three unknown functions.
2. One can also note that, the efforts involved in obtaining solutions using the present theory are only marginally higher as compared to those involved in the case of *CPT*.
3. With regards to the numerical results presented in Tables 1 through 6, the following can be noted:

a. For the case of square *SSSS* plate carrying a uniformly distributed load (Example 1), the results for non-dimensional transverse displacement ( $\overline{w}$ ), non-dimensional bending moment ( $\overline{M}_x$ ) and non-dimensional shear force ( $\overline{Q}_x$ ) are presented in Table 1.

• The non-dimensional transverse displacement ( $\overline{w}$ ) predicted by the present theory almost matches with corresponding results obtained using Mindlin plate theory (Lee et al. (2002)) and Levinson plate theory (Reddy et al. (2001)). Even for a square plate with $\mathrm{h/a} = 0.20$ , $\overline{w}$ obtained using the present theory is identical to the corresponding value obtained using Mindlin plate theory (Lee et al. (2002)) and Levinson plate theory (Reddy et al. (2001)). Whereas, *CPT* underestimates $\overline{w}$ by $17.14\ \%$ with respect to the present theory.

• The non-dimensional bending moment ( $\overline{M}_x$ ) predicted by the present theory matches exactly with corresponding results obtained using Mindlin plate theory (Lee et al. (2002)). Even for a square plate with $\mathrm{h/a} = 0.20$ , $\overline{M}_x$ obtained using the present theory is identical to the corresponding value obtained using Mindlin plate theory (Lee et al. (2002)).

• The non-dimensional shear force ( $\overline{Q}_x$ ) predicted by the present theory almost matches with corresponding results obtained using Mindlin plate theory (Lee et al. (2002)). Even for a square plate with $\mathrm{h/a} = 0.20$ , the percentage difference involved in predicting $\overline{Q}_x$ by the present theory and by Mindlin plate theory (Lee et al. (2002)) is $0.30\ \%$ with respect to the present theory. Whereas, *CPT* overestimates $\overline{Q}_x$ by $1.20\ \%$ with respect to the present theory.

b. For the case of rectangular *SCSC* plate ( $\mathrm{b/a} = 1.0$ and $\mathrm{b/a} = 2.0$ ) carrying a uniformly distributed load (Example 2), the results for non-dimensional transverse displacement ( $\overline{w}$ ) are presented in Table 2.

• The non-dimensional transverse displacement ( $\overline{w}$ ) predicted by the present theory matches exactly with corresponding results obtained using Mindlin plate theory (Lee et al. (2002)) for thin plates ( $\mathrm{h/a} = 0.01$ and $\mathrm{h/a} = 0.05$ ). Whereas, $\overline{w}$ predicted by the present theory using clamped edge: type 1 boundary condition almost matches with corresponding results obtained using Mindlin plate theory (Lee et al. (2002)) for thick plates ( $\mathrm{h/a} = 0.10$ and $\mathrm{h/a} = 0.20$ ). Even for a square plate with $\mathrm{h/a} = 0.20$ , the percentage difference involved in predicting $\overline{w}$ by the present theory using clamped edge: type 1 boundary condition and by Mindlin plate theory (Lee et al. (2002)) is $1.95\ \%$ with respect to the present theory. Whereas, *CPT* underestimates $\overline{w}$ by $37.66\ \%$ with respect to the present theory.

c. For the case of square *SFSF* plate carrying a uniformly distributed load (Example 3), the results for non-dimensional transverse displacement ( $\overline{w}$ ) are presented in Table 3.

• The non-dimensional transverse displacement ( $\overline{w}$ ) at the center of the plate predicted by the present theory matches exactly with corresponding results obtained using Mindlin plate theory (Lee et al. (2002)) for thin plates ( $\mathrm{h/a} = 0.01$ ). Whereas, $\overline{w}$ predicted by the present theory almost matches with corresponding results obtained using Mindlin plate theory (Lee et al. (2002)) for thick plates ( $\mathrm{h/a} = 0.10$ and $\mathrm{h/a} = 0.20$ ). Even for a square plate with $\mathrm{h/a} = 0.20$ , the percentage difference involved in predicting $\overline{w}$ by the present theory and by Mindlin plate theory (Lee et al. (2002)) is $0.90\ \%$ with respect to the present theory. Whereas, *CPT* underestimates $\overline{w}$ by $9.16\ \%$ with respect to the present theory.

d. For the case of rectangular *SCSS* plate ( $\mathrm{b/a} = 2.0$ ) carrying a uniformly distributed load (Example 4), the results for non-dimensional transverse displacement ( $\overline{w}$ ) are presented in Table 4.

• The non-dimensional transverse displacement ( $\overline{w}$ ) predicted by the present theory matches exactly with corresponding results obtained using mixed plate theory (Zenkour (2003)) for thin plates ( $\mathrm{h/a} = 0.001$ and $\mathrm{h/a} = 0.04$ ). Whereas, $\overline{w}$ predicted by the present theory using clamped edge: type 1 boundary condition almost matches with corresponding results obtained using mixed plate theory (Zenkour (2003)) for thick plates ( $\mathrm{h/a} = 0.10$ and $\mathrm{h/a} = 0.20$ ). Even for a rectangular plate with $\mathrm{h/a} = 0.20$ , the percentage difference involved in predicting $\overline{w}$ by the present theory using clamped edge: type 1 boundary condition and by mixed plate theory (Zenkour (2003)) is $0.28\ \%$ with respect to the present theory. Whereas, *CPT* underestimates $\overline{w}$ by $13.61\ \%$ with respect to the present theory.

e. For the case of rectangular *SCSF* plate ( $\mathrm{b/a} = 2.0$ ) carrying a uniformly distributed load (Example 5), the results for non-dimensional transverse displacement ( $\overline{w}$ ) are presented in Table 5.

• The non-dimensional transverse displacement ( $\overline{w}$ ) predicted by the present theory matches exactly with corresponding results obtained using mixed plate theory (Zenkour (2003)) for thin plates ( $\mathrm{h/a} = 0.001$ ). Whereas, $\overline{w}$ predicted by the present theory using clamped edge: type 1 boundary condition almost matches with corresponding results obtained using mixed plate theory (Zenkour (2003)) for thick plates ( $\mathrm{h/a} = 0.10$ and $\mathrm{h/a} = 0.20$ ). Even for a rectangular plate with $\mathrm{h/a} = 0.20$ , the percentage difference involved in predicting $\overline{w}$ by the present theory using clamped edge: type 1 boundary condition and by mixed plate theory (Zenkour (2003)) is $0.25\ \%$ with respect to the present theory. Whereas, *CPT* underestimates $\overline{w}$ by $12.46\ \%$ with respect to the present theory.

f. For the case of rectangular *SSSF* plate ( $\mathrm{b/a} = 2.0$ ) carrying a uniformly distributed load (Example 6), the results for non-dimensional transverse displacement ( $\overline{w}$ ) are presented in Table 6.

• The non-dimensional transverse displacement ( $\overline{w}$ ) predicted by the present theory matches exactly with corresponding results obtained using mixed plate theory (Zenkour (2003)) for thin plates ( $\mathrm{h/a} = 0.001$ and $\mathrm{h/a} = 0.04$ ). Whereas, $\overline{w}$ predicted by the present theory almost matches with corresponding results obtained using mixed plate theory (Zenkour (2003)) for thick plates ( $\mathrm{h/a} = 0.10$ and $\mathrm{h/a} = 0.20$ ). Even for a rectangular plate with $\mathrm{h/a} = 0.20$ , the percentage difference involved in predicting $\overline{w}$ by the present theory and by mixed plate theory (Zenkour (2003)) is $0.08\ \%$ with respect to the present theory. Whereas, *CPT* underestimates $\overline{w}$ by $10.58\ \%$ with respect to the present theory.

4. It should be noted that non-dimensional transverse displacement ( $\overline{w}$ ) at the center of the plate ( $x = a\ /\ 2\,, y = 0$ ) predicted by the present theory (Table 3) for the case of square *SFSF* plate (Example 3) has good agreement with the corresponding results available in the literature (for square *SFSF* plate with $\mathrm{h/a} = 0.20$ , maximum percentage difference of $0.90\ \%$ is observed in predicting $\overline{w}$ by the present theory and by Mindlin plate theory (Lee et al. (2002)) with respect to the present theory). However, percentage difference of $4.58\ \%$ in predicting $\overline{w}$ by the present theory and by Mindlin plate theory (Lee et al. (2002)) with respect to the present theory is observed for square *SFSF* plate with $\mathrm{h/a} = 0.20$ at location $x = a\ /\ 2\,, y = b\ /\ 2$ . With regards to this case, following points should be noted:

a. It can be observed that for the plate with *SFSF* boundary conditions, including more number of terms in series expansion of bending component ( $w_b$ ) of transverse displacement ( $w$ ) (beyond $m = 7$ ) while solving using the present theory have no significant influence on $\overline{w}$ as shown below in Figure 3.

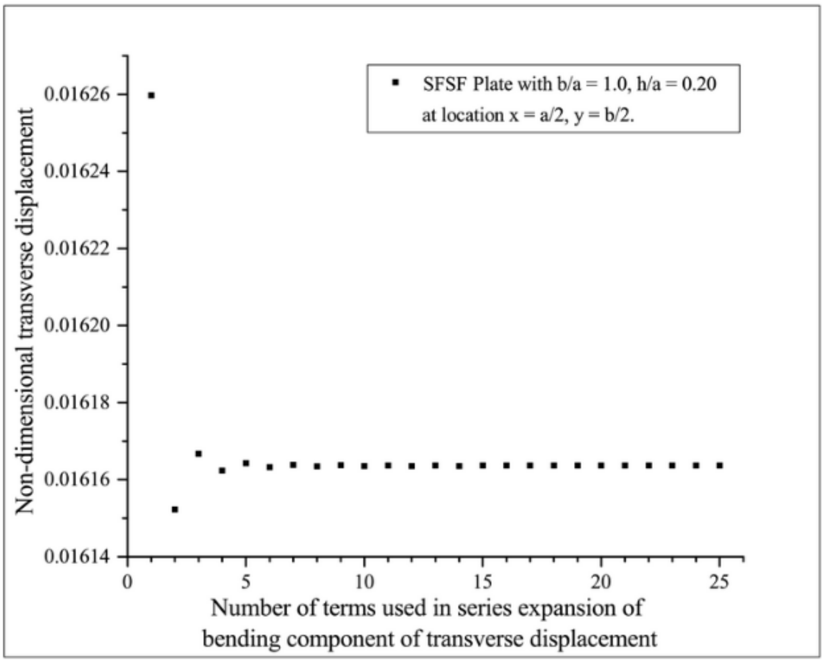


***Figure 3:*** *Effect of number of terms in series expansion of bending component* $w_b$ *of transverse displacement* $w$ *on non-dimensional transverse displacement* $\overline{w}$ *for the plate with SFSF boundary conditions (Example 3).*

Hence, percentage difference of $4.58\ \%$ for square *SFSF* plate with $\mathrm{h/a} = 0.20$ at location $x = a\ /\ 2$, $y = b\ /\ 2$ in predicting $\overline{w}$ by the present theory and by Mindlin plate theory (Lee et al. (2002)) with respect to the present theory is not due to lack of terms in series expansion of $w_b$.

b. Even for the simple case of plate clamped edge boundary conditions, it is interesting to note as quoted by Groh and Weaver (2015), that

"Essential boundary condition of vanishing Kirchoff rotation perpendicular to an edge ($w_{,x} = 0$ or $w_{,y} = 0$) is physically inaccurate, as the rotation at a clamped edge may in fact be non-zero due to the presence of transverse shear rotation."

As mentioed earlier, this leads to inconsistencies with regards to the shear force arising in the formulation of flexural behaviour of plates with clamped edge boundary conditions using a certian class of axiomatic shear deformation theories.

To the best of knowledge of authors, similar to the case of plate clamped edge boundary conditions, even plate free edge boundary conditions reported in the literature overall lack clarity in general.

c. Lee et al. (2002), Zenkour (2003), Thai and Choi (2013), Thai et al. (2013), Reddy et al. (2001), Salerno and Goldberg (1960) and Kant (1982) in which the work on flexural analysis of the plate with *SFSF* boundary conditions is reported, do not mention an exact solution for flexural analysis of the plate with *SFSF* boundary conditions obtained using three dimensional elasticity approach and authors are also unaware of such an exact solution for the plate with *SFSF* boundary conditions. Lack of such an exact solution restricts the comparison of the results obtained using the present theory with the corresponding results obtained using other shear deformation plate theories reported in the literature.

d. *CPT* requires specification of two bounary conditions per plate edge. It should be noted that for the case of plate free edge, one of the boundary conditions of *CPT* (Kirchoff shear force) is formulated using the combination of twisting moment along with shear force at that free edge (Timoshenko and Woinowsky-Krieger (1959), page no 83 through 88). Similar is the case with the present theory and it can be the possible reason behind percentage difference of $4.58\ \%$ for square *SFSF* plate with $\mathrm{h/a} = 0.20$ at location $x = a\ /\ 2, y = b\ /\ 2$ in predicting $\overline{w}$. Hence, the result for above-mentioned case may not be as accurate as that obtained using other shear deformation plate theories reported in the literature.

However, it should also be noted that, as far as the plate with $\mathrm{b/a} = 1.0$, $\mathrm{h/a} = 0.20$ is concerned, it is possible to debate on whether it qualifies as a plate or a stubby object. Percentage difference of $4.58\ \%$ for above-mentioned case needs to be construed by taking this point into account. The results for above-mentioned case are included in Table 3 due to availability of corresponding results in the literature. On the other hand, percentage difference in predicting $\overline{w}$ for square *SFSF* plate with $\mathrm{h/a} = 0.10$ at location $x = a\ /\ 2, y = b\ /\ 2$ is only $1.96\ \%$.

Hence from above discussion, it can be conceded that the results for the plate with *SFSF* boundary conditions obtained using the present theory are farely good. And the results for remaining plate examples (plate with *SSSS*, *SCSC*, *SCSS*, *SCSF* and *SSSF* boundary conditions) obtained using the present theory have very good agreement with the corresponding results available in the literature.

## 15 CONCLUDING REMARKS

In this paper, single variable new first-order shear deformation theory for flexure of an isotropic plate is presented. Important features of the present theory can be stated as follows:

**1.** The present theory is a displacement based first-order shear deformation plate theory and has single fourth-order partial governing differential equation involving only one unknown function. Whereas Mindlin plate theory, which is also a first-order shear deformation plate theory, involves three coupled partial governing differential equations and three unknown functions.

**2.** The displacement functions of the present theory give rise to constant transverse shear strains through the plate thickness. As is case of Mindlin plate theory, the present theory also requires specification of a shear correction factor.

**3.** In the present theory, transverse shear stresses are obtained using modified constitutive relations.

**4.** By utilizing gross equilibrium equations, the governing differential equation of the present theory is obtained. The boundary conditions have been obtained based on physical understanding.

**5.** To the best of authors' knowledge, the present theory is the only first-order shear deformation plate theory with two different types of physically meaningful clamped boundary conditions. Both clamped boundary conditions of the present theory are analogous to those discussed by Timoshenko and Goodier (1951) in the context of two-dimensional theory of elasticity approach for beam analysis. It is to be noted that the results obtained for plates with clamped edges using clamped edge: type 1 boundary condition of the present theory almost match with corresponding results available in the literature. In addition, the present theory specifies one more type of clamp boundary condition *i.e.* clamped edge: type 2, which has not been reported by any other first-order shear deformation plate theory available in the literature.

**6.** The expressions of the present theory have a striking resemblance to the corresponding expressions of classical plate theory in many aspects (e.g. governing differential equation, expressions for moments and shear forces). The only difference is that, bending component $w_b$ (of transverse displacement $w$) appears in case of the present theory, whereas in case of classical plate theory, in its place transverse displacement $w$ appears. Also, the expressions for shear forces obtained using the present theory are identical to those obtained using gross equilibrium equations.

**7.** Effectiveness of the present theory is demonstrated through illustrative examples for the plate flexure. The numerical results obtained are compared with corresponding results of other shear deformation plate theories available in the literature.

**8.** It is observed that, the efforts involved in obtaining solutions using the present theory are only marginally higher as compared to those involved in the case of classical plate theory.

In conclusion, the present plate theory is a simple and accurate first-order shear deformation plate theory for the flexure of isotropic plates.

## 16 DECLARATION OF CONFLICT OF INTEREST

The authors declare no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

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# APPENDIX A: RELATIONSHIP BETWEEN BENDING COMPONENT AND SHEAR COMPONENT OF TRANSVERSE DISPLACEMENT

Using displacement functions of the present theory as given by expressions (4), (5) and (6) in strain-displacement relations given by equation (1), expressions for transverse shear strains $\gamma_{yz}$ and $\gamma_{zx}$ can be written as follows:

$$\gamma_{yz} = \frac{\partial w_s}{\partial y} \tag{A.1}$$

$$\gamma_{zx} = \frac{\partial w_s}{\partial x} \tag{A.2}$$

Now, using expressions (A.1) and (A.2) in constitutive relations given by equations (17) and (18), expressions for transverse shear stresses $\tau_{yz}$ and $\tau_{zx}$ can be written as follows:

$$\tau_{yz} = \frac{E\,k}{2\,(1+\mu)}\frac{\partial w_s}{\partial y} \tag{A.3}$$

$$\tau_{zx} = \frac{E\,k}{2\,(1+\mu)}\frac{\partial w_s}{\partial x} \tag{A.4}$$

Now, using expressions (A.3) and (A.4) in equation (24), expressions for shear forces $Q_x$ and $Q_y$ can be written as follows:

$$Q_x = \frac{E\,k\,h}{2\,(1+\mu)}\frac{\partial w_s}{\partial x} \tag{A.5}$$

$$Q_y = \frac{E\,k\,h}{2\,(1+\mu)}\frac{\partial w_s}{\partial y} \tag{A.6}$$

Now, using expressions for moments $M_x$, $M_y$ and $M_{xy}$ as given by equations (25), (26) and (27) in gross equilibrium equations (34) and (35), expressions for shear forces $Q_x$ and $Q_y$ can also be written as follows:

$$Q_x = -D\frac{\partial}{\partial x}\left(\frac{\partial^2 w_b}{\partial x^2}+\frac{\partial^2 w_b}{\partial y^2}\right) \tag{A.7}$$

$$Q_y = -D\frac{\partial}{\partial y}\left(\frac{\partial^2 w_b}{\partial x^2}+\frac{\partial^2 w_b}{\partial y^2}\right) \tag{A.8}$$

Now, using expressions (A.5) and (A.6) in expressions (A.7) and (A.8) respectively, we can write as follows:

$$\frac{E\,k\,h}{2\,(1+\mu)}\frac{\partial w_s}{\partial x} = -D\frac{\partial}{\partial x}\left(\frac{\partial^2 w_b}{\partial x^2}+\frac{\partial^2 w_b}{\partial y^2}\right) \tag{A.9}$$

$$\frac{E\,k\,h}{2\,(1+\mu)}\frac{\partial w_s}{\partial y} = -D\frac{\partial}{\partial y}\left(\frac{\partial^2 w_b}{\partial x^2}+\frac{\partial^2 w_b}{\partial y^2}\right) \tag{A.10}$$

Now, using expressions (A.9) and (A.10), we can express shear component $w_s$ as follows:

$$w_s = -\frac{2\,D\,(1+\mu)}{E\,k\,h}\left(\frac{\partial^2 w_b}{\partial x^2} + \frac{\partial^2 w_b}{\partial y^2}\right) \tag{A.11}$$

Now, using expression (A.11) and expression (30), shear component $w_s$ can be expressed as follows:

$$w_s = -\frac{h^2}{6\,k\,(1-\mu)}\left(\frac{\partial^2 w_b}{\partial x^2} + \frac{\partial^2 w_b}{\partial y^2}\right) \tag{A.12}$$

# APPENDIX B: CONVERGENCE ANALYSIS FOR CASE OF THE PLATE WITH *SSSS* BOUNDARY CONDITIONS (EXAMPLE 1).

Effect of number of terms in series expansion of bending component $w_b$ of transverse displacement $w$ on non-dimensional transverse displacement $\bar{w}$, non-dimensional bending moment $\bar{M}_x$ and non-dimensional shear force $\bar{Q}_x$ of the plate with *SSSS* boundary conditions (Example 1) for various values of plate thickness ratio is depicted with the help of graphs as shown below in Figures B.4 through B.9.

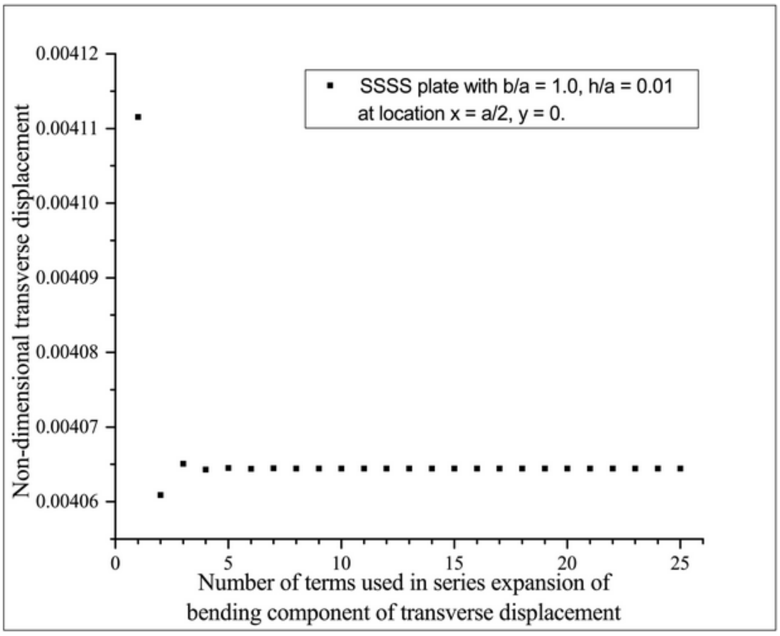


***Figure B.4:*** *Effect of number of terms used in series expansion of bending component $w_b$ of transverse displacement $w$ on non-dimensional transverse displacement $\bar{w}\left(=\frac{wD}{q_o a^4}\right)$ for the plate with SSSS boundary conditions (Example 1) and having $\mathrm{b/a} = 1.0$ and $\mathrm{h/a} = 0.01$.*

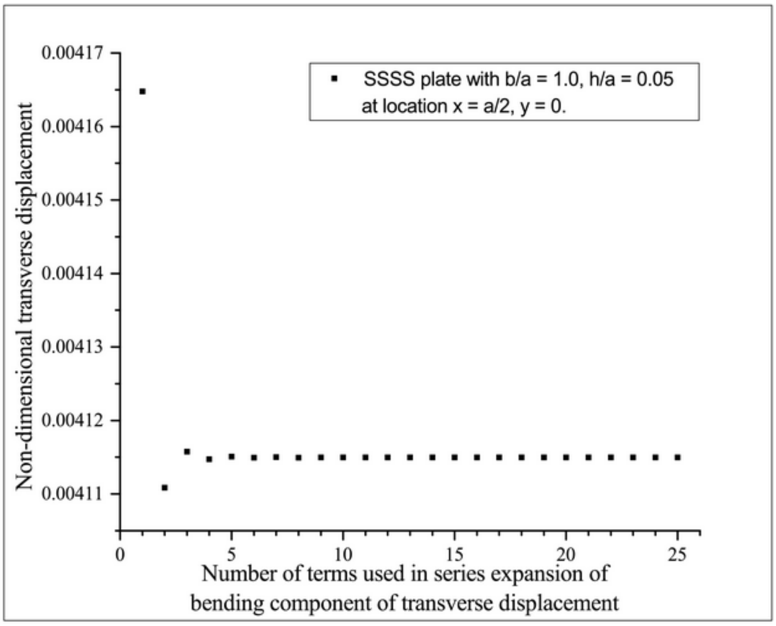


***Figure B.5:*** *Effect of number of terms used in series expansion of bending component* $w_b$ *of transverse displacement* $w$ *on non-dimensional transverse displacement* $\overline{w}\left(=\dfrac{wD}{q_o a^4}\right)$ *for the plate with SSSS boundary conditions (Example 1) and having* $\mathrm{b/a} = 1.0$ *and* $\mathrm{h/a} = 0.05$ *.*

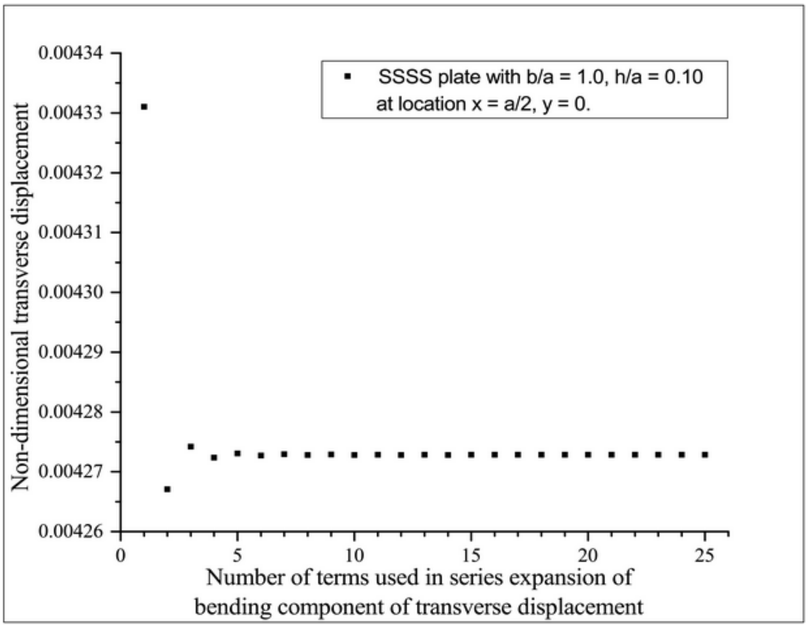


***Figure B.6:*** *Effect of number of terms used in series expansion of bending component* $w_b$ *of transverse displacement* $w$ *on non-dimensional transverse displacement* $\overline{w}\left(=\dfrac{wD}{q_o a^4}\right)$ *for the plate with SSSS boundary conditions (Example 1) and having* $\mathrm{b/a} = 1.0$ *and* $\mathrm{h/a} = 0.10$ *.*

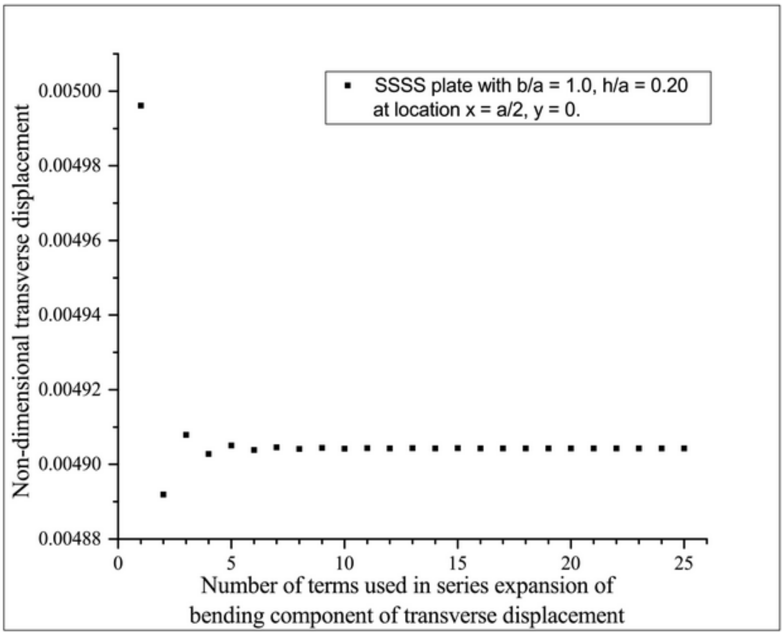


***Figure B.7:*** *Effect of number of terms used in series expansion of bending component* $w_b$ *of transverse displacement* $w$ *on non-dimensional transverse displacement* $\overline{w}\left(=\frac{wD}{q_o a^4}\right)$ *for the plate with SSSS boundary conditions (Example 1) and having* $\mathrm{b/a} = 1.0$ *and* $\mathrm{h/a} = 0.20$ *.*

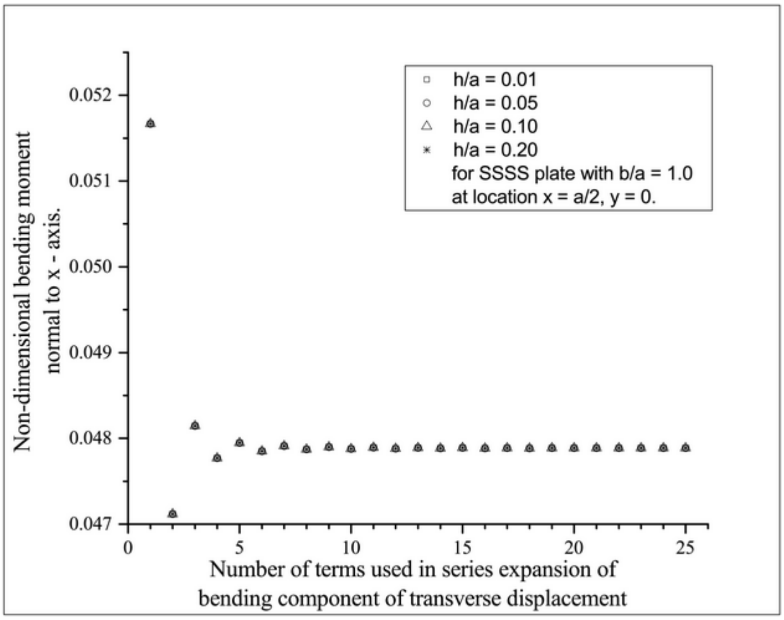


***Figure B.8:*** *Effect of number of terms used in series expansion of bending component* $w_b$ *of transverse displacement* $w$ *on non-dimensional bending moment* $\overline{M_x}\left(=\frac{M_x}{q_o a^2}\right)$ *for the plate with SSSS boundary conditions (Example 1) and having* $\mathrm{b/a} = 1.0$ *.*

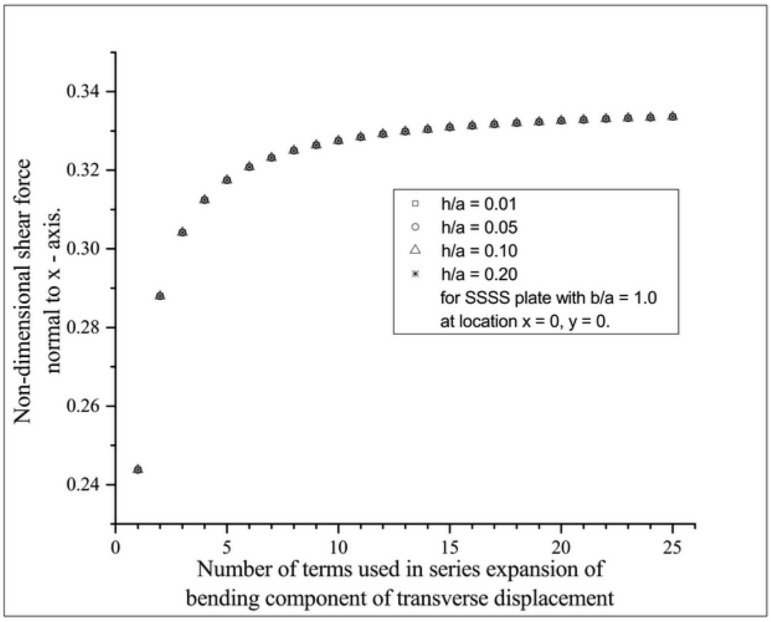


***Figure B.9:*** *Effect of number of terms used in series expansion of bending component $w_b$ of transverse displacement $w$ on non-dimensional shear force $\overline{Q_x}\left(=\dfrac{Q_x}{q_o a}\right)$ for the plate with SSSS boundary conditions (Example 1) and having* $\mathrm{b/a} = 1.0$.

From Figures B.4 through B.9, following observations can be made for the plate with *SSSS* boundary conditions:

1. It is evident from Figures B.4 through B.7 that including more number of terms in series expansion of bending component $w_b$ of transverse displacement $w$ (beyond $m = 7$ ) while solving using the proposed theory have no significant influence on non-dimensional transverse displacement $\overline{w}$ as transverse displacement $w$ is the primary unknown quantity.
2. On the other hand, inclusion of more number of terms in series expansion of bending component $w_b$ of transverse displacement $w$ is required to achieve convergence of derived unknown quantities such as $M_x$ and $Q_x$.

a. Series expansion of $w_b$ upto $m = 21$ is required to achieve convergence of $M_x$ (Figure B.8).

b. Series expansion of $w_b$ upto $m = 49$ is required to achieve convergence of $Q_x$ (Figure B.9).