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Application of Higher Order Hamiltonian Approach to the Nonlinear Vibration of Micro Electro Mechanical Systems

Abstract

This paper implements the higher order Hamiltonian method to analyze an electrostatically actuated nonlinear micro beam-based micro electro mechanical oscillator. First, second and third approximate solutions are obtained, and the frequency responses of the system are compared with energy balance method solution and previously solved Variational Approach (VA) and exact solution. After driving the equation of motion based on the Euler-Bernoulli beam theory, Galerkin method has been used to simplify the nonlinear equation of motion. Higher order Hamiltonian approach has been used to solve the problem and introduce a design strategy. Phase plane diagram of electrostatically actuated micro beam has plotted to show the stability of presented nonlinear system and natural frequencies are calculated to use for resonator design. According to the numerical results, the second approximate is more acceptable and results show that one could obtain a predesign strategy by prediction of effects of mechanical properties and electrical coefficients on the stability and free vibration of common electrostatically actuated micro beam.

Keywords

Higher order Hamiltonian, MEMS, Electrostatically actuated micro beam, Free Vibrations, Euler-Bernoulli theory, Galerkin method.

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1 INTRODUCTION

Micro Electro Mechanical Systems (MEMS) are used in various engineering fields such as aerospace, optical and biomedical engineering and are enormously used in applications such as micro-switches, transistors, accelerometers, pressure sensors, micro-mirrors, micro-pumps, micro-grippers and bio-MEMS.(Davis, Green et al. 1998, Wang and Musameh 2003, Atashbar, Bejcek et al. 2004, Lin, Taylor et al. 2004, Gu, Elkin et al. 2005, Osiander, Darrin et al. 2005, Balasubramanian and Burghard 2006, Allen, Kichambare et al. 2007, Yogeswaran and Chen 2008, Yogeswaran, Thiagarajan et al. 2008, Jain and Goodson 2011). MEMS are merged devices that join electrical and mechanical components. The study of dynamic and static behavior of atomic force microscope (AFM) cantilevers and vibration control of AFM cantilevers are one of the challenges that coupled electrical and mechanical components (Korayem, Sadeghzadeh et al. 2011, Korayem, Sadeghzadeh et al. 2012, Korayem, Homayooni et al. 2013, Korayem, Karimi et al. 2014). Ghalambaz et al.(Ghalambaz, Ghalambaz et al. 2015)studied the effects of the van der Waals attractions, Casimir force, the small size, the fringing field, the mid-plane stretching, and the axial load on the oscillation frequency of resonators.

Intrinsic intricacy of nonlinear vibration problem of MEMS forces numerical solutions instead of exact analytical responses. Shooting method(Abdel-Rahman, Younis et al. 2002), δ- perturbation method(He 2003), differential quadrature method(Kuang and Chen 2004) , Lindstedt–Poincaré method(He 2002), integral equation method(Pouya 1997), homotopy analysis method (HAM)(Beléndez, Beléndez et al. 2008), variational approach (VA) (He 2007), Max–Min approach (He 2008, Zeng 2009, Zeng and Lee 2009) and Energy Balance Method (He 2002)are some of the numerical and approximate analytical approaches could be addressed. Ganji, Azimi et al.(Ganji, Azimi et al. 2012) applied the Energy Balance Method (EBM) and Amplitude Frequency Formulation (AFF) to govern the approximate analytical solution for motion of two mechanical oscillators. They showed that in comparison with the fourth order Runge-Kutta method, their solution ismore comfortable and useful for solving strong non-linear oscillators. Ganji and Azimi. (Ganji and Azimi 2012) used the Max-Min Approach (MMA) and Amplitude Frequency Formulation (AFF) to derive the approximate analytical solution for motion of nonlinear free vibration of conservative, single degree of freedom systems, and they concluded that both methods have the same results. The results showed these methods are very convenient for solving nonlinear equations and also can be utilized for a wide range of time and boundary conditions for nonlinear oscillators. Yildirim, Saadatnia et al(Yildirim, Saadatnia et al. 2011) applied the Hamiltonian approach to obtain the natural frequency of the Duffing oscillator, the nonlinear oscillator with discontinuity and the quantic nonlinear oscillator. Obtained results were completely inagreement with the approximate frequencies and the exact solution. H. Askari et al. (Askari 2013)utilized the higher order Hamiltonian approach to elicit approximate solutions for the model of buckling of a column and mass-spring system. Y. Khan and M. Akbarzade. (Khan and Akbarzade 2012) used variational approach, Hamiltonian approach, and amplitude-frequency formulation to analysis of nonlinear oscillator equation arising in double-sided clamped microbeam-based electromechanical resonator. Qian, Ren et al.(Qian, Ren et al. 2012)utilized the homotopy analysis method (HAM) to derive analytical approximate solutions for nonlinear vibration of an electrostatically actuated microbeam and for verifying the accuracy of this approach, they compared their method with other analytical and exact solutions.

Fu et al.(Fu, Zhang et al. 2011)applied the Energy Balance Method (EBM) to study a nonlinear oscillation problem in the micro beam model. They governed equation of free vibration of a micro beam, based on the Euler- Bernoulli hypothesis and also compared the results with fourth-order Runge-Kutta method.

H. Rafieipour et al.(Rafieipour, Lotfavar et al. 2013)used the He's frequency amplitude method and presented an analytical closed form solution. Obtained results were in a good agreement with numerical methods.

Bayat et al.(Bayat, Bayat et al. 2014) investigated He's Variational Approach (VA) to solve nonlinear vibration of an electro statically actuated clamped- clamped micro beam that was equivalent to the first order of higher Hamiltonian method(Yildirim, Askari et al. 2012). They demonstrated that VA can be a good candidate for precise periodic solution of nonlinear systems. Final results of mentioned works are listed in table 1.

Fu et al (Fu, Zhang et al. 2011), 2011	$\omega_{EBM} = \sqrt{\frac{4a_4 + 3a_5A^2 + 7a_6A^4/3 + 15a_7A^6/8}{a_1A^4 + 2a_2A^2 + 4a_3}}$
H. Rafieipour et al (Rafieipour, Lotfavar et al. 2013), 2013	$\omega = \frac{\sqrt{2}}{4}\sqrt{\frac{64a_4 + 48A^2a_5 + 40A^4a_6 + 35A^6a_7}{5A^4a_1 + 6A^2a_2 + 8a_3}}$
Bayat et al (Bayat, Bayat et al. 2014), 2014	$\omega_{VA} = \frac{\sqrt{2}}{4}\sqrt{\frac{64a_4 + 48A^2a_5 + 40A^4a_6 + 35A^6a_7}{3A^4a_1 + 4A^2a_2 + 8a_3}}$

Table 1: Comparison of natural frequency of a micro beam from recent related works.

This research investigated high order approximate solutions by using higher order Hamiltonian method(He 2010) for solving a non-linear dynamic problem, in order to have a highly accurate numerical approximation. Contrarily to some recent researches such as (Bayat, Bayat et al. 2014), we showed that the second order is extremely close to the EBM solution and exact solution. The methodology of the higher order Hamiltonian for solving an ordinary differential equation with strong power nonlinearity is presented. Numerical comparisons and results were carried out to confirm the rightness and accuracy of the applied method.

To use higher order Hamiltonian approach, a clamped-clamped micro beam is modeled that placed between two completely fixed electrodes. Deriving the dimensionless equation of motion and separation with assumed mode method, first and higher order approximation of Hamiltonian of system have proposed and then, natural frequency calculated for each case. Finally, we show the effects of various parameters on the frequency of electrostatically actuated micro beam, concluding Hamiltonian approach is completely efficient and agreeable.

2 MATHEMATICAL MODEL

Figure 1 and 2 depict the clamped-clamped micro beam with length l, width h, constant thickness 'b', initial gap $g_0$and electrostatic applied voltage V. The micro beam is doubly clamped and placed between two completely fixed electrodes. Applied voltage is due to the electric field that could be divided into two parts; a DC polarization and an AC electric field.

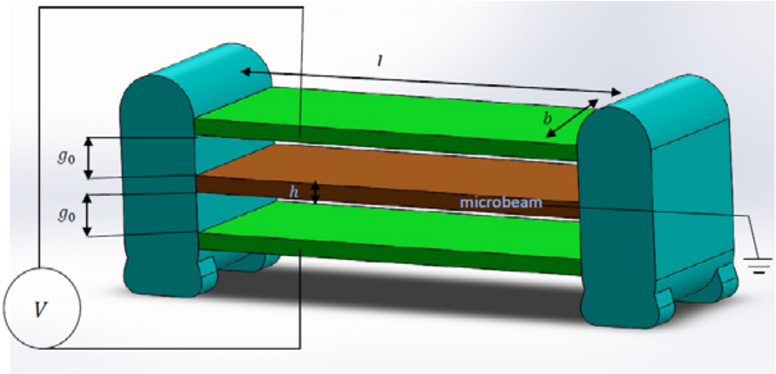

Figure 1: Schematics of clamped-free-clamped-free micro electromechanical resonator.

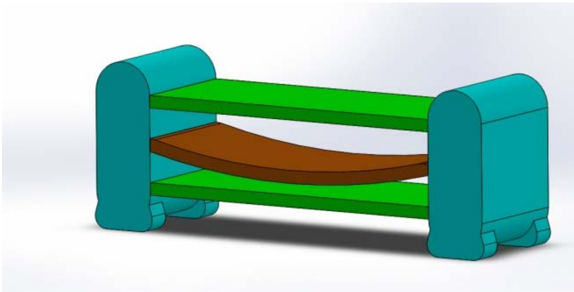

Figure 2: Schematics of deformed micro electromechanical resonator due to an electrostatic voltage.

Applying an AC electric field or a periodic mechanical load results in dynamic deflection and vibration of the micro beam (Younis and Nayfeh 2003). For more design options and facilities, computational studies are essential beside experiments. On the other hand, there are not exact (analytical) closed form solutions for all boundary conditions of mechanical systems. As a good alternative, by applying the Galerkin Method (GM) and utilizing the classical beam theory, the free vibration problem of MEMS could be solved.

The nonlinear partial differential equation of the transverse motion regarding the effect of mid-plane deformation could be expressed as (Rao 2007):

$$\bar{E}I\frac{\partial^4 w}{\partial x^4}+\rho S\frac{\partial^2 w}{\partial t^2}=\left[\bar{N}+\frac{\bar{E}S}{2l}\int_0^l\left(\frac{\partial w}{\partial x}\right)^2\right]\frac{\partial^2 w}{\partial x^2}+q(x,t) \tag{1}$$

Where $w\,(x,t\,)$ is the transverse deflection, $E$is the Young's modulus, $\upsilon$is the Poisson's ratio and $\bar{E}$ is the effective modulus of the micro beam. The quantity of Echanges with different thicknesses of the micro beam as follows (Rafieipour, Lotfavar et al. 2013):

$$\bar{E}=\begin{cases}\dfrac{E}{1-\upsilon^2} & \text{for wide micro beam } (b\geq\ 5h)\\ E & \text{for narrow micro beam } (b<\ 5h)\end{cases} \tag{2}$$

$\bar{N}$ symbolizes the tensile or compressive axial load and is related to the discrepancy of both thermal expansion coefficient and crystal lattice period between substrate and the micro beam. $q\,(x, t\,)$ is normalized motivating force that derived from electrostatic excitation as(Pelesko and Bernstein 2002):

$$q(x,t) = \frac{\varepsilon_v b v^2}{2}\left[\frac{1}{\left(g_0 - w(x,t)\right)^2} - \frac{1}{\left(g_0 + w(x,t)\right)^2}\right] \tag{3}$$

Where $\varepsilon_v = 8.85\, pF/m$is the dielectric constant of the interface. Boundary conditions are as follows;

$$w(0,t) = w(l,t) = 0 \tag{4}$$

$$\frac{\partial w}{\partial x}(0,t) = \frac{\partial w}{\partial x}(l,t) = 0 \tag{5}$$

The following dimensionless parameters are used to normalize equation (1);

$$\xi = \frac{x}{l}\,, W = \frac{w}{g_0}\,, \tau = t\sqrt{\frac{\bar{E}I}{\rho b h l^4}}\,, \alpha = 6\left(\frac{g_o}{h}\right)^2, N = \frac{\bar{N} l^2}{\bar{E}I}\,, V = \sqrt{\frac{24\varepsilon_v l^4 v^2}{\bar{E} h^3 g_0^3}} \tag{6}$$

Then, dimensionless boundary conditions could be written as:

$$w(0,\tau) = w(1,\tau) = 0 \tag{7}$$

$$\frac{\partial w}{\partial x}(0,\tau) = \frac{\partial w}{\partial x}(1,\tau) = 0 \tag{8}$$

Based on presented formulas, dimensionless equation of motion could be implemented for MEMS resonators by the following equation;

$$\frac{\partial^4 W}{\partial \xi^4} + \frac{\partial^2 W}{\partial \tau^2} = \left[N + \alpha \int_0^1 \left(\frac{\partial W}{\partial \xi}\right)^2\right]\frac{\partial^2 W}{\partial \xi^2} + \frac{V^2}{4}\left[\frac{1}{(1-W)^2} - \frac{1}{(1+W)^2}\right] \tag{9}$$

By using the assumed modes method, dimensionless deflection solution of Eq. (9) could be introduced as;

$$W(\xi,\tau) = \sum_{i=1}^{n} \phi_i(\xi) u_i(\tau) \tag{10}$$

Where $\phi_i(\xi)$ is the i$^{\text{th}}$ Eigen function of micro beam that fulfills the appropriate boundary conditions, $u_i(\tau)$is the i$^{\text{th}}$ time dependent deflection coordinate and $n$is the supposed degrees of freedom of the micro beam.

To solve Eq. (9), we consider a single degree of freedom model $(n = 1)$ and deflection function $W(\xi,\tau)$ is assumed to be as:

$$W(\xi,\tau) = \phi(\xi) u(\tau) \tag{11}$$

The trial function is

$$\phi(\xi) = 16\,\xi^2(1-\xi)^2 \tag{12}$$

This function satisfies the boundary conditions.

Then by substitution of presented functions to the dimensionless equation of motion and integrating from 0 to 1, dimensionless equation of motion changes to (Fu, Zhang et al. 2011):

$$\ddot{u}(a_1u^4 + a_2u^2 + a_3) + a_4u + a_5u^3 + a_6u^5 + a_7u^7 = 0 \quad under u(0) = A, \dot{u}(0) = 0 \tag{13}$$

Where

$$a_1 = \int_0^1 \phi^6 d\xi\,, \qquad a_2 = 2\int_0^1 \phi^4 d\xi\,, \qquad a_3 = \int_0^1 \phi^2 d\xi\,, \qquad a_4 = \int_0^1 \left(\phi^{''''}\phi - N\phi^{''}\phi - V^2\phi^2\right) d\xi$$
$$a_5 = \int_0^1 \left(-2\phi^{''''}\phi^3 + 2N\phi^{''}\phi^3 - \alpha\phi^{''}\phi\int_0^1 (\phi^{'})^2\, d\xi\right) d\xi$$
$$a_6 = \int_0^1 \left(\phi^{''''}\phi^5 - N\phi^{''}\phi^5 + 2\alpha\phi^{''}\phi^3\int_0^1 (\phi^{'})^2\, d\xi\right) d\xi \tag{14}$$
$$a_7 = -\int_0^1 \left(\alpha\phi^{''}\phi^5\int_0^1 (\phi')^2\, d\xi\right) d\xi$$

3 SOLUTION PROCEDURE

For the following general oscillator

$$\ddot{u} + f\big(u(t)\big) = 0 \qquad u(0) = A\dot{u}(0) = 0 \tag{15}$$

Where u and t are generalized dimensionless displacement and dimensionless time and$A$ is amplitude of oscillator. Based on the variational principle, by implementing the semi-inverse method(He 1997, He 2004) and He's method(He 2007, Bayat and Pakar 2011), variation parameter could be written as;

$$J(u) = \int_0^t \left\{-\frac{1}{2}\dot{u}^2 + F(u)\right\} dt \tag{16}$$

Where $T = 2\pi/\omega$is period of the oscillator and $\frac{\partial F}{\partial u} = f(u)$. Thus, Hamiltonian of presented problem could be expressed as;

$$\mathrm{H} = \frac{1}{2}\dot{\mathrm{u}}^2 + \mathrm{F(u)} = \mathrm{F(A)} \tag{17}$$

Then defining a new function as;

$$\mathrm{R(t)} = \frac{1}{2}\dot{\mathrm{u}}^2 + \mathrm{F(u)} - \mathrm{F(A)} \tag{18}$$

By choosing any arbitrary point like $\omega t = \pi/4$, and setting $R(t = \frac{\pi}{4\omega}) = 0$, an approximate frequency–amplitude relationship could be obtained. Such approach is much simpler and has been widely used(Jamshidi and Ganji 2010). The accuracy of such location method, however, strongly depends upon the chosen location point. To overcome the shortcomings of the energy balance method, a new approach based on Hamiltonian has been suggested (He 2010) . Differentiating the Hamiltonian leads to natural frequency of the system;

$$\frac{\partial H}{\partial A} = 0 \tag{19}$$

For more convenience, a new function $\widetilde{H}(u)$ defined as;

$$\widetilde{H}(u) = \int_{0}^{T/4} \left\{\frac{1}{2}\dot{u}^2 + F(u)\right\} dt = \frac{1}{4}TH \tag{20}$$

And then for natural frequencies of the system, one could use the following relation;

$$\frac{\partial}{\partial A}\left(\frac{\partial \widetilde{H}}{\partial T}\right) = 0 \text{ or } \frac{\partial}{\partial A}\left(\frac{\partial \widetilde{H}}{\partial \frac{1}{\omega}}\right) = 0 \tag{21}$$

From Eq. (21)we can obtain approximate frequency–amplitude relationship of a nonlinear oscillator(Shou 2009, He 2010). For current special problem, we have following Hamiltonian equation**:**

$$\widetilde{H}(u) = \int_{0}^{T/4} \frac{1}{2}(a_1 u^4 + a_2 u^2 + a_3)\dot{u}^2 + \left(\frac{1}{2}a_4 u^2 + \frac{1}{4}a_5 u^4 + \frac{1}{6}a_6 u^6 + \frac{1}{8}a_7 u^8\right) dt \tag{22}$$

3.1 First Order Hamiltonian Approach

With satisfying the initial conditions, utilizing $u = A\cos\omega t$ as the trial function into equation (**22**), we obtain;

$$\begin{aligned}\widetilde{H}(u) = \int_{0}^{\frac{T}{4}} &\frac{1}{2}(a_1 (A\cos\omega t)^{4} + a_2 (A\cos\omega t)^{2} + a_3)(-A\omega\sin\omega t)^2 \\ &+ \left(\frac{1}{2}a_4 (A\cos\omega t)^{2} + \frac{1}{4}a_5 (A\cos\omega t)^4 + \frac{1}{6}a_6 (A\cos\omega t)^6 + \frac{1}{8}a_7 (A\cos\omega t)^8\right) dt\end{aligned} \tag{23}$$

That leads to;

$$\begin{aligned}\widetilde{H}(u) = \int_{0}^{\frac{\pi}{2}} &\frac{1}{2}A^2\omega\sin^2 t\,(a_1 (A\cos t)^{4} + a_2 (A\cos t)^{2} + a_3) \\ &+ \frac{1}{\omega}\left(\frac{1}{2}a_4 (A\cos t)^{2} + \frac{1}{4}a_5 (A\cos t)^4 + \frac{1}{6}a_6 (A\cos t)^6 + \frac{1}{8}a_7 (A\cos t)^8\right) dt\end{aligned} \tag{24}$$

Then, the frequency–amplitude relationship can be obtained from;

$$\frac{\partial}{\partial A}\left(\frac{\partial \widetilde{H}}{\partial \frac{1}{\omega}}\right) = 0 \rightarrow A(-0.785\mathrm{a}_3 + 0.785\mathrm{a}_4\omega^2 + -0.392A^2\mathrm{a}_2 + 0.589A^2\mathrm{a}_5\omega^2 + -0.294A^4\mathrm{a}_1 + 0.490A^4\mathrm{a}_6\omega^2 + 0.429A^6\mathrm{a}_7\omega^2) = 0 \tag{25}$$

Therefore, after some approximations and simplifications, equation (**25**) could be solved and the natural frequency could be obtained as;

$$\omega \approx \sqrt{\frac{0.785\mathrm{a}_4 + 0.589A^2\mathrm{a}_5 + 0.490A^4\mathrm{a}_6 + 0.429A^6\mathrm{a}_7}{0.294A^4\mathrm{a}_1 + 0.392A^2\mathrm{a}_2 + 0.785\mathrm{a}_3}} \tag{26}$$

That is approximately equal to;

$$\omega \approx \frac{\sqrt{2}}{4}\sqrt{\frac{64\mathrm{a}_4 + 48A^2\mathrm{a}_5 + 40A^4\mathrm{a}_6 + 35A^6a_7}{3A^4\mathrm{a}_1 + 4A^2\mathrm{a}_2 + 8\mathrm{a}_3}} \tag{27}$$

The variational approach (Bayat, Bayat et al. 2014) and analytical approximate solution(Rafieipour, Lotfavar et al. 2013) resulted the same response for this problem.

3.2 Second Order Hamiltonian Approach

To improve the accuracy of this approach, a higher order periodic solution was assumed as time response function as;

$$u = a\cos\omega t + b\cos 3\omega t \tag{28}$$

Where the initial condition is

$$A = a + b \tag{29}$$

Substituting Eq. (29) into Eq. (22), we obtain:

$$\widetilde{H}(u) = \int_0^{\frac{\pi}{2}} \frac{1}{2}\omega(a_1(a\cos t + b\cos 3t)^4 + a_2(a\cos t + b\cos 3t)^2 + a_3)(-a\sin t - 3b\sin 3t)^2 + \frac{1}{\omega}\left(\frac{1}{2}a_4(a\cos t + b\cos 3t)^2 + \frac{1}{4}a_5(a\cos t + b\cos 3t)^4 + \frac{1}{6}a_6(a\cos t + b\cos 3t)^6 + \frac{1}{8}a_7(a\cos t + b\cos 3t)^8\right)dt \tag{30}$$

And then the frequency–amplitude relationship can be obtained from following equation;

$$0.343a^5\mathrm{a}_1 + 0.392a^3\mathrm{a}_2 + 3.239a^4\mathrm{a}_1b + 3.926a^2\mathrm{a}_2b + 7.068\mathrm{a}_3b + 4.417a^3\mathrm{a}_1b^2 + 11.191a^2\mathrm{a}_1b^3 + 3.534\mathrm{a}_2b^3 + 2.650\mathrm{a}_1b^5 + 0.196a^3\mathrm{a}_5\omega^2 + 0.245a^5\mathrm{a}_6\omega^2 + 0.257a^7\mathrm{a}_7\omega^2 + 0.785\mathrm{a}_4b\omega^2 + 1.178a^2\mathrm{a}_5b\omega^2 + 1.472a^4\mathrm{a}_6b\omega^2 + 1.803a^6\mathrm{a}_7b\omega^2 + 1.472a^3\mathrm{a}_6b^2\omega^2 + 3.865a^5\mathrm{a}_7b^2\omega^2 + 0.589\mathrm{a}_5b^3\omega^2 + 2.945a^2\mathrm{a}_6b^3\omega^2 + 7.731a^4\mathrm{a}_7b^3\omega^2 + 4.295a^3\mathrm{a}_7b^4\omega^2 + 0.490\mathrm{a}_6b^5\omega^2 + 5.154a^2\mathrm{a}_7b^5\omega^2 + 0.429\mathrm{a}_7b^7\omega^2 = 0 \tag{31}$$

To obtain natural frequency, substituting equation (**29**) in (**31**) as $b = A - a$, a second order algebraic equation set will be ready to solve to get the natural frequency and values of ‘a’, ‘b’ for various values of A and V, some of the results are listed in table 2.

(A, V)	a	b
(0. 3, 10)	0.29835	0.00164
(0. 4, 10)	0.39554	0.00445
(0. 5, 10)	0.49015	0.00984
(0. 6, 10)	0.58094	0.01905
(0. 7, 10)	0.66653	0.03346

Table 2: a, b parameters for different$A$ and Vvalues ($N = 10, \alpha = 24$).

3.3 Third Order Hamiltonian Approach

One could use a third order time response for micro beam as;

$$u = a\cos\omega t + b\cos 3\omega t + c\cos 5\omega t \tag{32}$$

Where the initial condition is

$$A = a + b + c \tag{33}$$

Same as the second order Hamiltonian approach, with some mathematical simplification, values of a, b *and* $c$could be obtained for various values of A and V such as what listed in table 3.

(A, V)	a	b	c
(0. 3, 10)	0.29830	0.00165	0.000046
(0. 4, 10)	0.39531	0.00446	0.000218
(0. 5, 10)	0.48932	0.00992	0.000754
(0. 6, 10)	0.57849	0.01937	0.002135
(0. 7, 10)	0.66036	0.03442	0.005215

Table 3: a, b, c parameters for some A and V ($N = 10, \alpha = 24$)

4 VALIDATIONS, RESULTS AND DISCUSSIONS

4.1 Computational Efficiency

Presented nonlinear algebraic equations are solved by using Wolfram Mathematica software on Intel(R) Core (TM) i5-3230M CPU @ 2.6 GHz processor, includes 6 GB installed memory on a 64-bit operating system. Required time for calculation of natural frequencies was 5 to 10 seconds, 30 to 40 seconds and 3.5 to 4 minutes for first, second and third order Hamiltonian approach respectively. In terms of accuracy and computational efficiency, second order solution was the best.

4.2 Validation

In comparison with previous works, where higher order approximations had not been used, more accurate dynamic response and natural frequencies are observed. Energy Balance Method (EBM) is the best criterion for comparisons. Figure 3 depicts comparison of the dynamic response of a micro beam under an electric excitation (V=24 Volt), with parameters $N = 10, a = 24$ and $A = 0.4$ obtained with the first, 2nd and 3rd order Hamiltonian approaches. Figure 4 repeated the comparisons with changing the A value to 0.5.

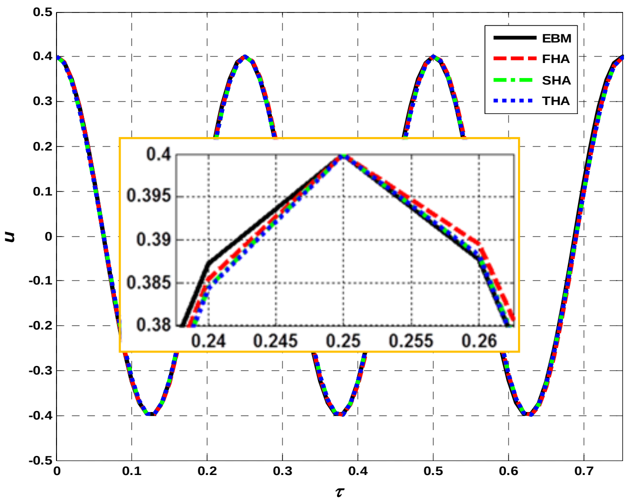

Figure 3: Comparison of dynamic response obtained with higher order Hamiltonian approaches and EBM solution ($\boldsymbol{N = 10, a = 24, V = 10, A = 0.4}$).

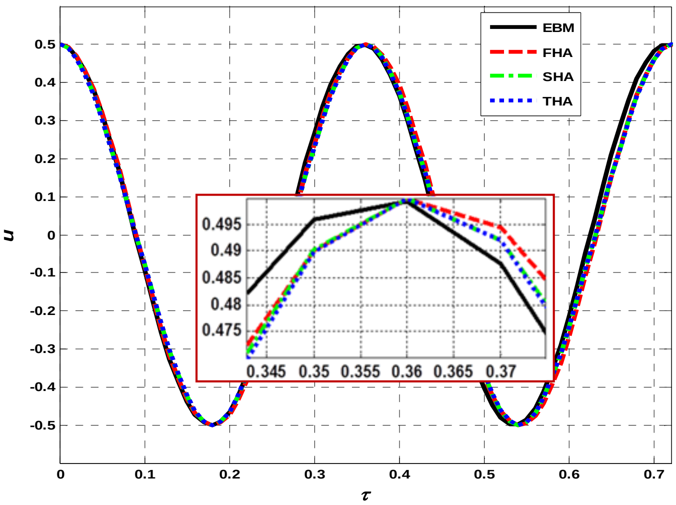

Figure 4: Comparison of dynamic response obtained with higher order Hamiltonian approaches and EBM solution ($\boldsymbol{N = 10, a = 24, V = 20, A = 0.5}$).

Table 4 compares frequency commensurate for different parameters of system, obtained from Hamiltonian method and energy balance method (EBM (Fu, Zhang et al. 2011)). The exact values for some cases are also reported. Accuracy increases with increasing the order of approximations. When the order increases more accurate results are achieved. Increasing applied voltage or initial amplitude leads to more errors. Thus, in the case of larger initial amplitude and applied voltage, higher order approximations would be more useful.

(A, V)	ω_{VA} (Relative Errors %)	$\omega_{Firstorder}$ (Relative Errors %)	$\omega_{Secondorder}$ (Relative Errors %)	$\omega_{Thirdorder}$ (Relative Errors %)	ω_{EBM}	ω_{ex} (Qian, Ren et al. 2012)
(0. 3, 0)	26. 3644 (0. 0845 %)	26. 3644 (0. 0845 %)	26. 3672 (0. 0739 %)	26. 3669 (0. 0750 %)	26. 3867	26.8372
(0. 3, 10)	24. 2526 (0. 0935 %)	24. 2526 (0. 0935 %)	24. 2547 (0. 0848 %)	24. 2543 (0. 0865 %)	24.2753	-
(0. 3, 20)	16. 3556 (0. 1666 %)	16. 3556 (0. 1666 %)	16. 3552 (0. 1690 %)	16. 3547 (0.1724%)	16. 3829	16.6486
(0. 4, 0)	27. 2053 (0. 2588 %)	27. 2053 (0. 2588 %)	27. 2214 (0. 1998 %)	27. 2195 (0. 2067 %)	27. 2759	-
(0. 4, 10)	25. 0500 (0. 2854 %)	25. 0500 (0. 2854 %)	25. 0639 (0. 2300 %)	25. 0621 (0. 2373 %)	25. 1217	-
(0. 4, 20)	17. 0187 (0. 4888 %)	17. 0187 (0. 4888 %)	17. 0238 (0. 4590 %)	17. 0219 (0. 4701 %)	17. 1023	-
(0. 5, 0)	28. 0019 (0. 6171 %)	28. 0019 (0. 6171 %)	28. 0657 (0. 3907 %)	27. 0605 (0. 4092 %)	28. 1758	-
(0. 5, 10)	25. 7611 (0. 6762 %)	25. 7611 (0. 6762 %)	25. 8203 (0. 4480 %)	25. 8155 (0. 4665 %)	25. 9365	-
(0. 5, 20)	17. 3839 (1. 1351 %)	17. 3839 (1. 1351 %)	17. 4270 (0. 8900 %)	17. 4241 (0. 9065%)	17. 5835	-
(0. 6, 0)	28. 5579 (1. 2612 %)	28. 5579 (1. 2612 %)	28. 7564 (0. 5749 %)	28. 7499 (0. 5974 %)	28. 9227	28.5382
(0. 6, 10)	26. 1671 (1. 3768 %)	26. 1671 (1. 3768 %)	26. 3600 (0. 6497 %)	26. 3562 (0. 6640 %)	26. 5324	-
(0. 6, 20)	17. 0940 (2. 3294 %)	17. 0940 (2. 3294 %)	17. 2901 (1. 2238 %)	17. 3013 (1. 1450 %)	17. 5017	18.5902

Table 4: Comparison of natural frequencies (rad/s) and relative errors for various parameters of system ($N = 10, \alpha = 24$).

Table 5 lists the effect of increasing the initial amplitude onto the natural frequencies obtained by various approximations.

(A, V)	ω_{VA} (Relative Errors %)	$\omega_{Firstorder}$ (Relative Errors %)	$\omega_{Secondorder}$ (Relative Errors %)	$\omega_{Thirdorder}$ (Relative Errors %)	ω_{EBM}
(0. 7, 5)	27. 9997 (2. 3761%)	27. 9997 (2. 3761%)	28. 5188 (0. 5662%)	28. 5410 (0. 4888%)	28. 6812
(0. 7, 15)	22. 2646 (2. 9691%)	22. 2646 (2. 9691%)	22. 8028 (0. 6236%)	22. 8612 (0. 3691%)	22. 9459

Table 5: Effect of increasing initial amplitude to natural frequencies obtained by using higher order Hamiltonian approaches

Dynamic response of micro beam is depicted in figure 4.By reduction of parameters a, b and c for each order, amplitude decreases and it decreases more by increasing the order of approximate solution. Furthermore, second and third order approximations have almost the same range of amplitudes.

4.3 Phase Diagram of Micro Beam

Simplifying the convolution of a nonlinear system to a linear wise model could provide a useful view on the stability and controllability of system. For example, assume that a nonlinear MEMS micro beam has a linear wise model as below that includes all nonlinearities on the second part of its dynamics;

$$\dot{x} = Ax + Bu + \Psi_{Nonlinear}(x, t, u) \tag{34}$$

Nonlinear term ($\Psi_{\mathrm{Nonlinear}}(\mathrm{x}, \mathrm{t}, \mathrm{u})$) could contain a high level of nonlinearity because of including space variables, time and also inputs as operands. It could exert some amazing variations on the dynamics of system. For more clarity, phase diagram of electrostatically actuated micro beam has plotted to demonstrate the effects of nonlinear part of MEMS micro beam. Figures 5 and 6 show compares phase diagram of obtained values by the higher Hamiltonian approach with the energy balance method (EBM). These figures depict the effects of A and V parameters on the phase plan of the system in second order Hamiltonian method.

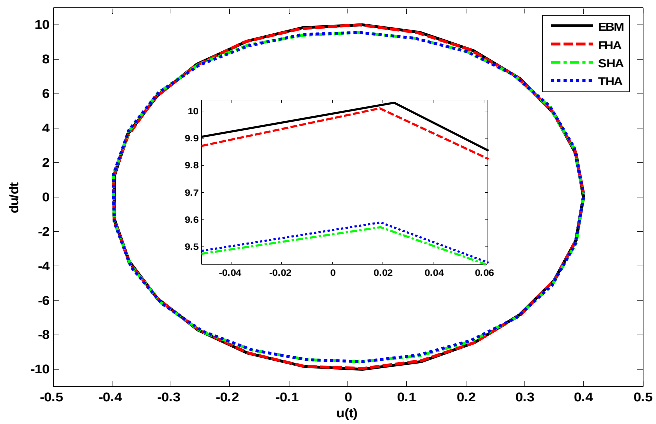

Figure 5: Comparison of phase diagram obtained from the higher order Hamiltonian method with the EBM solution ($\boldsymbol{N = 10, \alpha = 24, V = 10, A = 0.4}$).

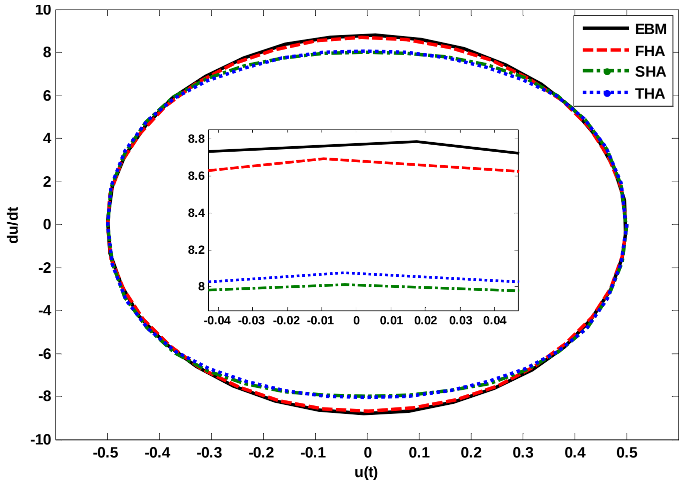

Figure 6: Comparison of phase diagram obtained from the higher order Hamiltonian method with the EBM solution ($\boldsymbol{N = 10, a = 24, V = 20, A = 0.5}$).

Figure 7 shows the effect of parameter A on the phase plan of system for $\mathrm{N = 10, a = 24, V = 5}$ simulated by using the second order of Hamiltonian method. As it can be seen on the figure, by increase in the order of Hamiltonian approach, amplitude parameters (a, b, c) decreased, thus overall amplitude decreases too. When micro beam resonates near the zero point as the basal condition, a notable reduction in the velocity of resonator is observable. This phenomenon disappears immediately after passing from basal condition. This means that dynamics of this nonlinear system also depends on the position of the point that is being measured on the MEMS micro beam. Same analysis figure 8, where shows the effect of V parameter on the phase plan of the system for $\mathrm{N = 10, a = 24, A = 0.6}$ simulated by using the second order of Hamiltonian method.

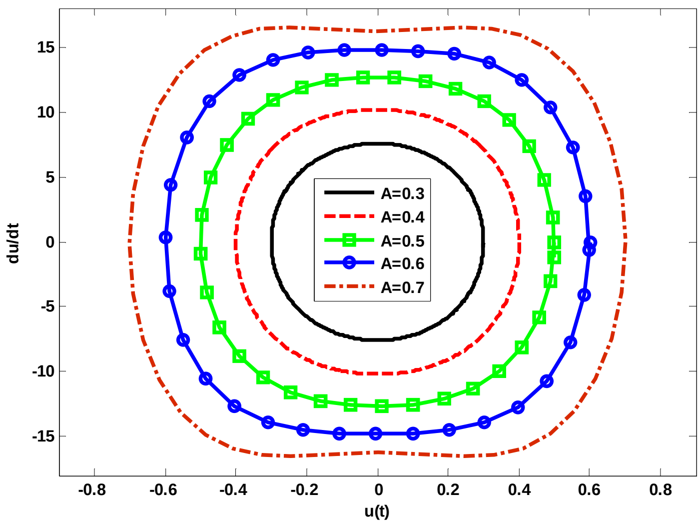

Figure 7: Effect of $\boldsymbol{A}$ on the phase plan of the system for $\boldsymbol{N = 10, a = 24, V = 5}$ simulated by using the second order of Hamiltonian.

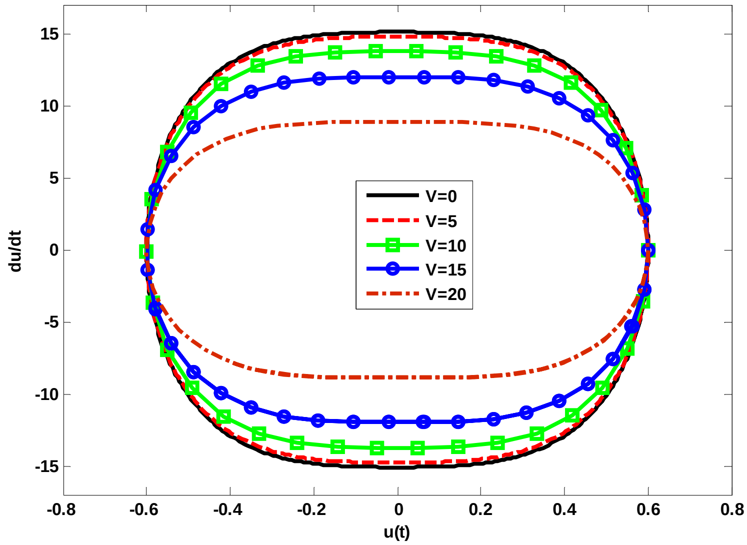

Figure 8: Effect of $\boldsymbol{V}$ parameter on the phase plan of the system for $\boldsymbol{N=10, a=24, A=0.6}$ simulated by using the second order of Hamiltonian.

4.4 Free Vibration

Figure 9 shows the effect of N parameter on natural frequency. It can be observed that the frequency is proportional with N. However, it decreases when initial amplitude (A) increases. The second order Hamiltonian has nearly same response in comparison to the EBM solution, even for higher values of N and amplitudes.

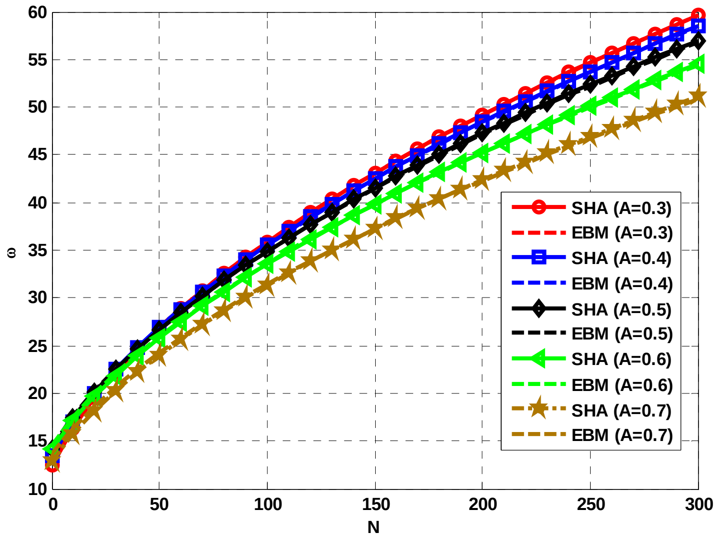

Figure 9: Effect of $\boldsymbol{N}$ parameter on the frequency of electrostatically actuated micro beam with $\boldsymbol{V=20, \alpha=24}$ and various values for A.

Figure 10 depicts the effect of parameter α on natural frequency of electrostatically actuated micro beam with parameters $N = 10, V = 20$and various values for A. It can be observed that the frequency increases with increasing α. Obtained results by the second order Hamiltonian are close to the EBM solution, especially for low amplitudes and α values.

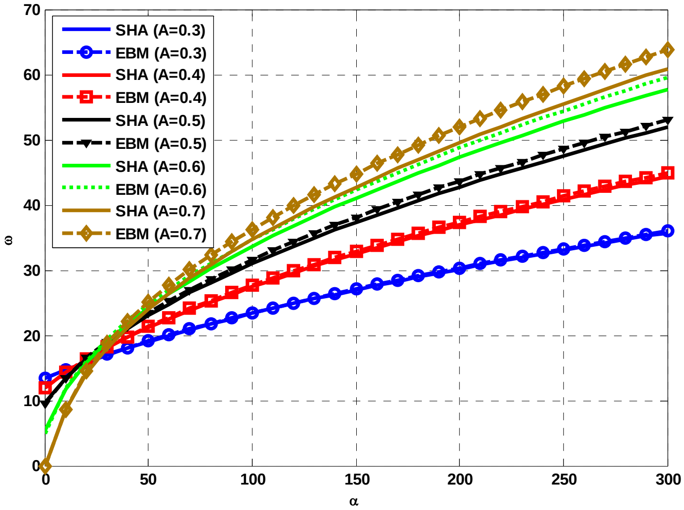

Figure 10: Effect of parameter α on natural frequency of electrostatically actuated micro beam with $N = 10, V = 20$ and various values of A.

Figure 11 shows the effect of applied voltage on the natural frequency of electrostatically actuated micro beam. It can be seen that the frequency is decreased with increase in the voltage. Second order Hamiltonian is extremely close to EBM solution but they have considerable discrepancies in high amplitudes and voltages.

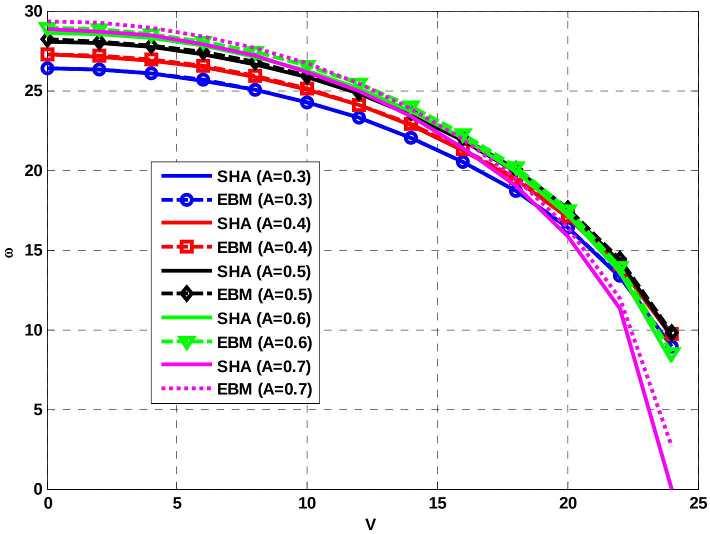

Figure 11: Effect of V parameter on natural frequency of electrostatically actuated micro beam with $N = 10, V = 20$ and various values of A.

Nonlinear behavior of system leads to abrupt falling on high applied voltages. Natural frequency decreases dramatically on high voltages. On the other hand, natural frequency increases also with increasing the amplitude. This is due to the effect of more hardening the equivalent linear system of electrostatically actuated micro beam when initial amplitude increases. It is also observed that for higher amplitudes, discrepancy is less than lower values.
Figure 12 shows the effect of modulus of elasticity of electrostatically actuated micro beam on the natural frequency with various initial amplitudes. As demonstrated clearly, natural frequency increases with increasing the modulus of elasticity.

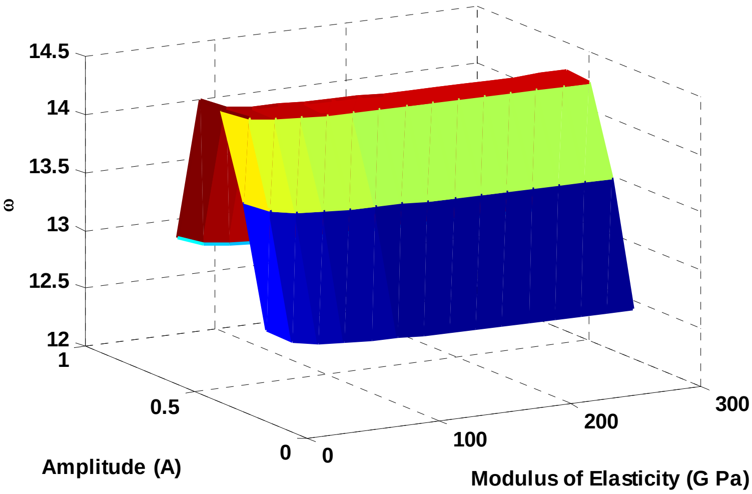

Figure 12: Variation of natural frequency due to initial amplitude (A) and modulus of elasticity (E).

Figure 13 shows the effect of thickness of electrostatically actuated micro beam on the natural frequency with various initial amplitudes. As demonstrated clearly, natural frequency decreases with increasing the thickness of micro beam.

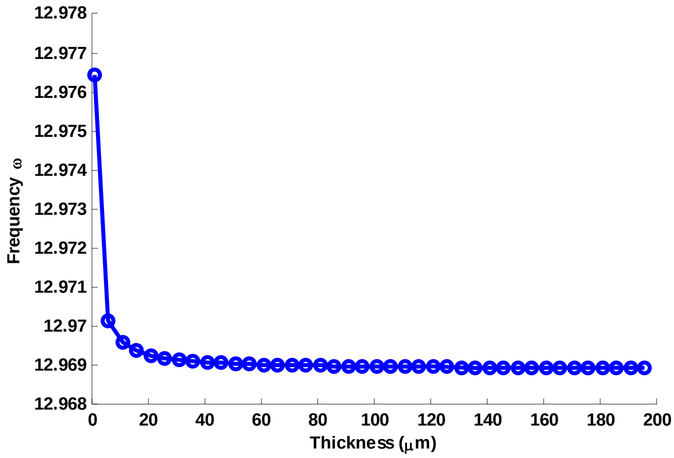

Figure 13: Variation of natural frequency due to thickness of micro beam (H).

Several simulations and plots could be introduced to consider the fundamental design rquirements before any manufacturing process. Based on the presented examples,the proposed nonlinear model based on Hamiltonian approach is completely efficient and acceptable to find the effects of parameters on the natural frequency and phase plane diagram of electrostatically actuated micro beam.

5 CONCLUSION

Applying the Hamiltonian approach to the nonlinear problem of electrostatically actuated micro beam, this paper studied the effect of various parameters on the dynamic response and phase diagram (stability) of the system. Due to the nonlinear manner of MEMS resonators on sensor design paradigms, this would be used in practical work for more efficient and low cost experiments. Utilized approximate solution converged to the exact solution and obviously demonstrated a good level of accuracy. It was presented that increasing the order of Hamiltonian approach, more agreeable results could be achieved. Briefly, presented approach resulted in below findings;

1. Time cost of presented approach was in an acceptable range (5 to 10 seconds, 30 to 40 seconds and 3.5 to 4 minutes for first, second and third order Hamiltonian approach, respectively).
2. In terms of accuracy and computational efficiency, second order solution was the best.
3. Increasing applied voltage or initial amplitude leads to more errors. Then, in the case of larger initial amplitude and applied voltage, higher order approximations will be useful.
4. By increase in the order of Hamiltonian approach, amplitude parameters (a, b, c) decreased, thus overall amplitude decreases too.
5. When micro beam resonates near the zero point as the basal condition, a few hardening manner could be observed that removed immediately after passing from basal condition. This means that dynamics of this nonlinear system also depends on the position of the point that is being measured on the MEMS micro beam.
6. Natural frequency increases with external load increasing. However, it decreases with increasing of initial amplitude (A). On the other hand, the second order Hamiltonian leads to extremely close response to the EBM solution, even for higher values of N and amplitudes.
7. Natural frequency increases with increasing α.
8. Natural frequency is decreasing with increase in the voltage.
9. Nonlinear behavior of system leads to abrupt falling on high applied voltages. Natural frequency decreases dramatically on high voltages. On the other hand, natural frequency increases also with increasing the amplitude. This is due to the effect of more hardening the equivalent linear system of electrostatically actuated micro beam when initial amplitude increases. It is also observed that for higher amplitudes, discrepancy is less than lower values.
10. Natural frequency increases with increasing the modulus of elasticity.
11. Natural frequency decreases with increasing the thickness of micro beam.

Finally, Based on presented examples to find the effects of parameters on the natural frequency and phase plane diagram of electrostatically actuated micro beam, it seems that this nonlinear model that was based on Hamiltonian approach is completely efficient and acceptable.

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