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A Semi-Analytical Model for Deflection Analysis of Laminated Plates with the Newton-Kantorovich-Quadrature Method

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A semi-analytical approach for analysis of laminated plates with general boundary conditions under a general distribution of loads is developed. The non-linear equations are solved by the Newton-Kantorovich-Quadrature (NKQ) method which is a combination of well-known Newton-Kantorovich method and the Quadrature method. This method attempts to solve a sequence of linear integral equations. The convergence of the proposed method is compared with other semi-analytical methods. The validation of the method is explored through various numerical examples and the results compared with finite element method (FEM) and experimental tests. Good agreement between the NKQ model, FEM and experimental results are shown to validate the model.

Keywords: *composites, laminates, semi-analytical model, Newton-Kantorovich-Quadrature method*

1. Introduction

In the last half century, the use of composite materials has grown rapidly. These materials are ideal for structural applications that require high strength and low weight. They have good fatigue characteristics and are resistant to corrosion[1].

Understanding the mechanical behaviour of composite plates is essential for efficient and reliable design and for the safe use of structural elements. The complex behaviour of laminated plate structures normally need a non-linear model to describe them. Moreover, anisotropic and coupled material behaviour add more non-linearity to analysis. In general, there is no closed-form exact solution for the non-linear problem of composite plates for large deformations with arbitrary boundary conditions. The non-linear analysis of laminated plates has been the subject of many research projects. Also, various semi-analytical and numerical methods for the description and response of laminated plates have been developed. A comprehensive summary of the solutions for the geometrically non-linear analysis of isotropic and composite laminated plates was recently given by Khandan et al.[2]. The common analytical non-linear theories for laminated composites such as classical laminated plate theory[3-9] and first shear deformation plate theory[9-11] generally use the Rayleigh-Ritz method[12] or the Galerkin method[13-15]. The accuracy of these analytical models depends on the trial functions which they choose and they have to satisfy at least the kinematic boundary conditions. For certain boundary conditions and out-of-plane loadings these methods are so complicated and time consuming[16]. There are also some numerical methods to analyse laminated plates for the large deflection including the finite strip method[17,18]; the differential quadrature technique[19]; the method of lines[20]; Finite Element Method (FEM)[21-32]. Different meshless methods are also presented to solve the equations for laminated composite plates[33]. The development of element-free or meshless methods and their applications in the analysis of composite structures have been reviewed by Liew et al.[34] recently.

Due to numerous computations and the number of unknown variables, numerical methods are needed to solve problem of laminated plates. However, the analytical and semi-analytical non-linear methods are an essential tool that provides perception to the physical non-linear behaviour of the composite plate structure. Furthermore, these methods normally present fast and reliable solutions during the preliminary design phase. They also provide a means of validating the numerical methods and enable the development of new computational models. Therefore, the development of the semi-analytical methods has been growing rapidly[16].

The aim of this work is to achieve a semi-analytical approach for the non-linear model of laminated plates with arbitrary boundary conditions for general out-of-plane loadings. A Newton-Kantorovich-Quadrature (NKQ) method was proposed recently, by Saberi-Najafi and Heidari[35], for solving nonlinear integral equations in the Urysohn form. This method is expanded and used in this paper to present a semi-analytical model for laminated composite plates. Different extended Kantorovich methods (EKM) have been used by researchers to analyse the free-edge strength of composite laminates[36], the bending of thick laminated plates[37], buckling of symmetrically laminated composite plates[38] and laminated rectangular plates under general out-of-plane loading[16]. The multi-term extended Kantorovich method assumes a solution of the two-dimensional problem in the form of a sum of products of functions in one direction and functions in the other direction. As a result, the problem is reduced to a set of non-linear ordinary differential equations in the second direction.

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The solution of the resulting one-dimensional problem is then used as the assumed functions and the problem is solved again for the first direction. These iterations are repeated until convergence is completed. Unlike most of the other semi-analytical methods the accuracy of the solution is independent of the initial chosen functions. This initial function, even if it does not satisfy any of the boundary conditions[39,40], does not affect the accuracy of the solution. The EKM was applied by Soong[41] to the large deflection analysis of thin rectangular isotropic plates subjected to uniform loading. Some solutions[40,41] used only one-term for expansion yielding of isotropic plates. However, it is showed that one term formulation is not enough to predict the behaviour of anisotropic plates[40]. In this study, the NKQ is used to overcome these shortcomings and the model for out-of-plane loading as well. The accuracy and convergence of the method has been investigated through a comparison with other semi-analytical solutions and with finite element analysis (FEA) using a number of numerical examples in order to validate the model[16].

2. Governing Equations

2.1. *General composite equations*

The state of stress at a point in a general continuum can be represented by nine stress components σ_{ij} (i,j = 1, 2, 3) acting on the sides of an elemental cube with sides parallel to the axes of a reference coordinate system (Figure 1).

In the most general case the stress and strain components are related by the generalised Hook's law as follows[1]:

$$\sigma_{ij} = C_{ijkl}\epsilon_{kl} \quad (i, j, k, l = 1,2,3) \tag{1}$$

where C_{ijkl} is the stiffness components[42]. Thus in general, it would require 81 elastic constants to characterize a material fully. However, by considering the symmetry of the stress and strain tensors and the energy relations, it is proven that the stiffness matrices are symmetric. Thus the state of stress (strain) at a point can be described by six components of stress (strain), and the stress-strain equations are expressed in terms of 21 independent stiffness constants[42].

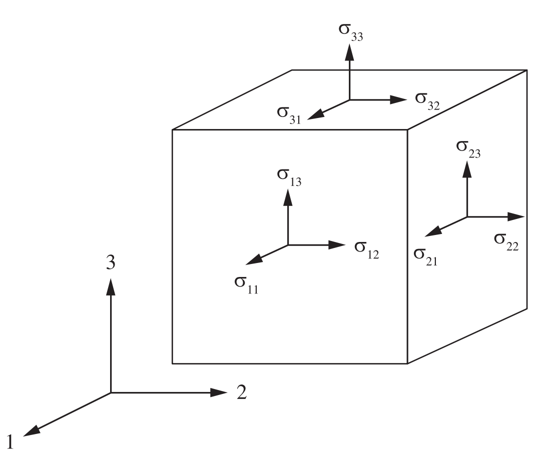

Figure 1. State of stress at a point in a general continuum[42].

2.1.1. In-plane stress

The classical laminate theory is used to analyze the mechanical behaviour of the composite laminate. It is assumed that plane stress components are taken as zero. The in-plane stress components are related to the strain components as:

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \tau_{xy}\end{bmatrix}_k = \begin{bmatrix}\bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16}\\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26}\\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66}\end{bmatrix}\begin{bmatrix}\epsilon_{xx}\\ \epsilon_{yy}\\ \gamma_{xy}\end{bmatrix} \tag{2}$$

where k is the lamina number, $\bar{Q}_{ij}$ are the off-axis stiffness components, which can be explained in terms of principal stiffness components, Q_{ij}, which are defined in Khandan et al.[1] and Daniel and Ishai[42].

Stress resultants, or forces per unit length of the cross section, are obtained as:

$$\begin{bmatrix}N_{xx}\\ N_{yy}\\ N_{xy}\end{bmatrix} = \int_{-h/2}^{h/2}\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \tau_{xy}\end{bmatrix}dz = 2\sum_{k=1}^{m} n_k t_0 \begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \tau_{xy}\end{bmatrix}_k \tag{3}$$

Here m is the number of distinct laminae, n_k is the number of plies in the kth lamina. Here, lamina is meant to be a group of plies with the same orientation angle. Substituting the stress–strain relation given by Equation 2 into Equation 3[1]:

$$\begin{bmatrix}N_{xx}\\ N_{yy}\\ N_{xy}\end{bmatrix} = \begin{bmatrix}A_{11} & A_{12} & A_{16}\\ A_{12} & A_{22} & A_{26}\\ A_{16} & A_{26} & A_{66}\end{bmatrix}\begin{bmatrix}\epsilon_{xx}\\ \epsilon_{yy}\\ \epsilon_{xy}\end{bmatrix} \tag{4}$$

where A_{ij}, components of extensional stiffness matrix, are given by:

$$A_{ij} = 2\sum_{k=1}^{m} n_k t_0 \left(\bar{Q}_{ij}\right)_k \tag{5}$$

Principal stress components can be obtained using the following transformation[29]:

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \tau_{12}\end{bmatrix} = \begin{bmatrix}\cos^2\theta_k & \sin^2\theta_k & 2\cos\theta_k\sin\theta_k\\ \sin^2\theta_k & \cos^2\theta_k & -2\cos\theta_k\sin\theta_k\\ -\cos\theta_k\sin\theta_k & \cos\theta_k\sin\theta_k & \cos^2\theta_k - \sin^2\theta_k\end{bmatrix}_k \begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \tau_{xy}\end{bmatrix} \tag{6}$$

2.1.2. Out-of-plane stress

In the classical laminate theory, it is assumed that straight lines normal to the middle surface remain straight and normal to that surface after deformation. These assumptions are not valid in the case of thicker laminates and laminates with low stiffness central plies undergoing significant transverse shear deformations. In the following, referred to as first-order shear deformation laminate plate theory, the assumption of normality of straight lines is removed. On the other hand straight lines normal to the middle surface remain straight but not normal to that surface after deformation[1,9]. Q_{ij} can be found in Khandan et al.[1] and Daniel and Ishai[42]:

$$\begin{bmatrix}N_{xx}\\N_{yy}\\N_{xy}\end{bmatrix}=\int_{-h/2}^{h/2}\begin{bmatrix}\sigma_{xx}\\\sigma_{yy}\\\tau_{xy}\end{bmatrix}dz$$

$$\begin{bmatrix}M_{xx}\\M_{yy}\\M_{xy}\end{bmatrix}=\int_{-h/2}^{h/2}\begin{bmatrix}\sigma_{xx}\\\sigma_{yy}\\\tau_{xy}\end{bmatrix}zdz \qquad (7)$$

$$\begin{bmatrix}V_q\\V_r\end{bmatrix}=\int_{-h/2}^{h/2}\begin{bmatrix}\tau_{xz}\\\tau_{yz}\end{bmatrix}dz$$

$$\begin{bmatrix}N_{xx}\\N_{yy}\\N_{xy}\\M_{xx}\\M_{yy}\\M_{xy}\\V_q\\V_r\end{bmatrix}=\begin{bmatrix}N_1\\N_2\\N_{12}\\M_1\\M_2\\M_{12}\\V_1\\V_2\end{bmatrix}=\begin{bmatrix}A_{11}&A_{12}&A_{13}&B_{11}&B_{12}&B_{13}&0&0\\A_{12}&A_{22}&A_{23}&B_{12}&B_{22}&B_{23}&0&0\\A_{13}&A_{23}&A_{33}&B_{13}&B_{23}&B_{33}&0&0\\B_{11}&B_{12}&B_{13}&D_{11}&D_{12}&D_{13}&0&0\\B_{12}&B_{22}&B_{23}&D_{12}&D_{22}&D_{23}&0&0\\B_{13}&B_{23}&B_{33}&D_{13}&D_{23}&D_{33}&0&0\\0&0&0&0&0&0&E_{11}&E_{12}\\0&0&0&0&0&0&E_{12}&E_{22}\end{bmatrix}\begin{bmatrix}\epsilon_1\\\epsilon_2\\\gamma_{12}\\k_1\\k_2\\k_{12}\\\gamma_{13}\\\gamma_{23}\end{bmatrix} \qquad (8)$$

where the components of this section stiffness matrix are given by:

$$\left(A_{ij},B_{ij},D_{ij}\right)=\int_{-h/2}^{h/2}\bar{Q}_{ij}^{m}\left(1,z,z^2\right)dz \quad (i,j=1,2,3)$$
$$E_{ij}=\int_{-h/2}^{h/2}\bar{Q}_{\alpha\beta}^{m}\,k_i k_j dz \quad (i,j=1,2\;and\;\alpha,\beta=i+4,j+4) \qquad (9)$$

The out-of-plane boundary conditions include three cases: simply supported (S), clamped (C), and free (F) edges. The four possible in-plane restraints along the plate edges are shown in Figure 2, and they are denoted by a subscript index[16].

2.2. *Basic NKQ equations*

The nonlinear integral equation in the Urysohn form is defined as[35]:

$$y(x)=f(x)+\int_{\Omega}K(x,t,y(t))dt \qquad a\le x\le b \qquad (10)$$

If $\Omega=(a,x)$, it is named a nonlinear Volterra integral equation and if $\Omega=(a,b)$, it is named the nonlinear Fredholm integral equation. To approximate the right-hand integral in Equation 10, the usual quadrature methods similar to the ones used to approximate the linear integral equations that lead to the following nonlinear systems for Fredholm and Volterra equations are used, respectively. For further information on quadrature methods in this respect, see references 35, 43-49.

$$y(x_i)=f(x_i)+\sum_{j-0}^{n}w_jK\left(x_i,x_j,y(x_j)\right) \quad i=0,1,2,\ldots,n \qquad (11)$$

$$\begin{cases}y(x_0)=f(x_0)\\ y(x_i)=f(x_i)+\sum_{j-0}^{i}w_{ij}K\left(x_i,x_j,y(x_j)\right) \quad i=0,1,2,\ldots,n\end{cases} \qquad (12)$$

where $w_{ij}s$ and w_js are weights of the integration formula.

In the Newton-Kantorovich method, an initial solution for $y(x)$ is considered. The following iteration method is used to solve the following sequence of linear integral equations instead of a nonlinear integral equation. For further information on the Newton-Kantorovich method, see Saberi-Nadjafi and Heidari[35], Appell et al.[50] and Polyanin and Manzhirov[51].

$$\begin{cases}y_k(x)=y_{k-1}(x)+\phi_{k-1}(x)\\ \phi_{k-1}(x)=\varepsilon_{k-1}(x)+\int_{\Omega}K'_y\left(x,t,y_{k-1}(t)\right)\phi_{k-1}(t)dt\\ \varepsilon_{k-1}(x)=f(x)-y_{k-1}(x)+\int_{\Omega}K\left(x,t,y_{k-1}(t)\right)dt\end{cases} \qquad (13)$$

where $K'_y(x,t,y)=\dfrac{\partial}{\partial y}K'(x,t,y)$.

In NKQ method which is used in this paper, Equations 11-13 are combined by Saberi-Najafi and Heidari[35] to solve the nonlinear integral equations.

3. Application of NKQ

As it is mentioned the general form of the nonlinear Volterra integral equations of the Urysohn form is:

$$y(x)=f(x)+\int_{a}^{x}K(x,t,y(t))dt \qquad a\le x\le b \qquad (14)$$

By considering the Equations 11-13 and by integrating $\phi_{k-1}(x)$ with $y_k(x)-y_{k-1}(x)$:

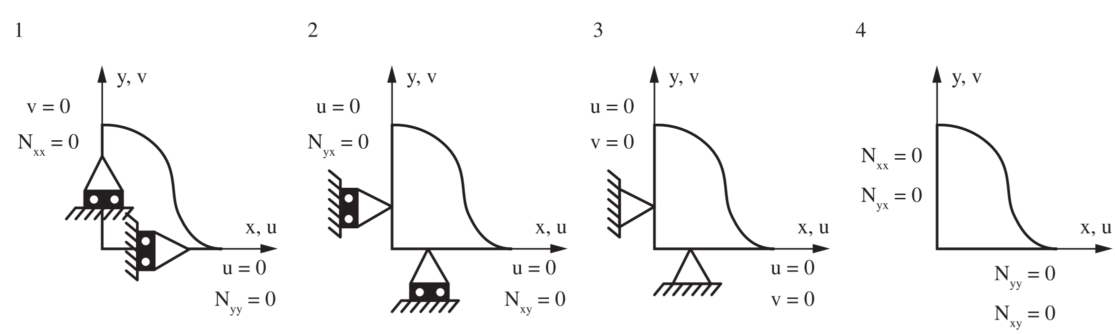

Figure 2. Boundary conditions[16].

$$\begin{cases} y_k\left(x_0\right) = f\left(x_0\right) \\ y\left(x_i\right) = f\left(x_i\right) + \sum\limits_{j-0}^{i} w_{ij} K\left(x_i, x_j, y_{k-1}\left(x_j\right)\right) + \\ \sum\limits_{j-0}^{i} w_{ij} K\left(x_i, x_j, y_{k-1}\left(x_j\right)\right)\left[y_k\left(x_j\right) - y_{k-1}\left(x_j\right)\right] \quad i = 1, 2, ..., n \end{cases} \tag{15}$$

Consider:

$$\left(F^{(k-1)}\right)_{i+1} = \begin{cases} f\left(x_0\right) \quad i = 0 \\ f\left(x_i\right) + \sum\limits_{j-0}^{i} w_{ij} K\left(x_i, x_j, y_{k-1}\left(x_j\right)\right) - \sum\limits_{j-0}^{i} w_{ij} K_y^{'} \\ \left(x_i, x_j, y_{k-1}\left(x_j\right)\right) y_{k-1}\left(x_j\right) \quad i = 1, 2, ..., n \end{cases} \tag{16}$$

$$\left(A^{(k-1)}\right)_{i+1 \ j+1} = \begin{cases} w_{ij} K_y^{'}\left(x_i, x_j, y_{k-1}\left(x_j\right)\right) \quad i = 1, 2, ..., n \\ \quad j = 0, 1, ..., i \\ 0 \qquad otherwise \end{cases} \tag{17}$$

$$(Y^{(k)})_{i+1} = y_k\,(x_i) \qquad\qquad i = 0,1,2,...,n \tag{18}$$

This equation can be solved by considering an initial solution $y_0(x)$ and constructing the $Y^{(0)}$, $A^{(0)}$, $F^{(0)}$ and also using the following repetition sequence (for further details see Saberi-Nadjafi and Heidari[35]):

$$(I - A^{(k-1)})Y^{(k)} = F^{(k-1)} \qquad\qquad k = 1,2,...,n \tag{19}$$

On the other hand, by considering an initial solution $y_0(x)$, $(Y^{(0)})_i$ would be $y_0(x_i)$ and by using Equation 16 and 17 $F^{(0)}$, $A^{(0)}$ are obtained respectively. Then by solving the system $(I - A^{(0)})Y^{(1)} = F^{(0)}$, $Y^{(1)}$ is obtained. By repeating this procedure and next using Equation 19, the values of $Y^{(1)}, Y^{(2)}, Y^{(3)}, ...Y^{(m)}$ are calculated for m N m is a constant value which can be increased for higher n. Depending on n an approximate solution for Equation 10 is presented. Noticeably, by increasing m, the solution tends to be more accurate with respect to n. However it is shown that to achieve good results it is not necessary to increase m significantly.

The general basic equations for laminated composite plate are[9]:

$$-\left(\frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y}\right) + I_0 \frac{\partial^2 u_0}{\partial t^2} + I_1 \frac{\partial^2 \phi_x}{\partial t^2} = 0 \tag{20}$$

$$-\left(\frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y}\right) + I_0 \frac{\partial^2 v_0}{\partial t^2} + I_1 \frac{\partial^2 \phi_y}{\partial t^2} = 0 \tag{21}$$

$$-\left(\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y}\right) - N\left(u_0, v_0, w_0\right) - q + I_0 \frac{\partial^2 w_0}{\partial t^2} = 0 \tag{22}$$

$$-\left(\frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{xv}}{\partial y}\right) + Q_x + I_2 \frac{\partial^2 \phi_x}{\partial t^2} + I_1 \frac{\partial^2 u_0}{\partial t^2} = 0 \tag{23}$$

$$-\left(\frac{\partial M_{xy}}{\partial x} + \frac{\partial M_{yy}}{\partial y}\right) + Q_y + I_2 \frac{\partial^2 \phi_y}{\partial t^2} + I_1 \frac{\partial^2 v_0}{\partial t^2} = 0 \tag{24}$$

where:

$$N\left(u_0, v_0, w_0\right) = \frac{\partial}{\partial x}\left(N_{xx} \frac{\partial w_0}{\partial x} + N_{xy} \frac{\partial w_0}{\partial y}\right) + \frac{\partial}{\partial y}\left(N_{xy} \frac{\partial w_0}{\partial x} + N_{yy} \frac{\partial w_0}{\partial y}\right) \tag{25}$$

$$\begin{Bmatrix} I_0 \\ I_1 \\ I_2 \end{Bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{Bmatrix} 1 \\ z \\ z^2 \end{Bmatrix} \rho_0 dz \tag{26}$$

And Equations 7-9 are simplified to[9]:

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2 \\ \frac{\partial v_0}{\partial y} + \frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2 \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y} \end{Bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial \phi_x}{\partial x} \\ \frac{\partial \phi_y}{\partial y} \\ \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \end{Bmatrix} \tag{27}$$

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial u_0}{\partial x} + \frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2 \\ \frac{\partial v_0}{\partial y} + \frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2 \\ \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x} + \frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y} \end{Bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \frac{\partial \phi_x}{\partial x} \\ \frac{\partial \phi_y}{\partial y} \\ \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x} \end{Bmatrix} \tag{28}$$

$$\begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = K_s \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} \begin{Bmatrix} \frac{\partial w_0}{\partial y} + \phi_y \\ \frac{\partial w_0}{\partial x} + \phi_x \end{Bmatrix} \tag{29}$$

where

$$\left(A_{ij},B_{ij},D_{ij}\right)=\int_{-h/2}^{h/2}\bar{Q}_{ij}\left(1,z,z^2\right)dz \qquad i,j=1,2,...,6 \tag{30}$$

where $\bar{Q}_{ij}\left(i,j=1,2,...6\right)$ are the transformed plane-stress stiffness coefficients.

By adopting the variation principle of virtual work and applying the NKQ the Equations 31-35 are derived:

$$\begin{aligned}
&\int_{\Omega^e}\left(\frac{\partial\delta u_0}{\partial x}\left\{A_{11}\left[\frac{\partial u_0}{\partial x}+\frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2\right]+A_{12}\left[\frac{\partial v_0}{\partial y}+\frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2\right]\right.\right.\\
&\left.+A_{16}\left[\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}+\frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y}\right]+B_{11}\frac{\partial\phi_0}{\partial x}+B_{12}\frac{\partial\phi_0}{\partial y}+B_{16}\left(\frac{\partial\phi_0}{\partial y}+\frac{\partial\phi_0}{\partial x}\right)\right\}\\
&+\frac{\partial\delta u_0}{\partial x}\left\{A_{16}\left[\frac{\partial u_0}{\partial x}+\frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2\right]+A_{26}\left[\frac{\partial v_0}{\partial y}+\frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2\right]\right.\\
&\left.\left.+\left\{A_{66}\left[\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}+\frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y}\right]+B_{16}\frac{\partial\phi_x}{\partial x}+B_{26}\frac{\partial\phi_y}{\partial y}+B_{66}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)\right\}\right\}\right)dxdy\\
&-\oint_{\Gamma^e}N_n\delta u_{0n}ds+\int_{\Omega^e}\left(I_0\frac{\partial^2 u_0}{\partial t^2}+I_1\frac{\partial^2\phi_x}{\partial t^2}\right)\delta u_0dxdy=0
\end{aligned}\tag{31}$$

$$\begin{aligned}
&\int_{\Omega^e}\left(\frac{\partial\delta v_0}{\partial y}\left\{A_{12}\left[\frac{\partial u_0}{\partial x}+\frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2\right]+A_{22}\left[\frac{\partial v_0}{\partial y}+\frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2\right]\right.\right.\\
&\left.+A_{26}\left[\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}+\frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y}\right]+B_{12}\frac{\partial\phi_0}{\partial x}+B_{22}\frac{\partial\phi_y}{\partial y}+B_{26}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)\right\}\\
&+\frac{\partial\delta v_0}{\partial x}\left\{A_{16}\left[\frac{\partial u_0}{\partial x}+\frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2\right]+A_{26}\left[\frac{\partial v_0}{\partial y}+\frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2\right]\right.\\
&\left.\left.+\left\{A_{66}\left[\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}+\frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y}\right]+B_{16}\frac{\partial\phi_x}{\partial x}+B_{26}\frac{\partial\phi_y}{\partial y}+B_{66}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)\right\}\right\}\right)dxdy\\
&-\oint_{\Gamma^e}N_s\delta u_{0s}ds+\int_{\Omega^e}\left(I_0\frac{\partial^2 v_0}{\partial t^2}+I_1\frac{\partial^2\phi_y}{\partial t^2}\right)\delta v_0dxdy=0
\end{aligned}\tag{32}$$

$$\begin{aligned}
&K_s\int_{\Omega^e}\left\{\frac{\partial\delta w_0}{\partial x}\left(A_{55}\left[\frac{\partial w_0}{\partial x}+\phi_x\right]+A_{45}\left[\frac{\partial w_0}{\partial y}+\phi_y\right]\right)+\frac{\partial\delta w_0}{\partial y}\left(A_{45}\left[\frac{\partial w_0}{\partial x}+\phi_x\right]\right)+A_{44}\left[\frac{\partial w_0}{\partial y}+\phi_y\right]\right\}dxdy\\
&\int_{\Omega^e}\left\{\frac{\partial\delta w_0}{\partial x}\left[\frac{\partial\delta w_0}{\partial x}\left(A_{11}\left[\frac{\partial u_0}{\partial x}+\frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2\right]+A_{12}\left[\frac{\partial v_0}{\partial y}+\frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2\right]\right.\right.\right.\\
&\left.+A_{16}\left[\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}+\frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y}\right]+B_{11}\frac{\partial\phi_x}{\partial x}+B_{12}\frac{\partial\phi_0}{\partial y}+B_{16}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)\right)\\
&+\frac{\partial\delta w_0}{\partial y}\left\{A_{16}\left[\frac{\partial u_0}{\partial x}+\frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2\right]+A_{26}\left[\frac{\partial v_0}{\partial y}+\frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2\right]\right.\\
&\left.\left.+A_{66}\left[\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}+\frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y}\right]+B_{16}\frac{\partial\phi_x}{\partial x}+B_{26}\frac{\partial\phi_y}{\partial y}+B_{66}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)\right\}\right]\\
&\frac{\partial\delta w_0}{\partial y}\left[\frac{\partial\delta w_0}{\partial y}\left(A_{12}\left[\frac{\partial u_0}{\partial x}+\frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2\right]+A_{22}\left[\frac{\partial v_0}{\partial y}+\frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2\right]\right.\right.\\
&\left.+A_{26}\left[\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}+\frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y}\right]+B_{12}\frac{\partial\phi_x}{\partial x}+B_{22}\frac{\partial\phi_y}{\partial y}+B_{26}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)\right)\\
&+\frac{\partial\delta w_0}{\partial x}\left\{A_{16}\left[\frac{\partial u_0}{\partial x}+\frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2\right]+A_{26}\left[\frac{\partial v_0}{\partial y}+\frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2\right]+A_{66}\left[\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}+\frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y}\right]\right.\\
&\left.\left.+B_{16}\frac{\partial\phi_x}{\partial x}+B_{26}\frac{\partial\phi_y}{\partial y}+B_{66}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)\right\}\right]-\oint_{\Gamma^e}V_n\delta w_0ds+\int_{\Omega^e}I_0\frac{\partial^2 w_0}{\partial t^2}\delta w_0dxdy=0
\end{aligned}\tag{33}$$

$$\begin{aligned}
&\int_{\Omega^e}\left(\frac{\partial\delta\phi_x}{\partial x}\left\{B_{11}\left[\frac{\partial u_0}{\partial x}+\frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2\right]+B_{12}\left[\frac{\partial v_0}{\partial y}+\frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2\right]\right.\right.\\
&\left.+B_{16}\left[\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}+\frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y}\right]+D_{11}\frac{\partial\phi_x}{\partial x}+D_{12}\frac{\partial\phi_y}{\partial y}+D_{16}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)\right\}\\
&+\frac{\partial\delta\phi_x}{\partial y}\left\{B_{16}\left[\frac{\partial u_0}{\partial x}+\frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2\right]+B_{26}\left[\frac{\partial v_0}{\partial y}+\frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2\right]\right.\\
&\left.+B_{66}\left[\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}+\frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y}\right]+D_{16}\frac{\partial\phi_x}{\partial x}+D_{26}\frac{\partial\phi_y}{\partial y}+D_{66}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)\right\}\\
&\left.+K_s\delta\phi_x\left(A_{55}\left[\frac{\partial w_0}{\partial x}+\phi_x\right]+A_{45}\left[\frac{\partial w_0}{\partial y}+\phi_y\right]\right)\right)dxdy\\
&-\oint_{\Gamma^e}M_n\delta\phi_n ds+\int_{\Omega^e}\left(I_2\frac{\partial^2\phi_x}{\partial t^2}+I_1\frac{\partial^2 u_0}{\partial t^2}\right)\delta\phi_x dxdy=0
\end{aligned}\tag{34}$$

$$\begin{aligned}
&\int_{\Omega^e}\left(\frac{\partial\delta\phi_y}{\partial y}\left\{B_{12}\left[\frac{\partial u_0}{\partial x}+\frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2\right]+B_{22}\left[\frac{\partial v_0}{\partial y}+\frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2\right]\right.\right.\\
&\left.+B_{26}\left[\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}+\frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y}\right]+D_{12}\frac{\partial\phi_x}{\partial x}+D_{22}\frac{\partial\phi_y}{\partial y}+D_{26}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)\right\}\\
&+\frac{\partial\delta\phi_y}{\partial x}\left\{B_{16}\left[\frac{\partial u_0}{\partial x}+\frac{1}{2}\left(\frac{\partial w_0}{\partial x}\right)^2\right]+B_{26}\left[\frac{\partial v_0}{\partial y}+\frac{1}{2}\left(\frac{\partial w_0}{\partial y}\right)^2\right]\right.\\
&\left.+B_{66}\left[\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}+\frac{\partial w_0}{\partial x}\frac{\partial w_0}{\partial y}\right]+D_{16}\frac{\partial\phi_x}{\partial x}+D_{26}\frac{\partial\phi_y}{\partial y}+D_{66}\left(\frac{\partial\phi_x}{\partial y}+\frac{\partial\phi_y}{\partial x}\right)\right\}\\
&\left.+K_s\delta\phi_y\left(A_{45}\left[\frac{\partial w_0}{\partial x}+\phi_x\right]+A_{44}\left[\frac{\partial w_0}{\partial y}+\phi_y\right]\right)\right)dxdy\\
&-\oint_{\Gamma^e}M_s\delta\phi_s ds+\int_{\Omega^e}\left(I_2\frac{\partial^2\phi_y}{\partial t^2}+I_1\frac{\partial^2 v_0}{\partial t^2}\right)\delta\phi_y dxdy=0
\end{aligned}\tag{35}$$

where the secondary variables of the formulation are:

$$\begin{aligned}&N_n\equiv N_{xx}n_x+N_{xy}n_y\\&N_s\equiv N_{xy}n_x+N_{yy}n_y\end{aligned}\tag{36}$$

$$\begin{aligned}&M_n\equiv M_{xx}n_x+M_{xy}n_y\\&M_s\equiv M_{xy}n_x+M_{yy}n_y\end{aligned}\tag{37}$$

$$\begin{aligned}V_n=&\left(Q_x+N_{xx}\frac{\partial w_0}{\partial x}+N_{xy}\frac{\partial w_0}{\partial y}\right)n_x\\&+\left(Q_y+N_{xy}\frac{\partial w_0}{\partial x}+N_{yy}\frac{\partial w_0}{\partial y}\right)n_y\end{aligned}\tag{38}$$

Equations 31-35 can be estimated by considering them as Urysohn form and an initial solution $y_0(x)$ and consequently the $Y^{(0)}$, $A^{(0)}$, $F^{(0)}$ and also repeating the sequences for Equation 19.

4. Verification Study

In order to verify the NKQ method a number of numerical examples are solved and compared with previous research. In the first example, a four layer glass/epoxy laminate $[0°, 90°]_s$ with ply properties[52] is studied:

$$\begin{aligned}&E_1=43.5\ Gpa,\quad E_2=E_3=11.5\ Gpa,\quad \nu_{12}=\nu_{13}=.27,\\&\nu_{23}=.4\quad G_{12}=G_{13}=3.45\ Gpa\quad G_{23}=4.12\ Gpa\end{aligned}\tag{39}$$

The plate is a square with 0.5 m length and 0.01 m thickness. A trigonometric function is chosen for initial guess ($y_0(x)=\sin(\varpi x/l)$). It is shown in Figure 3 that it is converged after five iterations. Aghdam and Falahatgar[37] used an Extended Kantorovich method EKM for analysing the thick composite plate. By choosing a trigonometric function as an initial guess, the model converges after 4 iterations.

In Table 1 the number of iterations, which are needed for convergence, for three different initial functions are shown. As it is mentioned earlier the initial guess does not have to satisfy the boundary conditions, so any initial function can be selected. Furthermore, as it is shown in Table 1 that the NKQ method is relatively quick and does not significantly depend on the initial value. The main advantage of this method, compared to EKM, is that it can be used for more complicated cases such as out-of-plane loading and different boundary conditions.

In the next study the material properties are:

$E_1 = 215Gpa \quad E_2 = E_3 = 23.6Gpa, \quad v_{12} = v_{13} = .17,$

$v_{23} = .28 \quad G_{12} = G_{13} = 5.4Gpa \quad G_{23} = 2.1Gpa$

The plate is a square and each length is 0.25 m, the thickness is 0.006 m and the lay ups are $[0°, 90°, 0°]_s$. In Table 2 the relative error between the NKQ method and FEM for different numbers of iterations are shown for a plate clamped on one side (C) and free (F) on the other three sides (CFFF). This example was then repeated for SSSS and CCCC boundary conditions and the results are shown in Tables 3, 4 respectively.

In Table 5, the dimensionless deflection at the centre of plate is compared between FEM, multi-term extended Kantorovich method (MTEKM) and NKQ method under different levels of load (patch out-of-plane load)[16]. As shown, the NKQ results generally show a reasonable agreement with FEM. Semi-analytical models (MTEKM and NKQ)

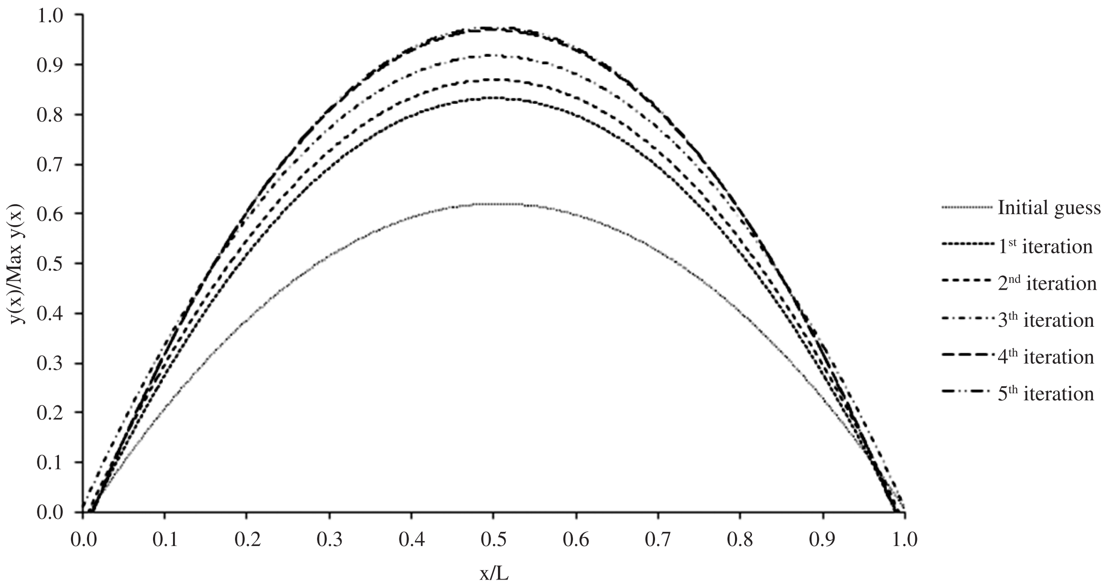

Figure 3. Convergence of NKQ method.

Table 1. Number of iteration for convergence of NKQ and EKM.

Initial guess	Number of iteration in order to converge for NKQ	Number of iteration in order to converge for EKM[37]
Trigonometric function $y_0(x) = \sin(x)$	5	4
Polynomial function $y_0(x) = 1 + x + x^2$	4	NA
Exponential function $y_0(x) = e(x)$	4	NA

Table 2. Relative error for CFFF boundary condition.

Number of iterations	1	2	3	4	5	6	7	8	9	10
Relative error (%) for u	29.4	8.3	3.4	1.9	1.1	0.6	0.3	0.1	0.0	0.0
Relative error (%) for v	34.5	9.2	5.1	2.1	1.2	0.8	0.3	0.1	0.0	0.0
Relative error (%) for w	53.4	11.3	5.4	2.1	1.2	0.8	0.4	0.2	0.0	0.0

Table 3. Relative error for SSSS boundary condition.

Number of iterations	1	2	3	4	5	6	7	8	9	10
Relative error (%) for u	25.4	9.1	3.9	1.9	1.2	0.7	0.4	0.1	0.0	0.0
Relative error (%) for v	47.2	11.0	5.6	2.5	1.5	1.0	0.5	0.2	0.1	0.0
Relative error (%) for w	76.6	21.2	7.0	2.9	1.5	0.9	0.5	0.2	0.1	0.1

Table 4. Relative error for CCCC boundary condition.

Number of iterations	1	2	3	4	5	6	7	8	9	10
Relative error (%) for u	92.2	30.4	14.1	8.3	5.3	3.2	1.8	1.0	0.4	0.1
Relative error (%) for v	63.8	19.2	9.2	5.1	3.7	1.9	0.9	0.4	0.1	0.0
Relative error (%) for w	77.1	24.5	11.1	6.6	4.9	2.2	1.3	0.7	0.3	0.1

Table 5. Dimensionless *W/h* for CFCC square laminated plate under different loads.

Q	MTEKM (W)[16]	NKQ (W)	ABAQUS	%MTEKM Error[16]	%NKQ Error
2488	.746	.741	.763	2.23	2.88
4975	1.092	1.090	1.115	2.05	2.24
7463	1.328	1.325	1.335	2.02	1.50
9950	1.512	1.559	1.541	1.81	1.16
12438	1.667	1.667	1.695	1.66	1.66
14925	1.800	1.802	1.827	1.54	1.36
17413	1.919	1.913	1.947	1.43	1.74
19900	2.027	2.025	2.053	1.33	1.36

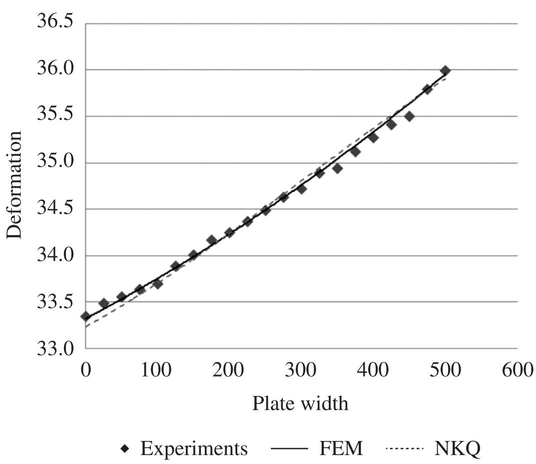

Figure 4. Deformation at free edge of anisotropic laminated plate.

illustrate less than 3% error. The structure is a square plate with CFCC boundary conditions. The angle-ply laminated plate has four symmetric layers [45, –45].

In the next example a square laminated plate with a length of 0.5 m under uniform loading is considered. The plate is clamped at one side and the deformation at the other edge is measured. Because of the different lay-ups (anisotropic) and out-of-plane loading there is an induced twist at the free edge of the plate. The aim of this example is to find out if the NKQ model can estimate this induced twist.

The results are shown for an experiment, FEM and NKQ method in Figure 4. Carbon fibre is used for all laminated experimental tests and the size of plate is 500*500 (mm). The experimental test results which are shown in Figure 4 are the average results of six identical plates under the constant load.

5. Conclusion

The constitutive equations of the laminated composite plates are non-linear. The semi-analytical non-linear methods are an essential tool that provides perception to the physical non-linear behaviour of the composite plate structure, present fast and reliable solutions during the preliminary design phase and also provide a means of validations the numerical methods and enable the development of new computational models. In this paper a semi-analytical approach for the analysis of laminated plates with general boundary conditions and distribution of loads is proposed. The non-linear equations are solved by Newton-Kantorovich-Quadrature (NKQ) method. This method breaks down the laminate composite plate equations into a series of sequential equations and attempts to solve iterative linear integral equations. The convergence of the proposed method is compared with other semi-analytical methods (EKM and MTEKM). Various numerical examples with different boundary conditions and loadings are studied. Good agreement between the NKQ model, FEM and experimental results are shown to validate the model.

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