*Original Articles*

# ON THE COMPARISONS OF LOGICS IN TERMS OF EXPRESSIVE POWER

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**Abstract:** This paper investigates the question "when is a logic more expressive than another?" In order to approach it, "logic" is understood in the model-theoretic sense and, contrary to other proposals in the literature, it is argued that relative expressiveness between logics is best framed with respect to the notion of expressing properties of models, a notion that can be captured precisely in various ways. It is shown that each precise rendering can give rise to a formal condition for relative expressiveness that has appeared in the literature. Five such conditions are exposed, tested for some properties and compared to each other. As the formal conditions for relative expressiveness have various levels of stringency, some results on lifting some conditions to stricter ones are explored. Finally, a discussion on the properties of these formal conditions is presented. Special attention is given to notion of meaning equivalence, and how one may consider that it holds or not, depending on the weight attributed to logical and non-logical constants in expressiveness comparisons.

## 1 Introduction

### *1.1 E, the main intuition for expressiveness relations*

This paper will study the relation "the logic $\mathcal{L}_2$ is at least as expressive/strong as (or includes) $\mathcal{L}_1$", whose intuitive basis could be said to be

$(E)$ Everything that can be said in $\mathcal{L}_1$ can also be said in $\mathcal{L}_2$.

A logic $\mathcal{L}$ will be understood here as a "model-theoretic logic", that is, as defined by a class of models, a class of sentences and a satisfaction relation on them. Moreover, all logics will be defined on the same class of models.

Formal comparisons of such logics in terms of $E$ exist at least since Lindström's famous characterization results for first-order logic (LINDSTRÖM, 1969).[1] In his works on expressive characterization, some formal conditions for $E$ were given, one of them is

$(\preccurlyeq_{DC})$ Every $\mathcal{L}_1$-definable class of structures is also $\mathcal{L}_2$-definable or, equivalently, for every $\mathcal{L}_1$-sentence, there is an $\mathcal{L}_2$-sentence having the same models.

[1] One of Lindström's results is that first-order logic has maximal expressiveness among the countably compact logics having a downward Löwenheim-Skolem theorem. For more, see (LINDSTRÖM, 1974) and (BARWISE; FEFERMAN, 1985).

Another condition requires additionally that there is an effective mapping from $\mathcal{L}_1$-sentences to $\mathcal{L}_2$-sentences.

Less stringent conditions have also been given to capture $E$. Makowsky (1980) considers a condition based on the concept of projective definability ($\preccurlyeq_{PC}$). His motivation is to allow for a more flexible approach on the role of non-logical symbols in expressiveness comparisons. For him, $\preccurlyeq_{PC}$ can eventually be a more natural notion for comparing expressiveness than $\preccurlyeq_{DC}$. Ebbinghaus (1985), besides the above ones, also considered a condition based on $\mathcal{L}$-equivalence ($\preccurlyeq_{EQ}$), that is, based on the ability of logics to distinguish models. Shapiro (1991) pointed out the strictness of $\preccurlyeq_{DC}$ and argued for the reasonableness of $\preccurlyeq_{PC}$ and an even wider condition, constructed in terms of the notion of relative-projective definability.

In the above works, the comments on relative expressiveness are brief and the investigation $E$ deserves is not given. To the best of our knowledge, the first such investigation appeared in (PETERS; WESTERSTÅHL, 2006). The authors study the various aspects involved in expressiveness comparisons, *e.g.* the related items, the maps between logics and some notions of synonymy that could be employed to base such comparisons. However, their approach is founded on a certain refinement of $E$, which may be adequate when one is interested in comparing natural languages, but it is not adequate for comparisons of expressiveness between logics, specially of the model-theoretic sort studied here. Another limitation of Peters and Westerståhl's work on expressivity is that it does not allow for comparisons neither involving $\mathcal{L}$-equivalence, nor projective definability.

Fernandes (2017) distinguishes three frameworks for expressiveness comparisons according to the kind of logics to be compared: for model-theoretic logics defined within the same class of models, for those defined within possibly different classes of models and for Tarskian or abstract logics. Though the above mentioned conditions, defined within the first framework, are exposed, the author focuses on those defined on the latter frameworks.

Kocurek (2018) explores these three frameworks. Formal conditions based on $\mathcal{L}$-equivalence and definability are studied and given brief explanations in terms of operations on classes of models. However, as in (PETERS; WESTERSTÅHL, 2006), comparisons involving projective definability are also absent and, moreover, would not even fit properly in the explanatory approach on expressiveness based on operations on classes of models.

The main purpose of this paper is to address the limitations of the above mentioned works. We will do this by providing a more inclusive and also coherent understanding of $E$ for model-theoretic logics. It will be seen that this proposal accommodates adequately the various formal measures of expressiveness that have appeared in the literature. Five such formal measures, including the one based on projective definability, will be compared to each other and tested with respect to some *prima facie* reasonable properties on expressiveness relations. The analysis of how each formal measure of expressiveness relate to each other and the verification of the properties each one satisfies will hopefully help clarifying the landscape of expressiveness comparisons.

### *1.2 Properties of expressiveness relations*

Given there are many formal conditions capturing $E$ precisely, it is of interest to compare them and check what properties relevant for expressiveness they satisfy. It is rather clear that an expressiveness relation should be a pre-order on logics (*i.e.* transitive and reflexive). All the formal conditions considered here satisfy this requirement. It is not clear what other properties they should satisfy, perhaps this could be properly answered only within some context of application. Rather than settling this, some *prima facie* reasonable properties are proposed for comparisons of expressiveness, and the formal conditions will be tested for them.[2] The properties are:

1. If $\mathcal{L}_2$ is at least as expressive as $\mathcal{L}_1$, then there is a fragment $\mathcal{L}_2^*$ of $\mathcal{L}_2$ such that $\mathcal{L}_2^*$ and $\mathcal{L}_1$ are equally expressive,

2. If $\mathcal{L}_2$ is at least as expressive as $\mathcal{L}_1$, and the same expressive tools are added both to $\mathcal{L}_2$ and $\mathcal{L}_1$, respectively obtaining $\mathcal{L}_2^e$ and $\mathcal{L}_1^e$, then it holds that $\mathcal{L}_2^e$ is at least as expressive as $\mathcal{L}_1^e$.

3. If $\mathcal{L}_2$ is at least expressive as $\mathcal{L}_1$, then $\mathcal{L}_1$-sentences have corresponding $\mathcal{L}_2$-sentences with equivalent meanings.

The proposition of property 1 was motivated by the discussion in (KOCUREK, 2018). It would seem to be equivalent with 3, but it is arguably stricter. Property 2 was motivated by the discussion in (FRENCH, 2019), concerning the notion of notational variance. Given the closeness of this concept with the intuitive concept of expressive equivalence, it is of interest to check how the various formal conditions on expressiveness fare with respect to it. As regards the property 3, whether or not it holds will depend on what "meaning equivalence" is to mean. It will be seen that this may hinge on how one deals with the relation between logical and non-logical terms in expressiveness comparisons.

[2]Their order should not be taken as an importance rank.

### *1.3 Overview of the paper*

This paper is structured as follows. Firstly, a refinement $E^*$ of $E$ will be proposed, so as to be closer to the notion of logic to be employed here. Next, it will be shown that the key term "expressing properties of models" contained in $E^*$ can be formally captured in various ways. Five such ways are explored in sequence, each of them giving rise to a distinct formal condition for relative expressiveness. Each such condition is then analysed with respect to the properties presented above, and then compared to each other. The proofs of the main remarks concerning satisfaction of properties and comparison of the conditions are placed in the appendix.

As there are cases where a condition $\preccurlyeq_X$ is stricter than another $\preccurlyeq_Y$ (*i.e.* if $\mathcal{L}_1 \preccurlyeq_X \mathcal{L}_2$, then $\mathcal{L}_1 \preccurlyeq_Y \mathcal{L}_2$ but not vice-versa), a natural question arises: are there properties $\Theta$ of logics such that whenever $\mathcal{L}_1 \preccurlyeq_Y \mathcal{L}_2$ and $\mathcal{L}_1, \mathcal{L}_2$ have some combination of properties $\Theta$, then it holds that $\mathcal{L}_1 \preccurlyeq_X \mathcal{L}_2$? Makowsky (1980) proposed an answer to the above question as regards $\preccurlyeq_{PC}$ and $\preccurlyeq_{DC}$, but later pointed out (MAKOWSKY, 1981) that it was mistaken. It also will be shown that the positive results proposed in (KOCUREK, 2018) for this question as regards $\preccurlyeq_{EQ}$ and $\preccurlyeq_{DC}$ are mistaken. Finally, we show that a slight modification of Makowsky's proposal still gives an interesting such result.

### *1.4 Notation*

As said above, in this paper, "logic" is to mean a model-theoretic logic *i.e.*

**Definition 1.4.1** (Model-theoretic Logic)**.** *A model-theoretic logic $\mathcal{L}$ is a sequence $(\mathcal{M}_\mathcal{L}, \mathcal{S}_\mathcal{L}, \Vdash_\mathcal{L})$, where $\mathcal{M}_\mathcal{L}$ and $\mathcal{S}_\mathcal{L}$ are classes and $\Vdash_\mathcal{L} \subseteq \mathcal{M}_\mathcal{L} \times \mathcal{S}_\mathcal{L}$.*

Here, $\mathcal{M}_\mathcal{L}$ is intended to be the class of models for

$\mathcal{L}$, $\mathcal{S}_{\mathcal{L}}$ the class of well formed sentences of $\mathcal{L}$ in every vocabulary, and $\Vdash_{\mathcal{L}}$ the corresponding satisfaction relation. The words "model" and "structure" will be used interchangeably. For the sake of simplicity, only single-sorted vocabularies will be considered.

The expressions $\mathcal{S}_{\mathcal{L}}[\tau]$ and also $\mathcal{L}[\tau]$ refer to the collection of $\mathcal{L}$-sentences in the vocabulary $\tau$, and $\mathcal{M}_{\mathcal{L}}[\tau]$ refers to the class of $\tau$-models belonging to $\mathcal{M}_{\mathcal{L}}$. Given that only logics defined within the same class of models will be dealt with in this paper, we shall drop the subscript "$\mathcal{L}$".

The property 2 uses the notion of adding the same expressive tools to a pair of logics. This notion is captured here by what will be called an *uniform extension*, which requires $\mathcal{S}_{\mathcal{L}}$ and $\Vdash_{\mathcal{L}}$ to be specifiable inductively by a collection $\mathbf{C}_{\mathcal{L}}$, containing clauses for sentence formation and satisfaction. For the sake of brevity, only an informal definition will be provided.

**Definition 1.4.2** (Uniform extensions)**.** *Let the logics $\mathcal{L}_1$ and $\mathcal{L}_2$ be generated by the collection of clauses $\mathbf{C}_1$ and $\mathbf{C}_2$, respectively. Let $e$ be a collection of sentence formation clauses (only total ones are allowed) and semantic clauses for a set of logical operators. Let $\mathbf{C}_1^e$ and $\mathbf{C}_2^e$ be, respectively, the extension of $\mathbf{C}_1$ and $\mathbf{C}_2$ with $e$. Then, the logics $\mathcal{L}_1^e = (\mathcal{M}, \mathcal{S}_1^e, \Vdash_{\mathcal{L}_1^e})$ and $\mathcal{L}_2^e = (\mathcal{M}, \mathcal{S}_2^e, \Vdash_{\mathcal{L}_2^e})$, generated by $\mathbf{C}_1^e$ and $\mathbf{C}_2^e$, respectively, are said to be* uniform extensions *of $\mathcal{L}_1$ and $\mathcal{L}_2$.*

Whenever $\mathcal{L}_1^e$ and $\mathcal{L}_2^e$ are mentioned together, it is assumed they uniformly extend $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. The following notation will also be used:

- $\mathscr{P}(X)$ — the power-set of $X$,
- $\tau, \tau', \tau'', \ldots$ — arbitrary signatures,
- $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ are arbitrary models, whose signatures will be specified by the context.

- $Mod^{\tau}_{\mathcal{L}}(\phi)$ and $Th^{\tau}_{\mathcal{L}}(\mathfrak{A})$ — the class of $\tau$-models satisfying $\phi$ in $\mathcal{L}$, and the class of $\tau$-formulas satisfied by $\mathfrak{A}$ in $\mathcal{L}$, respectively.

- $\mathfrak{A}^{\restriction\tau}$ and $\Vdash^{\restriction\mathcal{S}}$ — the reduct of $\mathfrak{A}$ to $\tau$, and the restriction of $\Vdash$ to $\mathcal{S}$, respectively.

- $\mathfrak{A}'$ — for a $\tau$-model $\mathfrak{A}$, $\mathfrak{A}'$ is an expansion to additional vocabulary $\tau' \supseteq \tau$, so that $\mathfrak{A}'^{\restriction\tau} = \mathfrak{A}$,

- $\mathcal{L}$-fragment — for a logic $\mathcal{L} = (\mathcal{S}, \mathcal{M}, \Vdash)$, an $\mathcal{L}$-fragment is any $\mathcal{L}^{\restriction\mathcal{S}^*} = (\mathcal{S}^*, \mathcal{M}, \Vdash^{\restriction\mathcal{S}^*})$, where $\mathcal{S}^* \subseteq \mathcal{S}$,

- $\equiv_{\mathcal{L}}$, $\preccurlyeq_X$ and $\approxeq_X$ — equivalence of models under $\mathcal{L}$, expressiveness relation on logics with respect to condition $X$, and equivalence of logics modulo $\preccurlyeq_X$, respectively.

It will be assumed that in every logic considered, interchange of logically equivalents does not change the meaning of resulting formulas. All systems $\mathcal{L}$ considered in this paper are supposed to satisfy the usual basic properties for model-theoretic logics, as listed in (EBBINGHAUS, 1985, p. 28).

## 2 First refinement of $E$

Peters and Westerståhl (2006) were, as far as we know, the first to investigate the comparisons of logics in terms of expressive power. Their approach is based on the following refinement of $E$ (*ibid*, p. 383):

($E^{pw}$) the basic concept [for expressiveness comparisons] is really that of an [$\mathcal{L}_1$]-sentence $\phi$ being translatable into [$\mathcal{L}_2$], in the sense that there is an [$\mathcal{L}_2$]-sentence saying the same thing.

Though perhaps sufficient when the intention is the application to natural languages, $E^{pw}$ does not fit well with the model-theoretic perspective on logic which is predominant in expressive comparisons. The inadequacy is readily seen in the fact that $E^{pw}$ cannot account for the already common practice of relating the expressive power of logics in terms of the ability of distinguishing models.[3] There are logics $\mathcal{L}_1$ and $\mathcal{L}_2$ such that, despite the fact that every pair of models distinguishable by $\mathcal{L}_1$ is also distinguishable by $\mathcal{L}_2$, it happens that there is an $\mathcal{L}_1$-formula not translatable to any $\mathcal{L}_2$-formula (see remark 3.3.6). Moreover, the most common inexpressibility results for logics,[4] contrary to Peters and Westerståhl's position (*ibid*, p. 413), is relative to the ability of distinguishing models. Thus such inexpressibility results are relative to an expressiveness relation that is not captured in their framework.

Another author that worked on expressivity with respect to model-theoretic logics is Kocurek. In (2018) his general perspective could be framed as the following refinement of $E$:

| | |
|---|---|
| $(E^k)$ | Every way of carving the class of models that can be done by $\mathcal{L}_1$, can also be done by $\mathcal{L}_2$ |

Then, "way of carving" could be further refined either as meaning "splitting" (*ibid*, p. 125), or as meaning "partitioning" (*ibid*, p. 131). These refinements would be the basis for the formal conditions mentioned above $\preccurlyeq_{DC}$ and $\preccurlyeq_{EQ}$, respectively. The problem of refining $E$ in terms of operations like splitting and partitioning the class of models is that there is no straightforward and reasonable way to base on them expressiveness comparisons using projecive definability and its derivates. This limitation would leave out of the

[3]For a recent example, see (Van De Putte; KLEIN, 2022, p. 501)

[4]E.g. (AGOTNES et al., 2010), (ARECES et al., 2011) and (TAMMINGA; DUIJF; PUTTE, 2021)

framework ways of comparing logics in terms of expressiveness that were used and defended by several authors since the beginnings of abstract model theory.[5] Hence, another understanding such comparisons must be sought.

Let us take a step back to see how some brief observations on expressivenes comparisons can lead us to a better refinement of $E$. A great number of expressiveness related results in the literature, *e.g.*, in finite model-theory or in modal logics, have as their main motivation the verification of what logics can "say" about a given collection of structures. For example, if they are able or not to "say" that a certain structure is finite, that it has an even number of elements, that there is a path between any of its elements, etc. Thus, one may reasonably hold that an expressive capacity of a logic consists in its ability to express properties of its models, where by "property of models" it is meant the following.

**Definition 2.0.1** (Property of models)**.** *Let $\mathcal{M}$ be a class of models. A property $P$ of models will be taken to be a subclass of $\mathcal{M}$.*

Given this, we are now in position to offer another and more adequate refinement of the relation $E$:

($E^*$) Every property of models expressible in $\mathcal{L}_1$, is also expressible in $\mathcal{L}_2$.

$E^*$ squares nicely in the model-theoretic point of view of logic, where one starts with a certain collection of structures of interest and takes a logic as a tool to describe and study them.[6]

Now there are certain ways in which one can understand when a property of models is expressible, and

[5] See references in subsection 3.5.

[6] One could also obtain a restricted relative measure of expressiveness, by selecting a set $\mathcal{P}$ of relevant properties of a model. However, in this paper we only analyse the unrestricted version.

thus to refine $E^*$ further. We start from the most straightforward and clear one, based on the notion of definability.

## 3 Further refinements

### 3.1 The condition $\preccurlyeq_{DC}$

Consider the following definition:

**Definition 3.1.1** (Definability)**.** *A class $P$ of $\tau$-models is definable in $\mathcal{L}$, if and only if there is an $\mathcal{L}[\tau]$-sentence $\phi$ such that $P = Mod^{\tau}_{\mathcal{L}}(\phi)$. Call the collection of all such classes "$DC_{\mathcal{L}}$".*

A definable class is also known in the literature (*e.g.* (BARWISE; FEFERMAN, 1985)) as an *elementary class*. Now, definability can clearly be taken as sufficient for expressing properties:

**Proposal 3.1.2** (Expressing properties: Definability)**.** *A property of models $P$ is expressible in $\mathcal{L}$ if $P \in DC_{\mathcal{L}}$*

Thus we have the first formal condition capturing $E^*$, which turns out to be the one employed in Lindström's seminal paper (1969):

**Definition 3.1.3** ($\preccurlyeq_{DC}$)**.** *$\mathcal{L}_1 \preccurlyeq_{DC} \mathcal{L}_2$ if and only if $DC_{\mathcal{L}_1} \subseteq DC_{\mathcal{L}_2}$.*

#### 3.1.4 On the properties of $\preccurlyeq_{DC}$

The following result is due to Kocurek (2018, p. 128):

**Remark 3.1.5** (Kocurek)**.** *Property 1 is satisfied by $\preccurlyeq_{DC}$.*

In order to check whether property 2 holds, some definitions will be given. It is assumed that formulas of the compared logics are built recursively from atomic formulas and logical operators, and that their semantics are compositional.

**Definition 3.1.6** (Schema)**.** *For* $1 \leq i \leq n$, *let* $R_i$ *be a relation symbol of arbitrary arity and let* $\vec{\mathbf{x}}_i$ *be an appropriate sequence of variables for* $R_i$. *Then, a schema* $\theta(R_1\vec{\mathbf{x}}_1, ..., R_n\vec{\mathbf{x}}_n)$ *is any formula having only the atomic formulas shown. Whenever it is clear from the context what is the schema at issue, the expression* $\theta(\phi_1, ..., \phi_n)$ *stands for the simultaneous substitution in the schema of* $\phi_i$ *for the respective* $R_i\vec{\mathbf{x}}_i$, $1 \leq i \leq n$.

If the source logic contains (generalized) quantifiers, they, together with an appropriate sequence of variables, will be treated as single operators of the appropriate arity.

**Definition 3.1.7** (Definitional translation)**.** *A translation* $\mathcal{T} : \mathcal{F}_1 \longrightarrow \mathcal{F}_2$ *is definitional iff*

- *For every n-ary relation symbol* $R$ *and terms* $t_1$, ..., $t_n$, *it holds that* $\mathcal{T}(Rt_1...t_n) = Rt_1...t_n$;
- *For every n-ary operator* $\circledast$ *of* $\mathcal{L}_1$ *and formulas* $\phi_1, ..., \phi_n$ *in* $\mathcal{F}_1$, *there is an* $\mathcal{L}_2$*-schema* $\theta^{\circledast}(R_1\vec{\mathbf{x}}_1, ..., R_n\vec{\mathbf{x}}_n)$ *for which we have* $\mathcal{T}(\circledast(\phi_1, ..., \phi_n)) = \theta^{\circledast}(\mathcal{T}(\phi_1),...,\mathcal{T}(\phi_n))$.

The following restriction of property 2 holds for $\preccurlyeq_{DC}$. The remark is a version for model-theoretic logics and uniform extensions, of an analogous remark for tarskian logic found on (FRENCH, 2019, p. 329):

**Remark 3.1.8.** *Suppose that* $\mathcal{L}_1 \preccurlyeq_{DC} \mathcal{L}_2$ *and that* $\mathcal{L}_1^e$, $\mathcal{L}_2^e$ *are uniform extensions. Let* $\mathcal{F}_1, \mathcal{F}_1^e, \mathcal{F}_2, \mathcal{F}_2^e$ *be their respective collections of well-formed formulas. Then, if there is a definitional translation* $\mathcal{T} : \mathcal{F}_1 \longrightarrow \mathcal{F}_2$, *there is a definitional translation* $\mathcal{T}^+ : \mathcal{F}_1^e \longrightarrow \mathcal{F}_2^e$ *such that* $\mathcal{L}_1^e \preccurlyeq_{DC} \mathcal{L}_2^e$.

As regards property 3, the criterion $\preccurlyeq_{DC}$ can be said to embody a particularly strong interpretation of what is it for $\mathcal{L}_1$ to have corresponding $\mathcal{L}_2$-sentences with equivalent meanings. Being thus, $\preccurlyeq_{DC}$ satisfies it straightforwardly.

The condition $\preccurlyeq_{DC}$ is very frequent in comparisons of logics in terms of expressive power, though it is a rather strict measure of relative expressiveness. As pointed out in the introduction, since the 1980s several authors considered broader conditions. In the sequence, some possible relaxations are studied, checked with respect to the satisfaction of the proposed properties and compared to each other.

### *3.2 The condition $\preccurlyeq_{DC^{\Delta}}$*

Consider the following wider notion of definability:

**Definition 3.2.1** ($\Delta$-definability)**.** *A property of $\tau$-structures is $\Delta$-definable in $\mathcal{L}$ if and only if, for some $\Delta \subseteq \mathcal{L}[\tau]$ it holds that $P = Mod^{\tau}_{\mathcal{L}}(\Delta)$ (for short $P \in DC^{\Delta}_{\mathcal{L}}$).*

Now one could consider $\Delta$-definability as sufficient for expressing properties:

**Proposal 3.2.2** (Expressing properties: $\Delta$-definability)**.** *A property $P$ is expressible in $\mathcal{L}$ if $P \in DC^{\Delta}_{\mathcal{L}}$*

Then, the respective formal condition capturing $E^*$ is:

**Definition 3.2.3** ($\preccurlyeq_{DC^{\Delta}}$)**.** *$\mathcal{L}_1 \preccurlyeq_{DC^{\Delta}} \mathcal{L}_2$ if and only if $DC_{\mathcal{L}_1} \subseteq DC^{\Delta}_{\mathcal{L}_2}$.*[7]

#### 3.2.4 On the properties of $\preccurlyeq_{DC^{\Delta}}$

**Remark 3.2.5.** *Property 1 is not satisfied by $\preccurlyeq_{DC^{\Delta}}$.*

**Remark 3.2.6.** *Property 2 is not satisfied by $\preccurlyeq_{DC^{\Delta}}$.*

[7] A definition using $DC^{\Delta}_{\mathcal{L}_1} \subseteq DC^{\Delta}_{\mathcal{L}_2}$ is also viable, but will not be analysed here.

The condition $\preccurlyeq_{DC^{\Delta}}$ embodies a arguably reasonable interpretation of the property 3. However, apart from very simple systems,[8] it will be difficult in practice to relate the systems in this way. This may explain the absence of comparisons using $\preccurlyeq_{DC^{\Delta}}$ in the literature. It seems more appropriate for comparing expressibility of logics with respect to a restricted set of properties.

Another direction of weakening $\preccurlyeq_{DC}$ can be pursued, by trying to capture expressibility of properties not in terms of definability. This is explored in the sequence.

### *3.3 The condition $\preccurlyeq_{EQ}$*

In this section, the expressibility of properties is approached in terms of the ability to distinguish pairs of structures having or not the property at issue:

**Proposal 3.3.1** (Expressing properties: distinguishing capacity)**.** *A property $P$ of $\tau$-structures is expressible in $\mathcal{L}$ if, for all $\tau$-structures $\mathfrak{A}$ and $\mathfrak{B}$, whenever $\mathfrak{A} \in P$ and $\mathfrak{B} \notin P$, then there is an $\mathcal{L}$-sentence $\phi$ that can distinguish $\mathfrak{A}$ and $\mathfrak{B}$,* i.e.*: $\mathfrak{A} \Vdash_{\mathcal{L}} \phi$ and $\mathfrak{B} \nVdash_{\mathcal{L}} \phi$.*

In this manner, another condition for capturing $E^*$ can be defined:

**Definition 3.3.2** ($\preccurlyeq_{EQ}$)**.** *$\mathcal{L}_1 \preccurlyeq_{EQ} \mathcal{L}_2$ iff for all $\tau$ and all $\mathfrak{A}, \mathfrak{B} \in \mathcal{M}[\tau]$, if $\mathfrak{A} \not\equiv_{\mathcal{L}_1} \mathfrak{B}$ then $\mathfrak{A} \not\equiv_{\mathcal{L}_2} \mathfrak{B}$.*

### 3.3.3 On the properties of $\preccurlyeq_{EQ}$

Kocurek (2018, p. 133) provides a counterexample for property 1, thus:

**Remark 3.3.4.** *(Kocurek) Property 1 is not satisfied by $\preccurlyeq_{EQ}$.*

[8] *E.g.* the case of $\mathcal{L}(Q_0)^{w*}$ and $\mathcal{L}_{\omega\omega}$ (proof of remark 3.2.5) and the case of $\mathcal{L}_{\omega\omega}^{atom}$ and $\mathcal{L}_{\omega\omega}^{conj}$ (proof of remark 4.0.13).

Moreover, it holds that:

**Remark 3.3.5.** *Property 2 is not satisfied by* $\preccurlyeq_{EQ}$.

**Remark 3.3.6.** *Property 3 is not satisfied by* $\preccurlyeq_{EQ}$.

### *3.4 The condition* $\preccurlyeq_{EQ^s}$

Kocurek (2018, p. 138) proposes a stronger version of $\preccurlyeq_{EQ}$ which is based on distinguishing possibly larger groups of structures. This stronger version can be said to embody the following stricter rendering of the notion of expressing properties of models.

**Proposal 3.4.1** (Expressing properties: strong distinguishing capacity)**.** *Let* $\mathcal{C}, \mathcal{D} \subseteq \mathcal{M}[\tau]$, *for some* $\tau$, *and define* $\mathcal{C} \equiv_{\mathcal{L}} \mathcal{D}$ *as* $\bigcap_{\mathfrak{A} \in \mathcal{C}} Th^{\tau}_{\mathcal{L}}(\mathfrak{A}) = \bigcap_{\mathfrak{B} \in \mathcal{D}} Th^{\tau}_{\mathcal{L}}(\mathfrak{B})$.

*A property* $P$ *of* $\tau$*-structures is expressible in* $\mathcal{L}$ *if, for all collections* $\mathcal{C}, \mathcal{D}$, *whenever every member of* $\mathcal{C}$ *has* $P$, *and some member of* $\mathcal{D}$ *does not have* $P$, *then* $\mathcal{C} \not\equiv_{\mathcal{L}} \mathcal{D}$.

From this, a new condition for $E^*$ can be defined:

**Definition 3.4.2** ($\preccurlyeq_{EQ^s}$)**.** $\mathcal{L}_1 \preccurlyeq_{EQ^s} \mathcal{L}_2$ *iff for every* $\tau$ *and all* $\mathcal{C}, \mathcal{D} \subseteq \mathcal{M}[\tau]$, *if* $\mathcal{C} \not\equiv_{\mathcal{L}_1} \mathcal{D}$, *then* $\mathcal{C} \not\equiv_{\mathcal{L}_2} \mathcal{D}$.

#### 3.4.3 On the properties of $\preccurlyeq_{EQ^s}$

As with $\preccurlyeq_{EQ}$, the condition $\preccurlyeq_{EQ^s}$ does not satisfy any of the properties:

**Remark 3.4.4.** *Property 1 is not satisfied by* $\preccurlyeq_{EQ^s}$.

**Remark 3.4.5.** *Property 2 is not satisfied by* $\preccurlyeq_{EQ^s}$.

**Remark 3.4.6.** *Property 3 is not satisfied by* $\preccurlyeq_{EQ^s}$.

The above weakenings of $\preccurlyeq_{DC}$ require the signature to remain the same when comparing logics. However, it often happens that, in order to match the expressivity of a given logic, one has to expand the non-logical vocabulary of the other one. In the sequence this sort of relaxation of $\preccurlyeq_{DC}$ will be studied.

*3.5 The condition $\preccurlyeq_{PC}$*

To motivate the introduction of a wider rendering of the notion of expressing properties of models, consider the class of infinite structures $\mathcal{I} \in \mathcal{M}[\emptyset]$.[9] As regards $\mathcal{L}(Q_0)$, that is, first-order logic ($\mathcal{L}_{\omega\omega}$) extended with the quantifier $Q_0$, meaning "there are infinitely many ...", the class $\mathcal{I}$ is straightforwardly definable with $Q_0 x(x = x)$. Concerning $\mathcal{L}_{\omega\omega}$, it is $\Delta$-definable but not definable.

However, if one expands the signature to $\tau = \{R\}$ where $R$ is a binary relation symbol, it is easy to find a $\mathcal{L}_{\omega\omega}[\tau]$-sentence $\phi$ such that every structure in $\mathcal{I}$ can be expanded into a model of $\phi$ (for example "$R$ is a non-reflexive, transitive and serial relation on the domain"). Thus, $\mathcal{I}$ *projective* in $\mathcal{L}_{\omega\omega}$. Let us consider a precise definition of this concept:

**Definition 3.5.1** (Projective definability)**.** *A property $P$ of $\tau$-structures is projectively definable in $\mathcal{L}$ (for short $P \in PC_{\mathcal{L}}$) if, for every $\tau$-structure $\mathfrak{A} \in P$, there is a $\tau'$-extension $\mathfrak{A}'$ with $\tau' \supseteq \tau$, and an $\mathcal{L}[\tau']$-formula $\phi$, such that $\mathfrak{A} \in P$ iff $\mathfrak{A}' \Vdash_{\mathcal{L}} \phi$.*

Several authors have defended projective definability as a means to compare expressiveness.[10] The proposal here is to interpret them as claiming that this wider notion of definability could be used as a way to express properties of models:

**Proposal 3.5.2** (Expressing properties: projective definability)**.** *A property $P$ of $\tau$-structures is expressible in $\mathcal{L}$ if $P \in PC_{\mathcal{L}}$.*

Thus, a corresponding relative measure of expressiveness is obtained:

[9] A structure $\mathfrak{A} \in \mathcal{M}[\emptyset]$ iff $\mathfrak{A} = (A)$ for some domain $A$.

[10] *E.g.* see (MAKOWSKY, 1980, p. 420), (TARLECKI, 1986, p. 358), (MESEGUER, 1989, p. 299), (SHAPIRO, 1991, p. 232) and (BRESOLIN; MUÑOZ-VELASCO; SCIAVICCO, 2016, p. 90).

**Definition 3.5.3** ($\preccurlyeq_{PC}$). *$\mathcal{L}_1 \preccurlyeq_{PC} \mathcal{L}_2$ iff $DC_{\mathcal{L}_1} \subseteq PC_{\mathcal{L}_2}$.*

Notice that the above definition can be given solely in terms of projective classes:[11]

**Remark 3.5.4.** *$DC_{\mathcal{L}_1} \subseteq PC_{\mathcal{L}_2}$ iff $PC_{\mathcal{L}_1} \subseteq PC_{\mathcal{L}_2}$.*

### 3.5.5 On the properties of $\preccurlyeq_{PC}$

It is easily seen that

**Remark 3.5.6.** *Property 1 does not hold for $\preccurlyeq_{PC}$.*

As regards property 2, Makowsky (1980, p. 414) has shown that it holds for a class of dynamic logics and operators. However, in a general setting, we have that

**Remark 3.5.7.** *Property 2 does not hold for $\preccurlyeq_{PC}$.*

Similarly with the other conditions seen above, a rather weak form naturally still holds:

**Remark 3.5.8.** *If $\mathcal{L}_1 \preccurlyeq_{PC} \mathcal{L}_2$ and every projective class in $\mathcal{L}_1^e$ is a projective class in $\mathcal{L}_1$, then $\mathcal{L}_1^e \preccurlyeq_{PC} \mathcal{L}_2^e$.*

Let us consider now property 3. Clearly, if two sentences *have equivalent meanings* only when they have the same class of models, then the property 3 does not hold for $\preccurlyeq_{PC}$.

However, one might require for equivalence of meaning only that the class of models of one sentence be an extension of the class of models of the other. The motivation would be to "distribute" better the role of logical and non-logical constants in same-saying relations. Taking again the examples of the beginning of

[11] However, by defining $\preccurlyeq_{PC}$ as $PC_{\mathcal{L}_1} \subseteq PC_{\mathcal{L}_2}$ the result 5.0.2 no longer holds. Given $\mathcal{K} \in PC_{\mathcal{L}_1}$, one would not be able to obtain by negation that $\overline{\mathcal{K}} \in PC_{\mathcal{L}_1}$, and thus cannot use the $\Delta$-interpolation property of $\mathcal{L}_2$.

the section, if the sentences $Q_0x(x = x)$ and "$R$ is a non-reflexive, transitive and serial relation on the domain" are considered equivalent, as essentially "saying the same thing", then one is allowing the logical strength embedded in $Q_0$ to be counterbalanced with the addition of the extra non-logical constant $R$. Thus, in this sense, the property 3 would still hold for $\preccurlyeq_{PC}$. This point will be discussed further in section 6 below.

## 4 Comparing the conditions for relative expressiveness

Among $\preccurlyeq_{DC}$, $\preccurlyeq_{DC^{\triangle}}$ and $\preccurlyeq_{EQ}$, the first is the strictest and the latter is the loosest.

**Remark 4.0.1.** *Both $\mathcal{L}_1 \preccurlyeq_{DC} \mathcal{L}_2$ and $\mathcal{L}_1 \preccurlyeq_{DC^{\triangle}} \mathcal{L}_2$ imply that $\mathcal{L}_1 \preccurlyeq_{EQ} \mathcal{L}_2$.*

The reverse implication does not hold:

**Remark 4.0.2.** *$\mathcal{L}_1 \preccurlyeq_{EQ} \mathcal{L}_2$ does not imply that $\mathcal{L}_1 \preccurlyeq_{DC^{\triangle}} \mathcal{L}_2$.*

As regards $\preccurlyeq_{EQ}$ and $\preccurlyeq_{EQ^s}$, we clearly have that

**Remark 4.0.3.** *If $\mathcal{L}_1 \preccurlyeq_{EQ^s} \mathcal{L}_2$, then $\mathcal{L}_1 \preccurlyeq_{EQ} \mathcal{L}_2$.*

A counterexample is provided in Kocurek (2018, p. 138) to show that

**Remark 4.0.4** (Kocurek)**.** *It is false that if $\mathcal{L}_1 \preccurlyeq_{EQ} \mathcal{L}_2$, then $\mathcal{L}_1 \preccurlyeq_{EQ^s} \mathcal{L}_2$.*

Kocurek (*ibid*, p. 139) also obtained the following two results.

**Remark 4.0.5** (Kocurek)**.** *$\mathcal{L}_1 \preccurlyeq_{DC^{\triangle}} \mathcal{L}_2$ implies that $\mathcal{L}_1 \preccurlyeq_{EQ^s} \mathcal{L}_2$.*

**Remark 4.0.6** (Kocurek)**.** *$\mathcal{L}_1 \preccurlyeq_{EQ^s} \mathcal{L}_2$ does not imply that $\mathcal{L}_1 \preccurlyeq_{DC^{\triangle}} \mathcal{L}_2$.*

Thus, we can conclude (also directly by the considerations on remark 3.4.6) that

**Remark 4.0.7.** *$\mathcal{L}_1 \preccurlyeq_{EQ^s} \mathcal{L}_2$ does not imply that $\mathcal{L}_1 \preccurlyeq_{DC} \mathcal{L}_2$.*

As regards the relation between $\preccurlyeq_{DC}$ and $\preccurlyeq_{PC}$, it clearly holds that

**Remark 4.0.8.** *If $\mathcal{L}_1 \preccurlyeq_{DC} \mathcal{L}_2$, then $\mathcal{L}_1 \preccurlyeq_{PC} \mathcal{L}_2$.*

The converse does not hold in general. A case where it does is second-order logic, where $DC = PC$.[12]

**Remark 4.0.9.** *It is false that if $\mathcal{L}_1 \preccurlyeq_{PC} \mathcal{L}_2$, then $\mathcal{L}_1 \preccurlyeq_{EQ} \mathcal{L}_2$.*

**Remark 4.0.10.** *It is false that if $\mathcal{L}_1 \preccurlyeq_{EQ} \mathcal{L}_2$, then $\mathcal{L}_1 \preccurlyeq_{PC} \mathcal{L}_2$.*

By remarks 4.0.3 and 4.0.9 we have that:

**Remark 4.0.11.** *It is false that if $\mathcal{L}_1 \preccurlyeq_{PC} \mathcal{L}_2$, then $\mathcal{L}_1 \preccurlyeq_{EQ^s} \mathcal{L}_2$.*

We have not been able to prove or disprove the converse of this remark. Finally, there follows the corresponding results as regards $\preccurlyeq_{DC^\Delta}$:

**Remark 4.0.12.** *It is false that if $\mathcal{L}_1 \preccurlyeq_{PC} \mathcal{L}_2$, then $\mathcal{L}_1 \preccurlyeq_{DC^\Delta} \mathcal{L}_2$.*

**Remark 4.0.13.** *It is false that if $\mathcal{L}_1 \preccurlyeq_{DC^\Delta} \mathcal{L}_2$, then $\mathcal{L}_1 \preccurlyeq_{PC} \mathcal{L}_2$.*

In the figure 1 all the relations among the conditions are drawn. The arrows obtained by transitivity are omitted. In connection with a comment made in section 2, notice that $\preccurlyeq_{EQ}$ is an interesting measure with respect to which to prove inexpressibility results.

[12]This so because whenever the characterization of a class of structures with the signature $\tau$ needs a formula $\phi$ having additional relation symbols $R_1, ..., R_n$, one can add existential quantifiers binding them and thus maintaining the signature of the resulting formula at $\tau$.

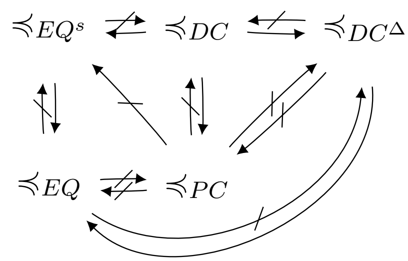


Figure 1: Relations between formal conditions for relative expressiveness.

## 5 Sufficient conditions for the existence of stricter relations of expressiveness

When we have a criterion $\preccurlyeq_X$ which is is stricter than another $\preccurlyeq_Y$, an interesting question is:

(i) If $\mathcal{L}_1 \preccurlyeq_Y \mathcal{L}_2$ are there any properties of $\mathcal{L}_1$ and $\mathcal{L}_2$ that are sufficient to conclude that $\mathcal{L}_1 \preccurlyeq_X \mathcal{L}_2$?

The case is easy for the pair $\preccurlyeq_{DC^\Delta}$ and $\preccurlyeq_{DC}$. Suppose $\mathcal{L}_1 \preccurlyeq_{DC^\Delta} \mathcal{L}_2$, then an immediate sufficient condition for $\mathcal{L}_1 \preccurlyeq_{DC} \mathcal{L}_2$ to hold, is that $\mathcal{L}_2$ has an infinitary conjunction $\bigwedge$.

As an answer to (i) involving $\preccurlyeq_{EQ}$ and $\preccurlyeq_{DC}$, Kocurek (2018, p. 134) proposed two properties for $\mathcal{L}_2$: infinitary conjunction and truth-functional negation.[13] However, this is incompatible with some of the results mentioned in the proof of remark 3.3.6, involving the logics $\mathcal{L}_{\infty\omega}$ and $\mathcal{L}_{\infty G}$. In the proof of his claim, the following sentence is used (where $\bigvee$ stands for infinitary disjunction, defined from $\bigwedge$ and $\neg$):

[13]There is a characterization of $\mathcal{L}_{\omega\omega}$ in (LINDSTRÖM, 1974, p.137) amounting to a related result: for a certain property $P$, if $\mathcal{L}_{\omega\omega} \preccurlyeq_{DC} \mathcal{L}$, $\mathcal{L} \preccurlyeq_{EQ} \mathcal{L}_{\omega\omega}$ and $\mathcal{L}$ has $P$, then $\mathcal{L} \preccurlyeq_{DC} \mathcal{L}_{\omega\omega}$.

$$\text{(ii)} \qquad \psi = \bigvee\{\bigwedge Th_{\mathcal{L}_2}(\mathfrak{A}) \mid \mathfrak{A} \Vdash_{\mathcal{L}_1} \phi\}$$

Keisler (1968) studied the logics $\mathcal{L}_{\infty G}$ and $\mathcal{L}_{\infty\omega}$ and had considered sentences such as (ii) above. He pointed out that the conflict with the results mentioned on the proof of remark 3.3.6 is only apparent, as a sentence of the form (ii) would not be an $\mathcal{L}_{\infty\omega}$-sentence, since the class $\Phi = \{\bigwedge Th_{\mathcal{L}_{\infty\omega}}(\mathfrak{A}) \mid \mathfrak{A} \Vdash_{\mathcal{L}_{\infty G}} \phi\}$ cannot in general be taken to be a set. Thus, Kocurek's proposal does not hold unless it is guaranteed that $\bigvee \Phi$ will be an $\mathcal{L}_2$-sentence.

As regards (i) with respect to $\preccurlyeq_{PC}$, $\preccurlyeq_{DC}$ and $\mathcal{L}_2$, it was proposed by Makowsky (1980) a version of Beth definability, called "occurrence normality",[14] but shortly afterwards the author pointed out that it would not work (MAKOWSKY, 1981).

Consider, however, the stronger property[15]

**Definition 5.0.1** ($\Delta$-interpolation)**.** *If $\mathcal{K}$ and $\overline{\mathcal{K}}$ are $PC_{\mathcal{L}}$, then $\mathcal{K}$ is $DC_{\mathcal{L}}$.*

Then we have that

**Remark 5.0.2.** *If $\mathcal{L}_1 \preccurlyeq_{PC} \mathcal{L}_2$, $\mathcal{L}_1$ has truth-functional negation and $\mathcal{L}_2$ has $\Delta$-interpolation, then $\mathcal{L}_1 \preccurlyeq_{DC} \mathcal{L}_2$.*

## 6 Discussion

The concept of expressing or capturing properties of models can be given many interpretations, as it was shown. In this sense, the choice of one interpretation over another is partly stipulative. However, some reasons for preference can be brought into consideration,

[14] Definition: let $\tau' = \tau \cup \{R\}$ and $\tau'' = \tau \cup \{S\}$, where $R, S$ are relation symbols not belonging to $\tau$ and having the same type. Consider a $\tau'$-formula $\phi(R)$ and let $\phi(S)$ be a $\tau''$-formula obtained from $\phi(R)$ replacing all occurrences of $R$ by $S$. Then $\mathcal{L}$ is *occurrence normal* whenever $\vDash_{\mathcal{L}} \phi(R) \leftrightarrow \phi(S)$ for every $R$ and $S$, implies that there is a $\tau$-formula $\theta$ such that $\vDash_{\mathcal{L}} \phi(R) \leftrightarrow \theta$.

[15] For more, see (MAKOWSKY; SHELAH; STAVI, 1976).

such as the properties the generated expressiveness relations have and also judgements of material adequacy.

### *6.1 Property 1*

This property will fail whenever the logic $\mathcal{L}_2$ is at least as expressive as $\mathcal{L}_1$, and the fragment of $\mathcal{L}_2$ required to express $\mathcal{L}_1$ already is capable of expressing properties of models that are not expressible in $\mathcal{L}_1$. When the notion of "expressing a property of models" is approached via proposals 3.2.2 and 3.4.1, this situation may happen with respect to the corresponding formal condition, as shown in remarks 3.2.5 and 3.4.4.

Requiring satisfaction of property 1 amounts to consider an expressiveness relation akin to an embedding of models. This would provide a guarantee of some sort of meaning preservation from one language to the other. It is, however, not necessary for the existence of meaning preservation, as it will be argued in subsection 6.3.

### *6.2 Property 2*

Requiring an expressiveness condition to satisfy property 2 could be motivated by a sort of "modular" view of expressiveness: adding the same expressive capacities to a pair of logics should preserve an eventual expressive equivalence between them. Despite its *prima facie* reasonableness, this modular view of expressiveness is rather restrictive: none of the studied conditions for expressiveness satisfy it, except for specific collections of logics and operators, as regards $\preccurlyeq_{PC}$, and a specific kind of associated translation (definitional), as regards $\preccurlyeq_{DC}$. That non-definitional translation can also be involved in a $\preccurlyeq_{DC}$-expressiveness relation is easily seen.[16]

[16] Take *e.g.* the following simple example from (PETERS; WESTERSTÅHL, 2006). Consider again the logic $\mathcal{L}(Q_0)$ and

### *6.3 Property 3*

There is a rather clear sufficient condition for satisfaction of this property: whenever for every sentence in one logic, there is a sentence in the other logic with the same class of models. One could also consider that the matching of each sentence with a (perhaps recursively definable) set of sentences with the same models is also sufficient for meaning equivalence and, thus, sufficient for the satisfaction of property 3. In both these cases, the related sentences/set of sentences are required to share the same class of models. In what follows it is argued this is not necessary for meaning equivalence.

Naturally, one can hardly hope to render the notion of meaning equivalence precise without leaving behind some or other salient feature. As an example of this, Peters and Westerståhl (2006) explore various "same-saying relations", including analytical, logical and cognitive equivalence, each highlighting an important aspect of meaning equivalence. The first is equivalence with respect to meaning postulates, the second is just the sharing of the same models and the third is the cognitively recognizable sharing of the same models. No same-saying relations allowing the use of extra non-logical symbols to express the "same thing" are in-

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let $\mathcal{L}(I)$ be first-order logic extended with Hartïg's equicardinality quantifier $I$, where $Ix(\psi_1, \psi_2)$ means "the cardinality of the set of things satisfying $\psi_1(x)$ is equal to the cardinality of the set of things satisfying $\psi_2(x)$", the free occurrences of $x$ in both $\phi_1$ and $\psi_2$ are bound by $I$. It is possible to express the infinity of $P$ in $\mathcal{L}(I)$ saying that subtracting one element does not alter the cardinality of $P$. This could be done with the translation $\mathcal{T} : \mathcal{L}(Q_0) \longrightarrow \mathcal{L}(I)$, whose main case is defined as

$$\mathcal{T}(Q_0 x \psi) = \exists y(\mathcal{T}^*(\psi) \wedge Ix(\mathcal{T}(\psi), \mathcal{T}(\psi) \wedge x \neq y)),$$

where $y$ is new and $\mathcal{T}^*(\psi)$ is the result of applying $\mathcal{T}$ to the formula obtained by substituting $y$ for the free occurrences of $x$ in $\psi$, that is, $\mathcal{T}^*(\psi) = \mathcal{T}(\psi^y_x)$. Note that $\mathcal{T}$ is not definitional, as $\mathcal{T}(Q_0 x \psi)$ is not defined with respect to $\mathcal{T}$ applied to the sub-formulas of $Q_0 x \psi$.

vestigated. Apparently, the reason is that projective definability is only "a particular form of definition in second-order logic" (*ibid*, p. 432).

Cognitively recognizable equivalence is perhaps a drastic example, but establishing whether two sentences have the same meaning even for much more definite cases can still be context dependent, as Shapiro (2019) argued. The remarks on projective definability by Peters and Westerståhl highlights another axis of dependence that should be considered: the weight attributed to logical and non-logical constants in considerations of meaning equivalence and expressive capacity. A crucial point of taking projective definability as a same-saying relation is exactly to consider that prefixing existential second order quantifiers in front of a formula does not add extra expressive capacity to it. This is a way to approach the eventual unbalance between what is already embedded in a logical constant in one logic and what needs to be constructed with the help of non-logical symbols in the other. On subsection 3.5 a simple example involving the class of infinite structures was given to illustrate this. It is of interest to consider the example used by Shapiro (1991) to defend the same point.

Let $\mathcal{L}(A)$ be an extension of $\mathcal{L}_{\omega\omega}$ with the addition of the ancestor operator $A$: for a formula $\phi$, $Ayz(\phi)c_1c_2$ means that $c_1$ is an ancestor of $c_2$ in the relation expressed by the formula $\phi(y,z)$, being the variables $y,z$ of $\phi$ bound by $A$.[17] Let $\Phi$ be the usual first-order axioms for ordered fields, in the vocabulary $\tau = \{+,\cdot,<,0,1\}$, for binary $+,\cdot,<$ and individual constants $0,1$. Let $\Phi_1$ be an $\mathcal{L}(A)$-sentence saying "for every element $x$ there is a greater one $y$ whose ancestral over the relation $w = z+1$ is 1". Finally, let $\mathcal{A}$ be the class of

[17] That is, there are elements $a_{i_1},...,a_{i_n}$ such that when assigned, respectively, to variables $x_{i_1},...,x_{i_n}$, the formulas $\phi(c_1,x_{i_1})$, $\phi(x_{i_1},x_{i_2})$ and ... and $\phi(x_{i_n},c_2)$ hold.

the Archimedean ordered fields.

It holds that $\mathcal{A}$ is definable in $\mathcal{L}(A)$ by the formula $\Phi \wedge \Phi_1$. But $\mathcal{A}$ is not definable in $\mathcal{L}(Q_0)$, and thus, $\mathcal{L}(A) \not\preccurlyeq_{DC} \mathcal{L}(Q_0)$. Concerning this (and analogous cases) Shapiro says

> I would suggest that the 'non-inclusions' here are artefacts of an unnatural restriction on the non-logical terminology. To illustrate this, we show that $\mathcal{L}(Q_0)$ can express the notion of an Archimedean field if the non-logical terminology is slightly expanded. (SHAPIRO, 1991, p. 232)

Let $N$ be an unary relation symbol, and let $\Phi_2$ be an $\mathcal{L}(Q_0)$-sentence saying "0 as well as every of its successors are $N$, and for every $x$ there are only finitely many $y$ such that $Ny$ and $y < x$." Shapiro continues remarking that every model of $\Phi \wedge \Phi_2$

> is an Archimedean field. Conversely, in every Archimedean field $F$, there is a set $P$ (namely, the 'natural numbers' of $F$) such that if $P$ is made the extension of $N$, then $P$ satisfies $[\Phi \wedge \Phi_2]$.(*ibid*)

Thus, the property of being an Archimedean field, though not definable, is projective in $\mathcal{L}(Q_0)$. Now, let $(\Phi_2)^X_N$ stand for the substitution of the predicate variable $X$ for $N$ in $\Phi_2$. In this manner, $\Phi \wedge \Phi_1$, $\Phi \wedge \Phi_2$ and $\exists X(\Phi \wedge (\Phi_2)^X_N)$, would be deemed to have equivalent meanings, using $PC$-equivalence as a measure. Indeed, if $\mathcal{L}^*$ is an extension of $\mathcal{L}$ with the addition of prenex second-order existential quantifiers, naturally we have that $\mathcal{L} \approxeq_{PC} \mathcal{L}^*$, so they would be considered expressively equivalent.

What can be a problem is the fact that $PC$-equivalence would not satisfy substitution *salva veritate* even in extensional contexts, such as in $\mathcal{L}(Q_0)$, since

$\neg Q_0 xx = x$ and $\neg$("$R$ is a non-reflexive, transitive and serial relation on the domain") are not $PC$-equivalent. This could be used to question the reasonableness of its use as a same-saying relation.

In any case, other authors also favoured $\preccurlyeq_{PC}$ as a measure of expressiveness: Makowsky (1980, p. 414) makes an analysis similar to Shapiro's as regards comparisons of dynamic logics in terms of expressive power:

> The aim of this chapter is to show that most of them [dynamic logics] are AP-equivalent [PC-Equivalent]. This shows us that differences in the expressive power of most dynamic logics are "accidental" in the sense that to show their equivalence one needs additional predicates. This means introducing "abbreviations" for certain procedures and is done freely in mathematics and programming.

Later, (*ibid*, p. 420) he defends the use of $PC$-classes for expressiveness comparisons:

> AP-reductibility [$\preccurlyeq_{PC}$] is a natural notion to compare expressive power of logics, eventually even more natural than reductibility [$\preccurlyeq_{DC}$]

Bresolin, Muñoz-Velasco and Sciavicco (2016, p. 94) also make similar remarks, when comparing $\preccurlyeq_{DC}$ and $\preccurlyeq_{PC}$ and the hierarchy of logics they generate, weak and strong, respectively:

> Adding new propositional letters to facilitate translations from a fragment to another is a common practice, for example, to prove that every $n$-ary clause in propositional logic can be transformed into an equi-satisfiable set of ternary clauses. In

> this sense, it can be argued that the weak hierarchy is less general; nonetheless, both the weak and the strong hierarchies contribute to the comprehension of the relative expressive power of sub-propositional fragments.

As a final consideration, we notice that some authors, *e.g.* Makowsky, Shelah and Stavi (1976, p. 156), Barwise and Feferman (1985, p. 18) and Mundici (1985, p. 216), seem to attribute to $\preccurlyeq_{PC}$ a measure of "implicit expressive power", as against the "explicit" measure which would be given by $\preccurlyeq_{DC}$. The remark 5.0.2 would support this claim, as the satisfaction by $\mathcal{L}_2$ of $\Delta$-interpolation implies a version of Beth definability (MAKOWSKY; SHELAH; STAVI, 1976, p. 163).

## 7 Conclusion

In this paper it was proposed that, for model-theoretic logics, the relation ($E$) "everything that can be said in $\mathcal{L}_1$ can also be said in $\mathcal{L}_2$" is best captured by the ($E^*$) "every property of models expressible in $\mathcal{L}_1$ is also expressible in $\mathcal{L}_2$". Five possible answers to the question "when is a property of models expressible?" were proposed, and it was argued that formal measures of expressiveness (or "conditions", for brevity) that appeared in the literature can be understood as issuing from each of these answers.

Some properties one might expect of expressiveness relations were analyzed, and each condition was tested for them. The table 1 contains the properties satisfied by each condition. It was seen that properties 1 and 2, contrary to what might be thought at first sight, were rather restrictive. Taking French's approach (2019) to notational variance a basis, then the results above on property 2 shows that notational variance and expressive equivalence are really distinct relations. The discussion on satisfaction of property 3 has highlighted the importance of considering the logical/non-logical divide when thinking about meaning equivalence.

Finally, it was checked how each condition is related to each other. Among them $\preccurlyeq_{DC}$ is the strictest and the two loosest are $\preccurlyeq_{EQ}$ and $\preccurlyeq_{PC}$. There are other conditions looser than $\preccurlyeq_{PC}$, which would be based on even laxer conceptions of "expressing properties of models", *e.g.*, those involving the concept of relative projective definability and their $\Delta$-variants.[18] These conditions based on relative projective definability have also has been used in expressiveness comparisons by some authors.[19] This, as well as further properties of the conditions analysed here, shall be object of further investigations.

| Condition | Property 1 | Property 2 | Property 3 |
|---|---|---|---|
| $\preccurlyeq_{DC}$ | $*$ | $*^{\dagger}$ | $*$ |
| $\preccurlyeq_{DC^{\Delta}}$ | – | – | $*^{\dagger}$ |
| $\preccurlyeq_{EQ}$ | – | – | – |
| $\preccurlyeq_{EQ^{s}}$ | – | – | – |
| $\preccurlyeq_{PC}$ | $*$ | – | $*^{\dagger}$ |

Table 1: List of conditions and properties. The dash indicates non-satisfaction, the asterisk indicates satisfaction and the superscripted asterisks refer to satisfaction of weak forms of the respective property.

[18] That is, variants allowing a set of formulas to express properties of models.

[19] *E.g.* in (EBBINGHAUS, 1985), (KRYNICKI; VÄÄNÄNEN, 1989) and (SHAPIRO, 1991).

## 8 Appendix: Proofs of some remarks

*8.0 Remark 3.1.5*

> If $\mathcal{L}_1 \preccurlyeq_{DC} \mathcal{L}_2$, then there is a fragment $\mathcal{L}_2^*$ of $\mathcal{L}_2$ such that $\mathcal{L}_1 \approxeq_{DC} \mathcal{L}_2^*$.

*Proof.* Take the associated mapping $\mathcal{T} : \mathcal{S}_1 \longrightarrow \mathcal{S}_2$ and let $\mathcal{T}[\mathcal{S}_1]$ be the image of $\mathcal{S}_1$ under $\mathcal{T}$. Define $\mathcal{L}_2^*$ to be $(\mathcal{T}[\mathcal{S}_1], \mathcal{M}, \Vdash_2^{\restriction \mathcal{T}[\mathcal{S}_1]})$. That $\mathcal{L}_1 \approxeq_{DC} \mathcal{L}_2^*$ is clear. □

*8.0 Remark 3.1.8*

> Suppose that $\mathcal{L}_1 \preccurlyeq_{DC} \mathcal{L}_2$ and that $\mathcal{L}_1^e$, $\mathcal{L}_2^e$ are uniform extensions. Let $\mathcal{F}_1, \mathcal{F}_1^e, \mathcal{F}_2, \mathcal{F}_2^e$ be their respective collections of well-formed formulas. Then, if there is a definitional translation $\mathcal{T} : \mathcal{F}_1 \longrightarrow \mathcal{F}_2$, there is a definitional translation $\mathcal{T}^+ : \mathcal{F}_1^e \longrightarrow \mathcal{F}_2^e$ such that $\mathcal{L}_1^e \preccurlyeq_{DC} \mathcal{L}_2^e$.

Let the hypotheses of the remark be satisfied. Define the following extension of $\mathcal{T}$:

**Definition 8.0.1.** $\mathcal{T}^+ : \mathcal{F}_1^e \longrightarrow \mathcal{F}_2^e$

- *If* $\phi \in \mathcal{F}_1$, *then* $\mathcal{T}^+(\phi) = \mathcal{T}(\phi)$
- *If* $\phi \in \mathcal{F}_1^e \setminus \mathcal{F}_1$, *then there are some cases:*

  (a) $\phi$ *is a* 0*-ary operator. Then* $\mathcal{T}^+(\phi) = \phi$.

  (b) $\phi$ *is of the form* $\divideontimes(\phi_1, ..., \phi_n)$, *where*

    (i) *the operator* $\divideontimes$ *is new. Then*
    $$\mathcal{T}^+(\divideontimes(\phi_1, ..., \phi_n)) = \divideontimes(\mathcal{T}^+(\phi_1), ..., \mathcal{T}^+(\phi_n)).$$

    (ii) *the operator* $\divideontimes$ *belongs to* $\mathcal{L}_1$. *Then*
    $$\mathcal{T}^+(\divideontimes(\phi_1, ..., \phi_n)) = \theta^{\divideontimes}(\mathcal{T}^+(\phi_1), ..., \mathcal{T}^+(\phi_n)),$$

*where $\theta^{\circledast}$ is the schema associated with $\circledast$ by $\mathcal{T}$.*

Now there follows the proof of remark 3.1.8:

*Proof.* The proof is by induction on the degree of formulas. Let the assumptions of the remark be satisfied. Let $\mathcal{S}_1$ and $\mathcal{S}_1^e$ be the collections of sentences of $\mathcal{L}_1$ and $\mathcal{L}_2^e$, respectively. If $\phi \in \mathcal{S}_1[\tau]$, or is a new 0-ary operator, the result follows directly. Take $\phi \in \mathcal{S}_1^e[\tau] \backslash \mathcal{S}_1[\tau]$ and consider the cases:

1. $\phi = \circledast(\psi_1, ..., \psi_n)$, such that $\circledast$ belongs to $\mathcal{L}_1$. The inductive hypothesis gives, for every $\mathfrak{A} \in \mathcal{M}[\tau]$,

   $$\text{(a)} \qquad \begin{array}{l} \mathfrak{A} \Vdash_{\mathcal{L}_1^e} \psi_i \text{ iff} \\ \qquad \mathfrak{A} \Vdash_{\mathcal{L}_2^e} \mathcal{T}^+(\psi_i), \text{ for } 1 \leq i \leq n. \end{array}$$

   Since $\mathcal{T}$ is definitional, then for atomic $\phi_1, ..., \phi_n \in \mathcal{F}_1[\tau]$ and for every $\mathfrak{A} \in \mathcal{M}[\tau]$:

   $$\text{(b)} \qquad \begin{array}{l} \mathfrak{A} \Vdash_{\mathcal{L}_1^e} \circledast(\phi_1, ..., \phi_n) \text{ iff} \\ \qquad \mathfrak{A} \Vdash_{\mathcal{L}_2^e} \theta^{\circledast}(\phi_1, ..., \phi_n). \end{array}$$

   Using (a) and (b), we have that

   $$\begin{array}{l} \mathfrak{A} \Vdash_{\mathcal{L}_1^e} \circledast(\psi_1,..., \psi_n) \text{ iff} \\ \qquad \mathfrak{A} \Vdash_{\mathcal{L}_2^e} \theta^{\circledast}(\mathcal{T}^+(\psi_1), ..., \mathcal{T}^+(\psi_n)), \end{array}$$

   where $\circledast(\psi_1, ..., \psi_n)$ is obtained from $\circledast(\phi_1, ..., \phi_n)$ by substituting $\psi_i$ for $\phi_i$; the sentence $\theta^{\circledast}(\mathcal{T}^+(\psi_1), ..., \mathcal{T}^+(\psi_n))$ is obtained from $\theta^{\circledast}(\phi_1, ..., \phi_n)$ by substituting $\mathcal{T}^+(\psi_i)$ for $\phi_i$ $(1 \leq i \leq n)$. Recall that, being $\mathcal{T}^+$ definitional, the free variables of $\psi_i$ are the same as those of $\mathcal{T}^+(\psi_i)$.

2. $\phi = \circledast(\psi_1, ..., \psi_n)$ such that $\circledast$ is new. Then the result follows directly from the inductive hypothesis.

$\square$

*8.0 Proof of remark 3.2.5*

> There are logics $\mathcal{L}_1, \mathcal{L}_2$ such that $\mathcal{L}_1 \preccurlyeq_{DC^\Delta} \mathcal{L}_2$, but there is no fragment $\mathcal{L}_2^*$ of $\mathcal{L}_2$ such that $\mathcal{L}_1 \approxeq_{DC^\Delta} \mathcal{L}_2^*$.

Let the logic $\mathcal{L}(Q_0)^w$ be built from atomic first-order language plus the quantifier $Q_0$ "there are infinitely many". Consider the fragment $\mathcal{L}(Q_0)^{w*}$ of $\mathcal{L}(Q_0)^w$, where there are no iterated occurrences of $Q_0$ and, for the sake of simplicity, only unary relations are allowed. Let $\mathcal{F}_{w*}$ be its collection of formulas and $\mathcal{F}$ the collection of first-order formulas.

**Lemma 8.0.2.** $\mathcal{L}(Q_0)^{w*} \preccurlyeq_{DC^\Delta} \mathcal{L}_{\omega\omega}$.

*Proof.* Define the translation $\mathcal{T} : F_{w*} \longrightarrow \mathscr{P}(\mathcal{F})$ as

- If $\phi$ is atomic, then $\mathcal{T}^\Delta(\phi) = \{\phi\}$
- If $\phi = Q_0 x Rx$, then, for some infinite ordinal $\kappa$,

  $\mathcal{T}^\Delta(\phi) =$
  $\{\exists x_1 ... x_n (\bigwedge_{1\leq i\leq n} (Rx)^{x_i}_x \bigwedge_{1\leq i<j\leq n} x_i \neq x_j) \,|\, n \in \kappa\}$.

Then, $\mathfrak{A} \in Mod_{\mathcal{L}(Q_0)^{w*}}(\phi)$ iff $|R^{\mathfrak{A}}|$ is infinite iff $\mathfrak{A} \in Mod_{\mathcal{L}_{\omega\omega}}(\mathcal{T}^\Delta(\phi))$. $\square$

**Lemma 8.0.3.** *For no fragment $\mathcal{L}^\dagger$ of $\mathcal{L}_{\omega\omega}$, it holds such that $\mathcal{L}(Q_0)^{w*} \approxeq_{DC^\Delta} \mathcal{L}^\dagger$.*

*Proof.* In order for it to hold that $\mathcal{L}(Q_0)^{w*} \preccurlyeq_{DC^\Delta} \mathcal{L}^\dagger$, for some fragment $\mathcal{L}^\dagger$ of $\mathcal{L}_{\omega\omega}$, it must hold that $\mathcal{F}_\dagger \subseteq \bigcup \mathcal{T}^\Delta[\mathcal{F}_{w*}]$. However, there would be no $\Gamma \subseteq \mathcal{F}_{w*}$, such that $Mod_{\mathcal{L}(Q_0)^{w*}}(\Gamma) = Mod_{\mathcal{L}^\dagger}(\exists x_1 x_2 (Rx_1 \wedge Rx_2 \wedge x_1 \neq x_2))$. Therefore, $\mathcal{L}^\dagger \not\preccurlyeq_{DC^\Delta} \mathcal{L}(Q_0)^{w*}$. $\square$

*8.0 Proof of remark 3.2.6*

> There are logics $\mathcal{L}_1, \mathcal{L}_2$ with uniform extensions $\mathcal{L}_1^e, \mathcal{L}_2^e$, such that $\mathcal{L}_1 \preccurlyeq_{DC^\Delta} \mathcal{L}_2$, but $\mathcal{L}_1^e \not\preccurlyeq_{DC^\Delta} \mathcal{L}_2^e$.

*Proof.* Consider the logic $\mathcal{L}(Q_0)^{w*}$ from the proof of remark 3.2.5. Let $\mathcal{L}^e$ be its extension with truth functional negation. Then, there is no $\Gamma \subseteq \mathcal{L}_{\omega\omega}$ such that $Mod_{\mathcal{L}^e}(\neg Q_0 xRx) = Mod_{\mathcal{L}_{\omega\omega}}(\Gamma)$.

□

*8.0 Proof of remark 3.3.5*

> There are logics $\mathcal{L}_1, \mathcal{L}_2$ with uniform extensions $\mathcal{L}_1^e, \mathcal{L}_2^e$, such that $\mathcal{L}_1 \preccurlyeq_{EQ} \mathcal{L}_2$, but $\mathcal{L}_1^e \not\preccurlyeq_{EQ} \mathcal{L}_2^e$.

Consider the logic, to be referred as $\mathcal{L}_{\exists at}$, consisting of the atomic fragment of first-order language together with existential quantification. Let $\mathcal{L}_{\exists d}$ be an extension of $\mathcal{L}_{\exists at}$ with the disjunction operator. Let $\mathcal{L}_{\exists d^e}$ and $\mathcal{L}_{\exists at^e}$ be the extensions of $\mathcal{L}_{\exists d}$ and $\mathcal{L}_{\exists at}$ with respect to truth-functional negation.

Some facts about $\preccurlyeq_{EQ}$:

1. If $\mathfrak{A} \not\equiv_{\mathcal{L}} \mathfrak{B}$, then $\mathfrak{A} \not\equiv_{\mathcal{L}^e} \mathfrak{B}$;
2. It is false that if $\mathfrak{A} \equiv_{\mathcal{L}} \mathfrak{B}$, then $\mathfrak{A} \equiv_{\mathcal{L}^e} \mathfrak{B}$.

The first item is is straightforward. As regards the second, consider again $\mathcal{L}_{\exists d}$ and its extension $\mathcal{L}_{\exists d^e}$. For $\tau = \{P_1, P_2\}$, where $P_1$ and $P_2$ are unary relation symbols, set the $\tau$-structures $\mathfrak{A} = (\{1,2,3\}, \{1,2\}, \{1,3\})$ and $\mathfrak{B} = (\{1,2,3\}, \{1,2\}, \{3\})$. There are, up to equivalence, four $\mathcal{L}_{\exists d}[\tau]$-formulas: $\exists x P_1 x$, $\exists x P_2 x$, $\exists x(P_1 x \vee P_2 x)$ and $\exists x P_1 x \vee \exists x P_2 x$. Thus it holds that $\mathfrak{A} \equiv_{\mathcal{L}_{\exists d}} \mathfrak{B}$. However, for the $\mathcal{L}_{\exists d^e}[\tau]$-formula $\phi = \exists x \neg(\neg P_1 x \vee \neg P_2 x)$ we have that $\mathfrak{A} \Vdash_{\mathcal{L}_{\exists d^e}} \phi$ and $\mathfrak{B} \nVdash_{\mathcal{L}_{\exists d^e}} \phi$.

The remark 3.3.5 follows from the next two lemmas.

**Lemma 8.0.4.** $\mathcal{L}_{\exists d^e} \not\preccurlyeq_{EQ} \mathcal{L}_{\exists at^e}$.

*Proof.* Consider again the vocabulary $\tau = \{P, Q\}$. Up to equivalence, there are eight $\mathcal{L}_{\exists at^e}[\tau]$-formulas: $\exists x Px$, $\exists x \neg Px$, $\neg\exists x Px$ and $\neg\exists\neg Px$ and the corresponding ones for $Q$. Thus, for the $\tau$-structures $\mathfrak{A}$ and $\mathfrak{B}$ defined above, it holds that $\mathfrak{A} \equiv_{\mathcal{L}_{\exists at^e}} \mathfrak{B}$. As it was shown that $\mathfrak{A} \not\equiv_{\mathcal{L}_{\exists d^e}} \mathfrak{B}$, it follows that $\mathcal{L}_{\exists d^e} \not\preccurlyeq_{EQ} \mathcal{L}_{\exists at^e}$. $\square$

**Lemma 8.0.5.** $\mathcal{L}_{\exists d} \preccurlyeq_{EQ} \mathcal{L}_{\exists at}$.

*Proof.* Suppose that for $\mathfrak{A}, \mathfrak{B} \in \mathcal{M}[\tau]$, for some $\tau$, it holds that $\mathfrak{A} \not\equiv_{\mathcal{L}_{\exists d}} \mathfrak{B}$. Then for some $\phi \in \mathcal{F}_{\exists d}[\tau]$, $\mathfrak{A} \Vdash_{\mathcal{L}_{\exists d}} \phi$ and $\mathfrak{B} \not\Vdash_{\mathcal{L}_{\exists d}} \phi$. Notice that $\phi$ can be either of the form (a) $Rx_1...x_n$, (b) $\exists x_1...x_n Rx_1...x_n$, (c) $\psi \vee \delta$ or (d) $\exists x_1...x_n(\psi \vee \delta)$. Given that $\mathcal{L}_{\exists d}$ formulas of the form (d) are equivalent to formulas of the form (c), we may suppose that all sub-formulas of $\phi$ are of the form (a), (b) or (c).

If $\phi$ is of the form (a) or (b), it is clear that $\mathfrak{A} \Vdash_{\mathcal{L}_{\exists at}} \phi$ and that $\mathfrak{B} \not\Vdash_{\mathcal{L}_{\exists at}} \phi$.

Now consider the case where $\phi$ is of the form (c) $\psi \vee \delta$:

- Given that $\mathfrak{A} \Vdash_{\mathcal{L}_{\exists d}} \psi \vee \delta$, it follows that $\mathfrak{A}$ satisfies at least one of them, call it $\xi$. If $\xi$ is of the form (a) or (b), then it follows that $\mathfrak{A} \Vdash_{\mathcal{L}_{\exists at}} \xi$. If $\xi$ is a disjunction, the process goes in the same way until it holds that $\mathfrak{A} \Vdash_{\mathcal{L}_{\exists d}} \xi'$, for a sub-formula $\xi'$ of $\xi$, having the form (a) or (b). In such a case, it also holds that $\mathfrak{A} \Vdash_{\mathcal{L}_{\exists at}} \xi'$.

- Given that $\mathfrak{B} \not\Vdash_{\mathcal{L}_{\exists d}} \psi \vee \delta$, then for all sub-formulas $\xi$ of $\psi \vee \delta$ having the form (a) or (b), it holds that $\mathfrak{B} \not\Vdash_{\mathcal{L}_{\exists d}} \xi$. Therefore, it also holds that $\mathfrak{B} \not\Vdash_{\mathcal{L}_{\exists at}} \xi$.

Then it holds that $\mathfrak{A} \not\equiv_{\mathcal{L}_{\exists at}} \mathfrak{B}$. $\square$

*8.0 Proof of remark 3.3.6*

> There are logics $\mathcal{L}_1$ and $\mathcal{L}_2$, with $\mathcal{L}_1 \preccurlyeq_{EQ} \mathcal{L}_2$, such that there is an $\mathcal{L}_1$-formula for which there is no $\mathcal{L}_2$-formula or set of formulas equivalent to it.

*Proof.* Let the logic $\mathcal{L}_{\infty\omega}$ be an extension of $\mathcal{L}_{\omega\omega}$, such that, if $\{\phi_1, \phi_2, ...\}$ is any set of first-order sentences, then $\bigwedge\{\phi_1, \phi_2, ....\}$ and $\bigvee\{\phi_1, \phi_2, ....\}$ are $\mathcal{L}_{\infty\omega}$-sentences. The semantics for such operators is the expected one. Let the logic $\mathcal{L}_{\infty G}$ be an extension of $\mathcal{L}_{\infty\omega}$, such that $\forall x_0 \exists x_1 \forall x_2 \exists x_3 ... \phi$ is a $\mathcal{L}_{\infty G}$-formula whenever $\phi$ is a $\mathcal{L}_{\infty\omega}$-formula. The quantifiers $\forall x_0 \exists x_1 \forall x_2 \exists x_3, ...$ are interpreted in terms of winning strategies.[20]

It is known that $\mathcal{L}_{\infty G} \approxeq_{EQ} \mathcal{L}_{\infty\omega}$ (BARWISE; FEFERMAN, 1985, p. 43). Nevertheless, it is not the case that every sentence of $\mathcal{L}_{\infty G}$ has counterparts in $\mathcal{L}_{\infty\omega}$ with an equivalent meaning. There is an $\mathcal{L}_{\infty G}$-sentence $\phi$ which is able to capture the common feature of having a well-ordered relation, that is shared by all well-ordered structures (KOLAITIS, 1985). On the other hand, there is no $\mathcal{L}_{\infty\omega}$-sentence equivalent to $\phi$ (LOPEZ-ESCOBAR, 1966), and no set of sentences either. Therefore, one can conclude that $\preccurlyeq_{EQ}$ does not satisfy property 3. $\square$

*8.0 Proof of remark 3.4.4*

> There are logics $\mathcal{L}_1$ and $\mathcal{L}_2$ with $\mathcal{L}_1 \preccurlyeq_{EQ^s} \mathcal{L}_2$, such that there is no fragment $\mathcal{L}_2^*$ of $\mathcal{L}_2$ for which it holds that $\mathcal{L}_1 \approxeq_{EQ^s} \mathcal{L}_2^*$.

The following counterexample is a variant of the one presented by Kocurek (2018) for remark 3.3.4.

*Proof.* Let $\mathcal{M} = \{\mathfrak{A}, \mathfrak{B}, \mathfrak{C}\}$ and $\mathcal{S} = \{p_1, p_2, p_3\}$. Define

[20] For more details, see (KOLAITIS, 1985).

- $Th_{\mathcal{L}_1}(\mathfrak{A}) = Th_{\mathcal{L}_1}(\mathfrak{B}) = \{p_1\}$,
  $Th_{\mathcal{L}_1}(\mathfrak{C}) = \{p_1, p_2, p_3\}$.
- $Th_{\mathcal{L}_2}(\mathfrak{A}) = \{p_1, p_2\}$, $Th_{\mathcal{L}_2}(\mathfrak{B}) = \{p_1, p_3\}$,
  $Th_{\mathcal{L}_2}(\mathfrak{C}) = \{p_1, p_2, p_3\}$.

For $\mathcal{C}, \mathcal{D} \subseteq \mathcal{M}$, recall that $\mathcal{C} \equiv_{\mathcal{L}} \mathcal{D}$ is defined as $\bigcap_{\mathfrak{A} \in \mathcal{C}} Th_{\mathcal{L}}(\mathfrak{A}) = \bigcap_{\mathfrak{B} \in \mathcal{D}} Th_{\mathcal{L}}(\mathfrak{B})$. It is easy to see that for all $\mathcal{C}, \mathcal{D} \subseteq \mathcal{M}$, if $\mathcal{C} \not\equiv_{\mathcal{L}_1} \mathcal{D}$, then $\mathcal{C} \not\equiv_{\mathcal{L}_2} \mathcal{D}$. Thus, it holds that $\mathcal{L}_1 \preccurlyeq_{EQ^s} \mathcal{L}_2$. However, there is no fragment $\mathcal{L}_2^*$ of $\mathcal{L}_2$ such that $\mathcal{L}_1 \approxeq_{EQ^s} \mathcal{L}_2^*$: Setting $\mathcal{L}_2^* = \mathcal{L}_2^{\restriction\{p_1\}}$, it holds that $\mathfrak{A} \not\equiv_{\mathcal{L}_1} \mathfrak{C}$, but $\mathfrak{A}^{\restriction\{p_1\}} \equiv_{\mathcal{L}_2^*} \mathfrak{C}^{\restriction\{p_1\}}$. For every other $\mathcal{S}^* \subset \mathcal{S}$, it will hold that $\mathcal{L}_2^* = \mathcal{L}_2^{\restriction\mathcal{S}^*}$ will distinguish pairs of structures not distinguished by $\mathcal{L}_1$.

□

*8.0 Proof of remark 3.4.5*

> There are logics $\mathcal{L}_1, \mathcal{L}_2$ with uniform extensions $\mathcal{L}_1^e, \mathcal{L}_2^e$, such that $\mathcal{L}_1 \preccurlyeq_{EQ^s} \mathcal{L}_2$, but $\mathcal{L}_1^e \not\preccurlyeq_{EQ^s} \mathcal{L}_2^e$.

*Proof.* Consider again the logics defined in the proof of remark 3.4.4. Let the 0-ary operator $\circledast$ have its truth conditions defined as follows, where $At_{\mathcal{L}}$ stands for the collection of atomic $\mathcal{L}$-sentences:

$$\mathfrak{A} \Vdash_{\mathcal{L}} \circledast \text{ iff } |\{p \in At_{\mathcal{L}} \mid \mathfrak{A} \Vdash_{\mathcal{L}} p\}| = 1$$

Let $\mathcal{L}_1^e$ and $\mathcal{L}_2^e$ be the uniform extensions of $\mathcal{L}_1$ and $\mathcal{L}_2$ with respect to $\circledast$.

It holds that $Th_{\mathcal{L}_1^e}(\mathfrak{A}) \cap Th_{\mathcal{L}_1^e}(\mathfrak{B}) = \{p_1, \circledast\}$ and that $Th_{\mathcal{L}_1^e}(\mathfrak{A}) \cap Th_{\mathcal{L}_1^e}(\mathfrak{B}) \cap Th_{\mathcal{L}_1^e}(\mathfrak{C}) = \{p_1\}$. Then,

$$Th_{\mathcal{L}_1^e}(\mathfrak{A}) \cap Th_{\mathcal{L}_1^e}(\mathfrak{B}) \cap Th_{\mathcal{L}_1^e}(\mathfrak{C}) \neq Th_{\mathcal{L}_1^e}(\mathfrak{A}) \cap Th_{\mathcal{L}_1^e}(\mathfrak{B}).$$

However,

$$Th_{\mathcal{L}_2^e}(\mathfrak{A}) \cap Th_{\mathcal{L}_2^e}(\mathfrak{B}) \cap Th_{\mathcal{L}_2^e}(\mathfrak{C}) = Th_{\mathcal{L}_2^e}(\mathfrak{A}) \cap Th_{\mathcal{L}_2^e}(\mathfrak{B}) = \{p_1\}.$$

□

*8.0 Proof of remark 3.4.6*

> There are logics $\mathcal{L}_1$ and $\mathcal{L}_2$, with $\mathcal{L}_1 \preccurlyeq_{EQ^s} \mathcal{L}_2$, such that there is an $\mathcal{L}_1$-formula for which there is no $\mathcal{L}_2$-formula or set of formulas equivalent to it.

*Proof.* Consider the proof of remark 3.4.4: for the formula $p_2$ in $\mathcal{L}_1$ it holds that $Mod_{\mathcal{L}_1}(p_2) = \{\mathfrak{C}\}$ and there is no $\mathcal{L}_2$-formula nor set of formulas equivalent to it. □

*8.0 Proof of remark 3.5.4*

> $DC_{\mathcal{L}_1} \subseteq PC_{\mathcal{L}_2}$ iff $PC_{\mathcal{L}_1} \subseteq PC_{\mathcal{L}_2}$.

*Proof.* From right to left it is immediate, as $DC_{\mathcal{L}} \subseteq PC_{\mathcal{L}}$.

From left to right: suppose that $DC_{\mathcal{L}_1} \subseteq PC_{\mathcal{L}_2}$. For $\mathcal{K} \subseteq \mathcal{M}[\tau]$, let $\mathcal{K} \in PC_{\mathcal{L}_1}$. Then, there is an $\mathcal{L}_1[\tau']$-formula $\psi$ with $\tau' \supseteq \tau$, such that $\mathcal{K} = \{\mathfrak{A}'^{\upharpoonright\tau} \mid \mathfrak{A}' \in Mod^{\tau'}_{\mathcal{L}_1}(\psi)\}$. By the hypothesis, there's a $\mathcal{L}_2[\tau'']$-formula $\delta$, for $\tau'' \supseteq \tau'$, such that $Mod^{\tau'}_{\mathcal{L}_1}(\psi) = \{\mathfrak{A}''^{\upharpoonright\tau'} \mid \mathfrak{A}'' \in Mod^{\tau''}_{\mathcal{L}_2}(\delta)\}$. Thus, $\mathcal{K} = \{\mathfrak{A}''^{\upharpoonright\tau} \mid \mathfrak{A}'' \in Mod^{\tau''}_{\mathcal{L}_2}(\delta)\}$ and $\mathcal{K} \in PC_{\mathcal{L}_2}$.

□

*8.0 Proof of remark 3.5.6*

> If $\mathcal{L}_1 \preccurlyeq_{PC} \mathcal{L}_2$, then there is a fragment $\mathcal{L}_2^*$ of $\mathcal{L}_2$ such that $\mathcal{L}_1 \approxeq_{PC} \mathcal{L}_2^*$.

The proof is analogous to the one for remark 3.1.5.

*8.0 Proof of remark 3.5.7*

> There are logics $\mathcal{L}_1, \mathcal{L}_2$ with uniform extensions $\mathcal{L}_1^e, \mathcal{L}_2^e$, such that $\mathcal{L}_1 \preccurlyeq_{PC} \mathcal{L}_2$, but $\mathcal{L}_1^e \not\preccurlyeq_{PC} \mathcal{L}_2^e$.

Consider the logic $\mathcal{L}(Q_0)^w$ from the proof of remark 3.2.5 and the following translation to first-order logic, $\mathcal{T} : \mathcal{F}_{\mathcal{L}(Q_0)^w} \longrightarrow \mathcal{F}_{\mathcal{L}_{\omega\omega}}$, where $R^\phi$ is a new binary relation symbol, and by $\mathcal{T}(\psi)^{x_i}_{x_j}$ it is meant the substitution of $x_i$ for the free occurrences of $x_j$ in $\mathcal{T}(\psi)$. Let $\phi \in \mathcal{L}(Q_0)^w$.

- $\phi$ is atomic, then $\mathcal{T}(\phi) = \phi$.
- $\phi = Q_0 x_1\, \psi$, then for new variables $x_2, x_3$ set $\mathcal{T}(\phi) = \forall x_1(\mathcal{T}(\psi) \to \neg R^\phi x_1 x_1) \wedge$ $\forall x_1 x_2 x_3(\mathcal{T}(\psi) \wedge \mathcal{T}(\psi)^{x_2}_{x_1} \wedge \mathcal{T}(\psi)^{x_3}_{x_1} \wedge R^\phi x_1 x_2 \wedge R^\phi x_2 x_3 \;\to\; R^\phi x_1 x_3) \wedge$ $\forall x_1(\mathcal{T}(\psi) \to \exists x_2(\mathcal{T}(\psi)^{x_2}_{x_1} \wedge R^\phi x_1 x_2))$.

**Lemma 8.0.6.** $\mathcal{L}(Q_0)^w \preccurlyeq_{PC} \mathcal{L}_{\omega\omega}$

*Proof.* Let $\tau' = \tau \cup \{R^{\phi_1}, R^{\phi_2}, ...\}$, where $R^{\phi_1}, R^{\phi_2}, ...$ are new binary relation symbols corresponding to each formula of $\mathcal{L}(Q_0)^w$. Let $Dom(\mathfrak{A})$ be the domain of $\mathfrak{A}$, and let $\mathfrak{A}^a_x$ be a model differing from $\mathfrak{A}$ at most in that $a \in Dom(\mathfrak{A})$ is assigned to the variable $x$. Then, for each $\phi \in \mathcal{L}(Q_0)^w[\tau]$, there is an $\mathcal{L}_{\omega\omega}[\tau']$-sentence $\psi$ such that $Mod^{\mathcal{T}}_{\mathcal{L}(Q_0)^w}(\phi) = \{\mathfrak{A}^{\restriction\tau} \mid \mathfrak{A} \in Mod^{\mathcal{T}'}_{\mathcal{L}_{\omega\omega}}(\psi)\}$. Take the main case where $\phi = Q_0 x_1\, \psi$. This formula is true in a model just in case $|\{a \in Dom(\mathfrak{A}) \mid \mathfrak{A}^a_{x_1} \Vdash_{\mathcal{L}(Q_0)^w} \psi\}|$ is infinite. The same happens with the formula $\mathcal{T}(Q_0 x_1\, \psi)$: it is true at $\mathfrak{A}$ in first-order logic if and only if $|\{a \in Dom(\mathfrak{A}) \mid \mathfrak{A}^a_{x_1} \Vdash_{\mathcal{L}_{\omega\omega}} \mathcal{T}(\psi)\}|$ is infinite. This is because there is a non-reflexive, transitive and serial[21] relation on $\{a \in Dom(\mathfrak{A}) \mid \mathfrak{A}^a_{x_1} \Vdash_{\mathcal{L}_{\omega\omega}} \mathcal{T}(\psi)\}$. □

Now there follows the proof of remark 3.5.7:

*Proof.* Let $\mathcal{L}(Q_0)^{we}$ and $\mathcal{L}_{\omega\omega}{}^e$ be the uniform extensions of both logics with respect to truth-functional negation. Given that $\mathcal{L}_{\omega\omega}$ already has it, $\mathcal{L}_{\omega\omega}{}^e \approx_{EC}$

[21] By "serial" we mean the property $\forall x_1 \exists x_2 R^\phi x_1 x_2$.

$\mathcal{L}_{\omega\omega}$. As the class of finite structures $\mathcal{F}$ is definable in $\mathcal{L}(Q_0)^{we}$, and given that $\mathcal{F}$ is not projective in $\mathcal{L}_{\omega\omega}$, it is not so in ${\mathcal{L}_{\omega\omega}}^{e}$, thus it holds that $\mathcal{L}(Q_0)^{we} \not\preccurlyeq_{PC} {\mathcal{L}_{\omega\omega}}^{e}$. □

*8.0 Proof of remark 4.0.2*

> $\mathcal{L}_1 \preccurlyeq_{EQ} \mathcal{L}_2$ does not imply that $\mathcal{L}_1 \preccurlyeq_{DC^{\Delta}} \mathcal{L}_2$.

*Proof.* Consider again the logics $\mathcal{L}_{\infty\omega}$ and $\mathcal{L}_{\infty G}$. Let $\mathcal{W}$ be the class of well-orderings and let $\phi$ be the $\mathcal{L}_{\infty G}[\tau]$-sentence such that $Mod^{\tau}_{\mathcal{L}_{\infty G}}(\phi) = \mathcal{W}$. We have that $\mathcal{L}_{\infty G} \approxeq_{EQ} \mathcal{L}_{\infty\omega}$, but by the result of Lopez-Escobar (1966), $\mathcal{W} \notin DC_{\mathcal{L}_{\infty\omega}}$, which implies that $\mathcal{W} \notin DC^{\Delta}_{\mathcal{L}_{\infty\omega}}$.

□

*8.0 Proof of remark 4.0.9*

> It is false that if $\mathcal{L}_1 \preccurlyeq_{PC} \mathcal{L}_2$, then $\mathcal{L}_1 \preccurlyeq_{EQ} \mathcal{L}_2$.

*Proof.* Let the signature $\tau$ to be $\{0, 1, +, \cdot, <\}$ and consider two $\tau$-structures $\mathfrak{A}$ and $\mathfrak{B}$ which are models of the first-order axioms for an ordered field, $\mathfrak{A}$ being Archimedean and $\mathfrak{B}$ not. We have that $\mathcal{L}(A) \preccurlyeq_{PC} \mathcal{L}(Q_0)$ and that $\mathfrak{A} \not\equiv_{\mathcal{L}(A)} \mathfrak{B}$ (SHAPIRO, 1991, p. 231). However, by a result of Cowles (1979) we have that $\mathfrak{A} \equiv_{\mathcal{L}(Q_0)} \mathfrak{B}$. □

*8.0 Proof of remark 4.0.10*

> It is false that if $\mathcal{L}_1 \preccurlyeq_{EQ} \mathcal{L}_2$, then $\mathcal{L}_1 \preccurlyeq_{PC} \mathcal{L}_2$.

*Proof.* Take the logics $\mathcal{L}_{\infty\omega}$ and $\mathcal{L}_{\infty G}$ mentioned above. As we saw, we have that $\mathcal{L}_{\infty G} \preccurlyeq_{EQ} \mathcal{L}_{\infty\omega}$, and that there is an $\mathcal{L}_{\infty G}[\tau]$-sentence $\phi$ such that the class of well-orderings $\mathcal{W} = Mod^{\tau}_{\mathcal{L}_{\infty G}}(\phi)$. If it were the case that $\mathcal{L}_{\infty G} \preccurlyeq_{PC} \mathcal{L}_{\infty\omega}$, then $\mathcal{W} \in PC_{\mathcal{L}_{\infty\omega}}$, but it is known that $\mathcal{W} \notin PC_{\mathcal{L}_{\infty\omega}}$ (BARWISE; FEFERMAN, 1985, p. 274). □

*8.0 Proof of remark 4.0.12*

> It is false that if $\mathcal{L}_1 \preccurlyeq_{PC} \mathcal{L}_2$, then $\mathcal{L}_1 \preccurlyeq_{DC^{\Delta}} \mathcal{L}_2$.

*Proof.* Consider $\mathcal{L}(A)$ and $\mathcal{L}(Q_0)$. It is the case that $\mathcal{L}(Q_0) \preccurlyeq_{PC} \mathcal{L}(A)$. Suppose that $\mathcal{L}(Q_0) \preccurlyeq_{DC^{\Delta}} \mathcal{L}(A)$. Take $\tau$ to be a monadic vocabulary and consider the $\mathcal{L}(Q_0)[\tau]$-sentence $\neg Q_0 c\, Pc$. Then by hypothesis, there is a set $\Delta$ of $\mathcal{L}(A)[\tau]$-sentences for which it holds that $Mod^{\tau}_{\mathcal{L}(Q_0)}(\neg Q_0 c\, Pc) = Mod^{\tau}_{\mathcal{L}(A)}(\Delta)$. However, by a result of Shapiro (1991, p. 232), $\mathcal{L}(A) \approxeq_{DC} \mathcal{L}_{\omega\omega}$ under monadic vocabulary. Thus each sentence in $\Delta$ would be equivalent to a first-order sentence. By compactness, there is no set of first-order sentences defining finiteness. □

*8.0 Proof of remark 4.0.13*

> It is false that if $\mathcal{L}_1 \preccurlyeq_{DC^{\Delta}} \mathcal{L}_2$, then $\mathcal{L}_1 \preccurlyeq_{PC} \mathcal{L}_2$.

*Proof.* Take the conjunction fragment of $\mathcal{L}_{\omega\omega}$, and the atomic fragment, to be represented as $\mathcal{L}^{conj}_{\omega\omega}$ and $\mathcal{L}^{atom}_{\omega\omega}$, respectively. It is clear that $\mathcal{L}^{conj}_{\omega\omega} \preccurlyeq_{DC^{\Delta}} \mathcal{L}^{atom}_{\omega\omega}$, but it is false that $\mathcal{L}^{conj}_{\omega\omega} \preccurlyeq_{PC} \mathcal{L}^{atom}_{\omega\omega}$. □

*8.0 Proof of remark 5.0.2*

> If $\mathcal{L}_1 \preccurlyeq_{PC} \mathcal{L}_2$, $\mathcal{L}_1$ has truth-functional negation and $\mathcal{L}_2$ has $\Delta$-interpolation, then $\mathcal{L}_1 \preccurlyeq_{EC} \mathcal{L}_2$.

*Proof.* Let the hypotheses of the remark be satisfied. Take $\mathcal{K} \in DC_{\mathcal{L}_1}$. Then, $\overline{\mathcal{K}} \in DC_{\mathcal{L}_1}$. Thus, $\mathcal{K} \in PC_{\mathcal{L}_2}$

and $\overline{\mathcal{K}} \in PC_{\mathcal{L}_2}$. By $\Delta$-interpolation, it follows that $\mathcal{K} \in DC_{\mathcal{L}_2}$. $\square$

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