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Application of the rigid finite element method to modelling ropes

Abstract

Two-dimensional motion of a rope fixed at one end is considered. The Rigid Finite Element Method (RFEM) is reviewed and applied to obtain a model of the rope, including its elastic and dissipative properties. Equations of motion are derived without the small displacement assumption, using the Lagrange equations. The resulting model is compared to another one, being derived within the framework of standard analytical mechanics methods and the Lagrange formalism. Advantages of the RFE approach are discussed from a computational point of view. The presented, alternative model can be a basis for e cient numerical simulations, which seem to be useful in further, comparative studies of the rope dynamics.

Keywords

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1 INTRODUCTION

Using discrete models to approximate continuous systems is a common procedure. It can be extremely convenient when dealing with mechanics of a deformable solid body submitted to large displacements and deformations, e.g. ropes, cables or belts. A discrete model may be produced by applying various theoretical formulations. The slender bodies, for instance, can be simply represented as chains of rigid elements and described with use of some analytical mechanics methods. Otherwise, the discretization may be an imminent feature of a certain computational technique, e.g. the Finite Element Method (FEM). The former approach has been previously applied to describe and simulate motion of a hanging rope (see [1{3]). However, the resulting mathematical model consists of the implicit system of ordinary differential equations (ODEs):

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} = \mathbf{f}(t, \mathbf{q}, \dot{\mathbf{q}}) \tag{1}$$

with time-dependent mass matrix on the left-hand side. Consequently, numerical integration is a cumbersome process, involving very sophisticated strategies. The difficulties can be avoided when using the Rigid Finite Element Method (RFEM).

This method was developed by Kruszewski et al. [4] and should be distinguished from the classical FEM. In the RFE approach, a physical model is composed of rigid (nondeformable) bodies connected by massless elastic-dissipative elements. In this paper, the methodology is outlined and employed in modeling of the rope. Perhaps the earliest studies of the rope motion were done by D. Bernoulli (1732) and Euler (1781). They considered and solved the problem of small vibrations of a perfectly flexible, uniform rope which is fixed at one end [6]. Nowadays many researchers derive and improve models of various slender bodies. Usually continuum approach is applied, e.g. in case of ropes [10, 13], cables [9], y lines [7] and even chains [10, 11]. Probably dynamics of a whip is most spectacular, which has been investigated both theoretically and experimentally. The contemporary theory of whip motion was developed by Goriely and McMillen [8]. When it comes to a discrete approach, the chain-like model of Pieranski and Tomaszewski [12] should be mentioned, which was crucial for our previous works on the classical problem of rope dynamics.

2 PHYSICAL MODEL

The RFE approach has been thoroughly described in [4, 5]. Let us consider a rope of length L and mass M which is initially suspended between two supports (see Fig. 1a). Physical model of the system is created in two steps. First, the slender body is divided into $\tilde{n}$ sections of equal length $\Delta l = L / \tilde{n}$. Elastic-dissipative properties of each section are concentrated in its center and represented by a spring-damping element (SDE). Due to the secondary division, the system consists of $n = \tilde{n} + 1$ rigid finite elements (RFEs) interconnected, and connected to foundation, via the SDEs as illustrated in Fig. 1b. The SDEs are assumed to be massless and characterized by stiffness and damping coefficients. Every RFE, in turn, is described by mass and inertial moments.

The two additional rigid elements (RFE_0 and RFE_{n+1}) can be regarded as a foundation or can be used for realization of non-stationary constraints (movable supports). The classical fall of a folded rope can be observed as the parameters of SDE_{n+1} have zero values.

3 MATHEMATICAL DESCRIPTION

In case of the plane system, every RFE has three degrees of freedom and its position can be specified with use of the generalized coordinate vector $\mathbf{q}_i = \left[q_{xi}, q_{yi}, q_{\varphi i} \right]^T$. As illustrated in Fig. 2, there is a local coordinate system ${}^i x {}^i y$ connected to the ith rigid element. However,
note that $\mathbf{q}_i$ related to the global, reference system *xy*. Similarly, the generalized forces acting on the RFE can be specified by the vector $\mathbf{P}_i = \left[P_{xi}, P_{yi}, P_{\varphi i} \right]^T$.

Assuming that the rope is a homegenuous prismatic bar of diameter d, length of the RFEs can be written as $l_1 = l_n = \Delta l / 2$ and $l_i = \Delta l$ for $i = 2,3,\ldots,n-1$. Consequently, the mass $m_i = M l_i / L, i = 1,2,\ldots,n$. Since each rigid element is a cylinder, its moment of inertia with respect to the central principal axis of inertia ${}^{i}z$ (perpendicular to the plane of motion) is given by $J_i = \frac{m_i}{4}\left(\frac{l_i^2}{3} + \frac{d^2}{4}\right), \qquad i = 1,2,\ldots,n$.

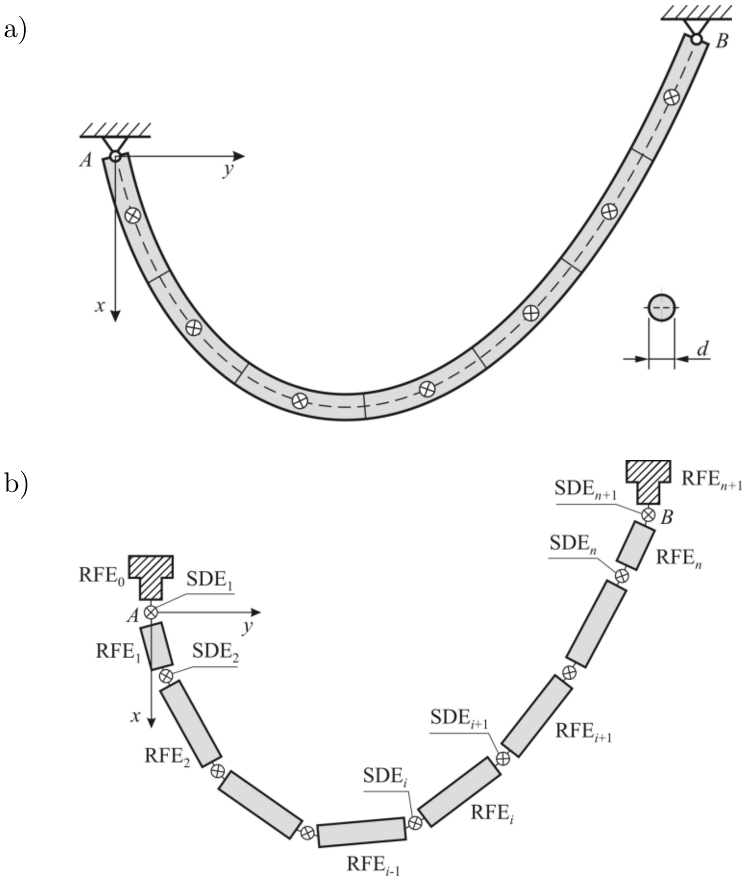

Fig. 1 Division of a rope: a primary division into equal segments, b secondary division: RFEs connected by SDEs

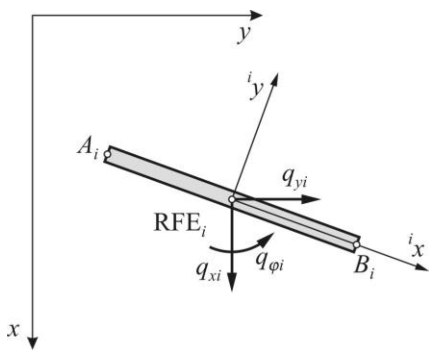

Fig. 2 Local coordinate system and generalized coordinates of a RFE

The system has N = 3n degrees of freedom. In the RFE approach, equations of motion are derived using the Lagrange equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\mathbf{q}}}\right)-\frac{\partial T}{\partial \mathbf{q}}+\frac{\partial D}{\partial \dot{\mathbf{q}}}+\frac{\partial V}{\partial \mathbf{q}}=\mathbf{P}, \tag{2}$$

where T denotes kinetic energy of the system, V is potential energy of the system, D is the dissipation function. The vectors q and P are composed of the subvectors $\mathbf{q}_i$ and $\mathbf{P}_i$, respectively:

$$\mathbf{q}=\begin{bmatrix}\mathbf{q}_1\\ \mathbf{q}_2\\ \vdots\\ \mathbf{q}_n\end{bmatrix}, \qquad \mathbf{P}=\begin{bmatrix}\mathbf{P}_1\\ \mathbf{P}_2\\ \vdots\\ \mathbf{P}_n\end{bmatrix}, \tag{3}$$

The first two terms of Eq. (2) have the form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{\mathbf{q}}}\right)-\frac{\partial T}{\partial \mathbf{q}}=\mathbf{A}\,\ddot{\mathbf{q}}, \tag{4}$$

where A is a diagonal mass matrix ($N \times N$), defined as

$$\mathbf{A}_{kk}=\begin{cases} m_i & \text{for } k=3i-2, \quad i=1,2,\ldots,n\\ m_i & \text{for } k=3i-1, \quad i=1,2,\ldots,n\\ J_i & \text{for } k=3i, \quad i=1,2,\ldots,n\end{cases} \tag{5}$$

Now, consider two neighbouring RFEs (see Fig. 3). The ith SDE is assumed to have the same orientation as the ith RFE. Deformation of the SDEi can be written as follows:

$$\Delta\mathbf{w}_i=\begin{bmatrix}\Delta w_{xi}\\ \Delta w_{yi}\\ \Delta w_{\varphi i}\end{bmatrix}=\begin{bmatrix}{}^{i-1}x_{Ai}-{}^{i-1}x_{Bi-1}\\ {}^{i-1}y_{Ai}-{}^{i-1}y_{Bi-1}\\ q_{\varphi i}-q_{\varphi i-1}\end{bmatrix}, \tag{6}$$

where position of the points A_i and B_{i-1} are expressed in the same coordinate system; the subscripts x, y, φ correspond to tension, shearing and bending, respectively. The potential energy and the dissipation function of the system become

$$V=\frac{1}{2}\sum_{i=1}^{n+1}\left[C_{xi}\Delta w_{xi}^2+C_{yi}\Delta w_{yi}^2+C_{\varphi i}\Delta w_{\varphi i}^2\right]-\sum_{i=1}^{n}m_i g q_{xi}, \tag{7}$$

$$D = \frac{1}{2}\sum_{i=1}^{n+1}\Big[B_{xi}\Delta\dot{w}_{xi}^2 + B_{yi}\Delta\dot{w}_{yi}^2 + B_{\varphi i}\Delta\dot{w}_{\varphi i}^2 \Big] . \tag{8}$$

In the above formulas C_{xi}, C_{yi}, $C_{\varphi i}$ are stuffiness coefficients and B_{xi}, B_{yi}, $B_{\varphi i}$ are damping coefficients, which can be expressed as follows [4]:

$$C_{xi} = \frac{EA}{\Delta l}, \qquad C_{yi} = \frac{GA}{\chi \Delta l}, \qquad C_{\varphi i} = \frac{EI}{\Delta l} , \tag{9}$$

$$B_{xi} = \frac{\eta A}{\Delta l}, \qquad B_{yi} = \frac{\overline{\eta} A}{\chi \Delta l}, \qquad B_{\varphi i} = \frac{\eta I}{\Delta l} , \tag{10}$$

where A denotes the cross-sectional area of the rope, I is the second area moment of the rope's cross-section, E denotes the Young's modulus, G is the shear modulus, χ denotes the shape factor, η and $\overline{\eta}$ are material constants of normal and tangential damping, respectively. Unlike in the case of small vibrations, $\Delta \mathbf{w}_i$ and $\Delta \dot{\mathbf{w}}_i$ have much more complex forms and the derivatives $\partial V / \partial \mathbf{q}$ and $\partial D / \partial \dot{\mathbf{q}}$ cannot be linearly expressed in terms of $\mathbf{q}$ and $\dot{\mathbf{q}}$. Finally, the equations of motion are given by

$$\mathbf{A}\,\ddot{\mathbf{q}} = \mathbf{F}(t, \mathbf{q}, \dot{\mathbf{q}}) , \tag{11}$$

where F is a non-linear vector function, including the components resulting from spring deformation, gravity, dissipation and external forces. Considering that A is diagonal, Eq. (11) can be easily transformed to

$$\ddot{\mathbf{q}} = \mathbf{F}(t, \mathbf{q}, \dot{\mathbf{q}}) . \tag{12}$$

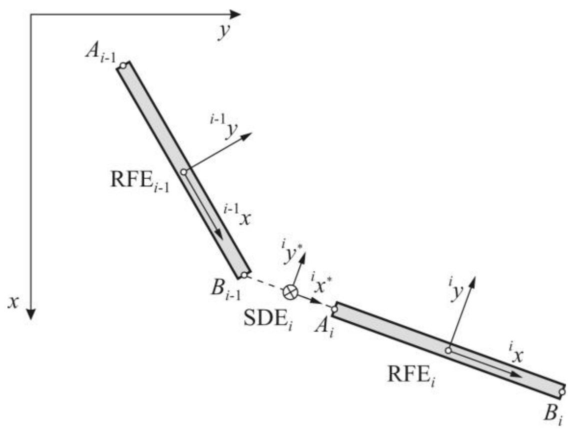

Fig. 3 Two neighbouring RFEs connected by a SDE

Compared with Eq. (1), the mathematical model (12) is a system of ordinary differential equations in the standard (explicit) form. Therefore, solving initial values problems for the equations seems to be significantly simpler and straightforward: many sophisticated techniques, necessary in the former case (especially for computation and processing of the left-hand side matrix), become redundant. Using an appropriate solver can lead to very efficient numerical simulations, which plays a crucial role in studies of long-term behaviour of physical systems (e.g. in chaos identification).

4 NUMERICAL EXPERIMENT

To verify the numerical efficiency of the presented approach, a series of simple numerical experiments have been performed with use of two different models and the results have been compared. Consider motion of the rope which is initially deflected aside, which means that the deflection angle is equal for all the rigid elements: $q_{\varphi i}(0) = \alpha$, where $i = 1,2,...,n$ and $\alpha = 75^{\mathrm{o}}$. Parameters of the rope are presented in Tab. 1. It should be noticed that the damping material constants fulfil the relation [4, 5]:

$$\frac{\overline{\eta}}{\eta} = \frac{G}{E}\,. \tag{13}$$

Quantity	Value	Unit
Rope density, ρ	6000	$\mathrm{kg/m^3}$
Rope length, L	1.0	m
Rope diameter, d	0.005	m
Young's modulus, E	$35 \cdot 10^{-6}$	Pa
Shear modulus, G	$14 \cdot 10^{-6}$	Pa
Material constant of normal damping, η	10^3	$\mathrm{Ns/m^2}$
Material constant of tangential damping $\overline{\eta}$	$4 \cdot 10^{-2}$	$\mathrm{Ns/m^2}$

For purposes of the comparative analysis, a combination of the models described in [2,3] has been used. This system can be regarded as a multiple physical pendulum (MPP) with some additional features. Its members are identical and consist of two connected parts: a rigid rod and a spring. Such whole members are connected by elastic-dissipative joints. Thus, the rope is extensible by involving a simple spring-mass conception; it also includes bending stiffness and damping via non-ideal joints. Mathematical model of the system was derived in Lagrange formalism.

Since the two models are based on various theoretical conceptions, the problem of parameters matching arises. Some properties of the MPP (mass, spring constant, bending stiffness) can be easily calculated using the values from Tab. 1. However, selection of the damping values has been performed by trial and error to ensure possibly highest agreement of the two systems motion.

The problem of the rope dynamics has been solved several times for different discretization density. More precisely, the number of RFEs and the number of the pendulum members have been varied. As with the problem analyzed in [1{3], we decided to apply the MEBDFV solver designed by Abdulla and Cash (Imperial College, London). The code implements the modified extended backward differentiation formulas (MEBDF) developed by Cash (1980). In each case all the solver parameters have been set identically. The calculations have been carried out using PC with Phenom X4 3.0 GHz processor.

In the experiments motion lasting 30 s has been considered. As other works indicate, the time interval is long enough to go beyond a transient phase of the rope's behaviour. Figure 4 shows computation time for various numbers of elements n. As can be seen, in most cases the RFEM approach leads to considerably shorter time of calculations. The MPP model is numerically more efficient only for small n.

Detailed comparative studies focused on dynamics itsels are beyond the scope of this paper. However, to illustrate an agreement rate of the two systems, their total energy E = T+V is shown in Fig. 5. To make the both cases fully comparable, the initial energy E0 is regarded as the zero level. In more systematic analysis the difference between energy of the systems should be minimized by more sophisticated selection of the parameters.

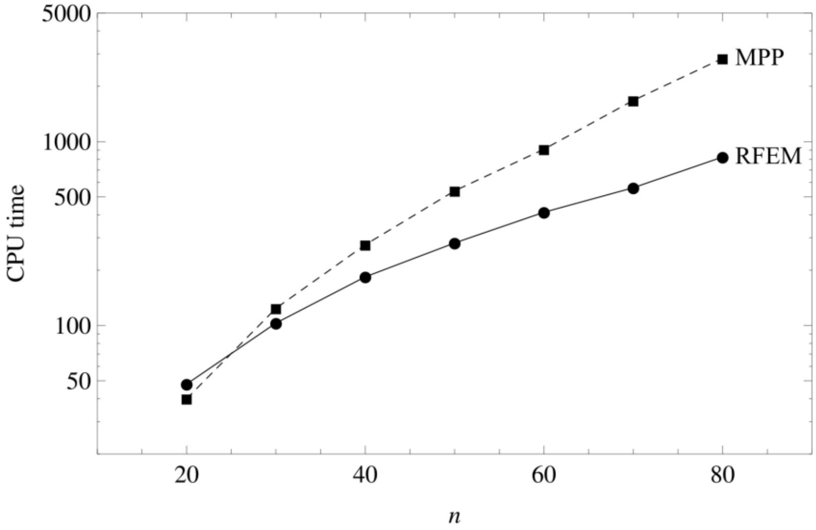

Fig. 4 Number of elements versus computation time in seconds

5 CONCLUSIONS

The rigid finite element method has been applied to obtain mathematical model of a rope. Because the equations of motion have not been formulated under the assumption of small vibrations, the model appears to be non-linear. Nevertheless, its form is advantagous from the computational point of view, when compared with the other models presented in the previous works. The numerical experiment has shown that parameters matching is not straightforward when studying behaviour of two models based on different formulation. However, the performed selection of the parameters values has led to quite compatible solutions. What is more, the obtained results prove that the RFEM approach is numerically more efficient.

All in all, the RFEM is a well-developed and effective approach, which is based on simple conceptions and can be used to analyze both small and large deformations. The presented, numerically efficient model will be useful in further, comparative studies of the rope dynamics.

a)

b)

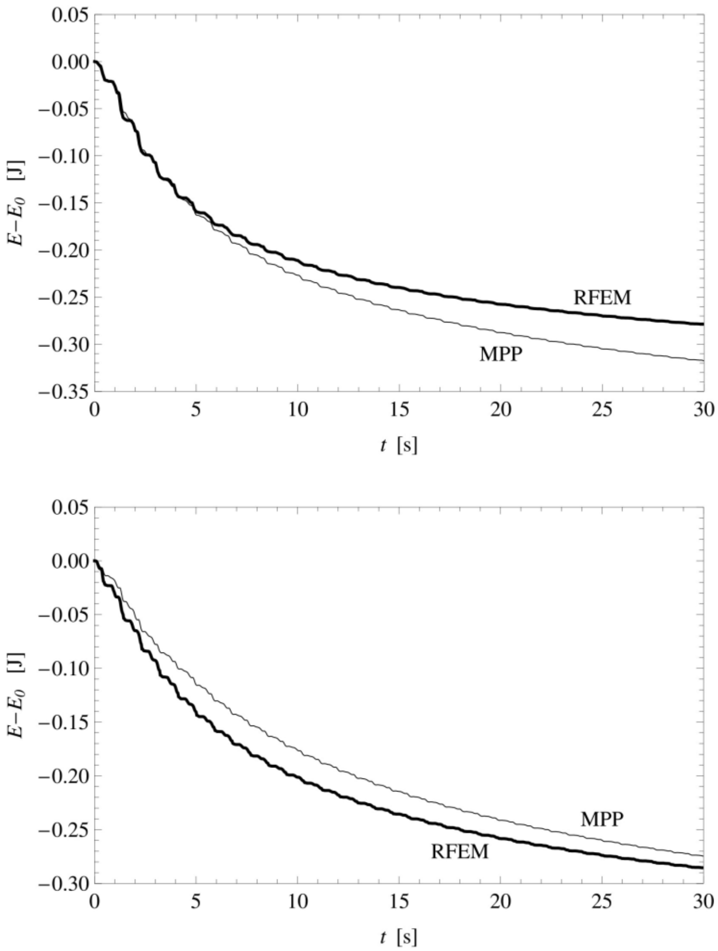

Fig. 5 Total energy of the two systems: a $n = 30$, b $n = 70$

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