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ARTICLES

SOME NOVEL FIXED POINT RESULTS FOR (Ω,Δ)-WEAK CONTRACTION CONDITION IN COMPLETE FUZZY METRIC SPACES

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ABSTRACT. In the present article, some fixed point theorems are investigated for two pairs of weakly compatible maps through (Ω,Δ)-type weak contractive maps in the framework of fuzzy metric spaces. The results studied in this workpiece are generalizations of some recent results existing in literature. Also, some illustrative examples are presented in last section to check the authenticity of our results.

Keywords: fuzzy metric space, fixed points, (Ω,Δ)-weak contractive map, altering distance function.

1 INTRODUCTION

Contraction principle given by Banach (1922) is the most eminent result in the era of metrical fixed point theory. Though this principle requires the continuity of the mapping, still it works as the back-bone even for the recent results in different metric spaces. An open question on the continuity of the mapping in Banach principle is answered by many authors. In 1968, Kannan (1968) settled this problem in a robust way by introducing the following inequality:

$$\hat{d}(T\rho,T\sigma)\leq\beta[\hat{d}(\rho,T\rho)+\hat{d}(\sigma,T\sigma)] \quad \text{for all } \rho,\sigma\in U \text{ and } \beta\in\left(0,\frac{1}{2}\right). \tag{1}$$

Later on, Rakotch (1962), Boyd & Wong (1969) extended the contraction inequality due to Banach (1922) by characterizing a control function stated below:

$$\hat{d}(T\rho,T\sigma)\leq\alpha(\rho,\sigma)\hat{d}(\rho,\sigma) \quad \text{for all } \rho,\sigma\in U \text{ and } \alpha:[0,\infty]\to[0,1]$$

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and

$$\hat{d}(T\rho,T\sigma) \leq \phi(\hat{d}(\rho,\sigma)) \quad \text{for all } \rho,\sigma \in U,$$

where $\phi : [0,\infty] \to [0,\infty]$ is a non-decreasing continuous function such that $\phi(t)$ vanishes at $t = 0$.

On the other hand, Alber & Guerre-Delabriere (1997) introduced a modified contractive condition in Hilbert spaces which was further elaborated by Rhoades (2001) as follows:

If a mapping $T : U \to U$ satisfies the following condition:

$$\hat{d}(T\rho,T\sigma) \leq \hat{d}(\rho,\sigma) - \Delta(\hat{d}(\rho,\sigma)) \quad \text{for all } \rho,\sigma \in U,$$

then T possesses a fixed point.

Zhang & Song (2009) proved unique common fixed point results for hybrid generalized Δ-weak contractive mappings in complete metric spaces whereas Doric (2009) established some related theorems using control functions. This work was an extension to the results due to Zhang & Song (2009). Then, Murthy et al. (2015) proved some results using weak contractive condition on two pairs of discontinuous weakly compatible mappings.

In 1975, the concept of fuzzy metric space is initiated by Kramosil & Michalek (1975) with the concept of t-norm. Later on, George & Veeramani (1994) extended the notion of fuzzy metric space by defining the Hausdorff topology in this framework. After that, Mihet (2008) introduced the fuzzy version of Banach's result and introduced fuzzy ψ-contractive type mapping in non-Archimedean fuzzy environment. A key distinction between a fuzzy metric and a classical metric is that the latter contains a parameter in its definition. This concept has been used successfully in engineering applications including colour picture filtering and perceived colour disparities. (For details, one can refer to the study of Camarena et al. (2008), Camarena et al. (2010), Morillas et al. (2009), Morillas et al. (2007), Morillas et al. (2005), Morillas et al. (2008a), Morillas et al. (2008b)).

It has been demonstrated, in particular, that the class of topological spaces that are fuzzy metrizable matches with the class of topological spaces that may be metrized and then some traditional metric completeness and compactness theorems have been modified for fuzzy metric spaces. (See George & Veeramani (1995), Gregori & Romaguera (2000)). However, compared to the traditional theories of metric completion, the theory of fuzzy metric completion is significantly distinct. In actuality, some fuzzy metric spaces are non-completable (See Gregori (2002)). The example below demonstrates the existence of a fuzzy metric space that forbids fuzzy metric completion.

Example 1 (Gregori (2002)). *The continuous t-norm defined on* $[0,1]\times[0,1]$ *is indicated by the symbol* $*$ *and is defined as*

$$l * m = max\{0, l+m-1\}$$

for each $l,m \in [0,1]$.

Now, let $\{u_m\}_{m=3}^{\infty}$ *and* $\{v_m\}_{m=3}^{\infty}$ *be two arbitrary sequences of non-identical points such that* $U \cap V = \phi$*, where* $U = \{u_m : m \geq 3\}$ *and* $V = \{v_m : m \geq 3\}$.
Put $W = U \cup V$*. Let* M *be a real-valued function defined on* $W \times W \times (0,\infty)$ *as:*

$$\begin{aligned} M(u_m,u_n,t) &= M(v_m,v_n,t) \\ &= 1 - \left[\frac{1}{m \wedge n} - \frac{1}{m \vee n}\right] \\ M(u_m,v_n,t) = M(v_n,u_m,t) &= \frac{1}{m} + \frac{1}{n}, \end{aligned}$$

for every $m,n \geq 3$.
We firstly claim that $(M,*)$ *is a fuzzy metric on* W.
Observe that the first four characteristics are nearly evident. (for $m,n \geq 3$*):*

1. $0 < M(u,v,t) \leq 1$ *for all* $u,v \in W$, $t > 0$;

2. $M(u,v,t) = 1$ *if and only if* $u = v$;

3. $M(u,v,t) = M(v,u,t)$ *for all* $u,v \in W$, $t > 0$;

4. *For every* $u,v \in W$; $M(u,v,.)$ *is a continuous function on* $(0,\infty)$.

Also, a straightforward calculation reveals that, for every $m,n,p \geq 3$ *and* $s,t > 0$;

$$M(u_m,u_n,s) * M(u_n,u_p,t) \leq M(u_m,u_p,s+t)$$

and

$$M(v_m,v_n,s) * M(v_n,v_p,t) \leq M(v_m,v_p,s+t).$$

Finally, the relationships listed below are simple:

$$M(u_m,u_n,s) * M(u_n,v_p,t) \leq M(u_m,v_p,s+t)$$

Similarly,

$$M(u_m,v_n,s) * M(v_n,v_p,t) \leq M(u_m,v_p,s+t)$$

and

$$M(u_m,v_n,s) * M(v_n,u_p,t) \leq M(u_m,u_p,s+t)$$

Thus, for every $u,v,w \in W$ *and* $s,t > 0$*, we have*

$$M(u,v,s) * M(v,w,t) \leq M(u,w,s+t).$$

Hence, $(M,*)$ *is a fuzzy metric on* W.
Next, we assert that in the fuzzy metric space $(W,M,*)$; $\{u_m\}_{m=3}^{\infty}$ *is a Cauchy sequence.*

For fixed $\varepsilon \in (0,1)$ and $s > 0$, there exists $m_0 \geq 3$ such that $|\frac{1}{m} - \frac{1}{n}| < \varepsilon$ for all $m,n \geq m_0$.
Let $n \geq m$. Then,

$$M(u_m,u_n,s) = 1 - \left(\frac{1}{m} - \frac{1}{n}\right) > 1 - \varepsilon$$

for $m,n \geq m_0$. Thus, $\{u_m\}_{m=3}^{\infty}$ is a Cauchy sequence in $(W,M,*)$. Similarly, $\{v_m\}_{m=3}^{\infty}$ is also a Cauchy sequence in $(W,M,*)$.
Although, $\{u_m\}_{m=3}^{\infty}$ and $\{v_m\}_{m=3}^{\infty}$ do not converge in W w.r.t. the topology ς_M induced by $(M,*)$. Actually, ς_M is the discrete topology on W as for every $m \geq 3$ and each $s > 0$, we have for $B_M(u,\varepsilon,s) = \{u \in W : M(u,v,s) > 1-\varepsilon\}$ (where B_M is the base with family of open sets of the form $\{B_M(u,\varepsilon,s) : u \in W, 0 < \varepsilon < 1, t > 0\}$)

$$B_M\left(u_m, \frac{1}{m(m+1)}, s\right) = \{u_m\} \ \ \text{and}$$

$$B_M\left(v_m, \frac{1}{m(m+1)}, s\right) = \{v_m\}.$$

To demonstrate the two prior equality claims, it is sufficient to observe that for $m,n \geq 3$, with $m \neq n$, and $s > 0$, we have

$$\begin{aligned} M(u_m,u_n,s) &= 1 - \left[\frac{1}{m \wedge n} - \frac{1}{m \vee n}\right] \\ &\leq 1 - \left[\frac{1}{m} - \frac{1}{m+1}\right] \\ &= 1 - \frac{1}{m(m+1)}. \end{aligned}$$

Similarly,

$$M(v_m,v_n,s) = 1 - \frac{1}{m(m+1)}$$

and for $m,n \geq 3$ and $s > 0$, we have

$$\begin{aligned} M(u_m,v_n,s) &= \frac{1}{m} + \frac{1}{n} \\ &\leq 1 - \left(\frac{1}{m} - \frac{1}{m+1}\right). \end{aligned}$$

Similarly,

$$M(v_m,u_n,s) = 1 - \left(\frac{1}{m} - \frac{1}{m+1}\right).$$

Hence, $(W,M,*)$ is not complete.

The main intent of our work is to extend and generalize (Δ,Ω)-weak contraction due to Murthy et al. (2015) to fuzzy metric spaces. The authenticity of the results is further verified with some illustrative examples.

Theorem 1. *[Murthy et al. (2015)] Let (U,d) be a metric space equipped with completeness, and C,D,E and T be the self mappings defined on U satisfying*

$$\Omega(\hat{d}(C\rho,D\sigma)) \leq \Omega(M(\rho,\sigma)) - \Delta(N(\rho,\sigma))$$

for all $\rho,\sigma \in U$, with $\rho \neq \sigma$ and

$$M(\rho,\sigma) = \max\left\{\hat{d}(E\rho,T\sigma), \frac{1}{2}(\hat{d}(E\rho,C\rho)+\hat{d}(T\sigma,D\sigma)), \frac{1}{2}(\hat{d}(E\rho,D\sigma)+\hat{d}(T\sigma,C\rho))\right\}$$

and

$$N(\rho,\sigma) = \min\left\{\hat{d}(E\rho,T\sigma), \frac{1}{2}(\hat{d}(E\rho,C\rho)+\hat{d}(T\sigma,D\sigma)), \frac{1}{2}(\hat{d}(E\rho,D\sigma)+\hat{d}(T\sigma,C\rho))\right\},$$

$C(U) \subset T(U)$ and $D(U) \subset E(U)$, (C,E) and (D,T) are weakly compatible pairs.

$\Delta : [0,\infty] \to [0,\infty)$ is such that $\Delta(t) > 0$, which is lower semi-continuous for all $t > 0$ and Δ is discontinuous at $t = 0$ with $\Delta(0) = 0$, $\Omega : (0,\infty) \to [0,\infty)$ is an altering distance. Then C,D,E and T have a unique common fixed point in U.

2 PRELIMINARIES

Definition 1 (George & Veeramani (1994)). *Let $* : [0,1]\times[0,1] \to [0,1]$ be a binary operation. $*$ is a continuous t-norm if it satisfies the postulates stated below:*

1. *$*$ is commutative as well as associative;*
2. *$*$ is a continuous binary operation;*
3. *$a * 1 = a \ \forall\, a \in [0,1]$;*
4. *$a*b \leq c*d$ provided $a \leq c$ and $b \leq d \ \forall\, a,b,c,d \in [0,1]$.*

Definition 2 (George & Veeramani (1994)). *$(U,M,*)$ is named a fuzzy metric space if U is any non-empty set, M is a fuzzy set on $U^2 \times [0,\infty)$ and '$*$' is a continuous t-norm, satisfying the following axioms $\forall\ \sigma_1,\sigma_2,\sigma_3 \in U$ and $t,s > 0$:*

1. *$M(\sigma_1,\sigma_2,t)$ is positive;*
2. *$M(\sigma_1,\sigma_2,t) = 1 \ \forall\, t > 0 \Leftrightarrow \sigma_1 = \sigma_2$;*
3. *$M(\sigma_1,\sigma_2,t) = M(\sigma_2,\sigma_1,t)$;*

4. *$M(\sigma_1,\sigma_2,t) * M(\sigma_2,\sigma_3,s) \le M(\sigma_1,\sigma_3,t+s)$;*

5. *$M(\sigma_1,\sigma_2,t,\cdot) : [0,\infty) \to [0,1]$ is left continuous.*

Here, $M(\sigma_1,\sigma_2,t)$ signifies the degree of nearness between two elements σ_1 and σ_2 w.r.t. t. These spaces are referred as GV-spaces.

Lemma 1 (George & Veeramani (1994)). *if $(U,M,*)$ is a fuzzy metric space (FMS), then $M(\rho,\sigma,\cdot)$ is non-decreasing $\forall\ \rho,\sigma \in U$.*

Definition 3 (George & Veeramani (1994)). *Let $(U,M,*)$ be a FMS. Then,*

1. *any sequence $\{\rho_n\}$ in U is convergent to a point $\rho \in U$ if $\forall\ t>0, \lim_{n\to\infty} M(\rho_n,\rho,t) = 1$.*

2. *any sequence $\{\rho_n\}$ in U is named a Cauchy sequence if $\forall\ t>0$ and for each $\varepsilon \in]0,1), \exists n_0 \in N$ such that $M(\rho_n,\rho_m,t) > 1-\varepsilon\ \ \forall\ n,m \ge n_0$.*

3. *A fuzzy metric space in which every Cauchy sequence convergent in it, is named as complete fuzzy metric space.*

3 MAIN RESULTS

Theorem 2. *Let $(U,M,*)$ be a complete fuzzy metric space, and let Θ, D, E and $T : U \to U$ be four mappings satisfying*

$$\Omega(M(\Theta\rho,D\sigma,t)) \ge \Omega(\kappa_1(\rho,\sigma,t)) + \Delta(\kappa_2(\rho,\sigma,t)) \tag{2}$$

for all $\rho,\sigma \in U$, with $\rho \ne \sigma$ and

$$\kappa_1(\rho,\sigma,t) = \min\{M(E\rho,T\sigma,t), M(E\rho,\Theta\rho,t) * M(T\sigma,D\sigma,t), \\ M(E\rho,D\sigma,t) * M(T\sigma,\Theta\rho,t)\}$$

and

$$\kappa_2(\rho,\sigma,t) = \max\{M(E\rho,T\sigma,t), M(E\rho,\Theta\rho,t) * M(T\sigma,D\sigma,t), \\ M(E\rho,D\sigma,t) * M(Ty,\Theta\rho,t)\},$$

$$\Theta(U) \subset T(U) \text{ and } D(U) \subset E(U), \tag{3}$$

$$(\Theta,E) \text{ and } (D,T) \text{ are weakly compatible pairs,} \tag{4}$$

where

$$\Delta : [0,1] \to [0,1] \text{ is an upper semi-continuous mapping and } \Delta(t) \text{ is less than } 1\ \forall\ t<1,$$

$$\Delta \text{ is discontinuous at } t=1 \text{ with } \Delta(1)=0, \tag{5}$$

$$\Omega : [0,1] \to [0,1] \text{ is a non-decreasing and continuous function with } \Omega(t)=1 \Leftrightarrow t=1. \tag{6}$$

Then Θ, D, E and T possess a unique common fixed point.

Proof. Let ρ_0 be any arbitrary point in U. As $\Theta(U) \subset T(U)$ and $D(U) \subset E(U)$, therefore, there exists another point $\rho_1 \in U$ for which $\Theta\rho_0 = T\rho_1$, and in the similar manner, for the point $\rho_1 \in U$, there exists a point $\rho_2 \in U$ for which $D\rho_1 = E\rho_2$. Following the same pattern, we can set up a sequence $\{\sigma_n\}$ such that

$$\begin{aligned}
\sigma_{2n+1} &= \Theta\rho_{2n} = T\rho_{2n+1}\,, \\
\sigma_{2n+2} &= D\rho_{2n+1} = E\rho_{2n+2}\,, \text{ for } n = 0, 1, 2, \ldots
\end{aligned}$$

Let us suppose that for all $n \in N \cup \{0\}$,

$$\sigma_{2n} \neq \sigma_{2n+1}\,. \tag{7}$$

Now, we show that $M(\sigma_{2n}, \sigma_{2n+1}, t) \to 1$ as n tends to ∞ $\forall$ $n \in N \cup \{0\}$. Assume that $\rho = \rho_{2n}$ and $\sigma = \sigma_{2n+1}$ in (2).

$$\begin{aligned}
\Omega(M(\Theta\rho_{2n}, D\rho_{2n+1}, t)) &= \Omega(M(\sigma_{2n+1}, \sigma_{2n+2}, t)) \\
&\geq \Omega(\kappa_1(\rho_{2n}, \rho_{2n+1}, t)) + \Delta(\kappa_2(\rho_{2n}, \rho_{2n+1}, t)),
\end{aligned} \tag{8}$$

where

$$\begin{aligned}
\kappa_1(\rho_{2n}, \rho_{2n+1}, t) = \min\{&M(\sigma_{2n}, \sigma_{2n+1}, t), M(\sigma_{2n}, \sigma_{2n+1}, t) * M(\sigma_{2n+1}, \sigma_{2n+2}, t), \\
&M(\sigma_{2n}, \sigma_{2n+1}, t) * M(\sigma_{2n+1}, \sigma_{2n+1}, t)\}
\end{aligned}$$

and

$$\begin{aligned}
\kappa_2(\rho_{2n}, \rho_{2n+1}, t) = \max\{&M(\sigma_{2n}, \sigma_{2n+1}, t), M(\sigma_{2n}, \sigma_{2n+1}, t) * M(\sigma_{2n+1}, \sigma_{2n+2}, t)), \\
&M(\sigma_{2n}, \sigma_{2n+2}, t) * M(\sigma_{2n+1}, \sigma_{2n+1}, t)\}.
\end{aligned}$$

Then by triangle inequality, we have

$$\begin{aligned}
\kappa_1(\rho_{2n}, \rho_{2n+1}, t) \geq \min\Big\{&M(\sigma_{2n}, \sigma_{2n+1}, t), M(\sigma_{2n}, \sigma_{2n+1}, t) * M(\sigma_{2n+1}, \sigma_{2n+2}, t) \\
&M\left(\sigma_{2n}, \sigma_{2n+1}, \frac{t}{2}\right) * M\left(\sigma_{2n+1}, \sigma_{2n+2}, \frac{t}{2}\right) * 1\Big\}
\end{aligned}$$

and

$$\begin{aligned}
\kappa_2(\rho_{2n}, \rho_{2n+1}, t) = \max\{&M(\sigma_{2n}, \sigma_{2n+1}, t), M(\sigma_{2n}, \sigma_{2n+1}, t) * M(\sigma_{2n+1}, \sigma_{2n+2}, t)), \\
&M(\sigma_{2n}, \sigma_{2n+2}, t) * M(\sigma_{2n+1}, \sigma_{2n+1}, t)\}.
\end{aligned}$$

If

$$M(\sigma_{2n}, \sigma_{2n+1}, t) \geq M(\sigma_{2n+1}, \sigma_{2n+2}, t), \tag{9}$$

then we obtain

$$\kappa_1(\rho_{2n}, \rho_{2n+1}, t) \geq M(\sigma_{2n+1}, \sigma_{2n+2}, t) \tag{10}$$

and (8) implies

$$\Omega(M(\sigma_{2n+1},\sigma_{2n+2},t)) \geq \Omega(\kappa_1(\rho_{2n},\rho_{2n+1},t)) + \Delta(\kappa_2(\rho_{2n},\rho_{2n+1},t)),$$

and so,

$$\Omega(M(\sigma_{2n+1},\sigma_{2n+2},t)) \geq \Omega(\kappa_1(\rho_{2n},\rho_{2n+1},t)).$$

Using monotonically increasing property of Δ and Ω functions, we have

$$M(\sigma_{2n+1},\sigma_{2n+2},t) \geq \kappa_1(\rho_{2n},\rho_{2n+1},t). \tag{11}$$

From (10) and (11), we get

$$\kappa_1(\rho_{2n},\rho_{2n+1},t) = M(\sigma_{2n+2},\sigma_{2n+1},t). \tag{12}$$

Since

$$\begin{aligned} 1 &\geq (M\left(\sigma_{2n+1},\sigma_{2n+2},\frac{t}{2}\right) * M\left(\sigma_{2n},\sigma_{2n+1},\frac{t}{2}\right)) \\ &\geq M(\sigma_{2n+2},\sigma_{2n},t), \end{aligned} \tag{13}$$

we have, $\kappa_2(\rho_{2n},\rho_{2n+1},t) < 1$, then from (8), (12) and the properties of Δ and Ω functions, one can get,

$$\begin{aligned} \Omega(M(\sigma_{2n+1},\sigma_{2n+2},t)) &\geq \Omega(M(\sigma_{2n+1},\sigma_{2n+2},t) + \Delta(\kappa_2(\rho_{2n},\rho_{2n+1},t) \\ &> \Omega(M(\sigma_{2n+1},\sigma_{2n+2},t)), \end{aligned}$$

this is a contradiction, thus we have

$$M(\sigma_{2n+1},\sigma_{2n+2},t) \geq M(\sigma_{2n},\sigma_{2n+1},t). \tag{14}$$

So, we obtain the following

$$\kappa_1(\rho_{2n},\rho_{2n+1},t) = M(\sigma_{2n},\sigma_{2n+1},t), \tag{15}$$

$$\kappa_2(\rho_{2n},\rho_{2n+1},t) = M(\sigma_{2n},\sigma_{2n+2},t). \tag{16}$$

Now putting (15) and (16) in (8), we have

$$\Omega(M(\sigma_{2n+1},\sigma_{2n+2},t)) \geq \Omega(M(\sigma_{2n},\sigma_{2n+1},t)) + \Delta(M(\sigma_{2n},\sigma_{2n+2},t)), \tag{17}$$

$$\geq \Omega(M(\sigma_{2n},\sigma_{2n+1},t)), \tag{18}$$

As Ω is a non-decreasing function, therefore, we get,

$$M(\sigma_{2n+1},\sigma_{2n+2},t) \geq M(\sigma_{2n},\sigma_{2n+1},t).$$

This shows that $M(\sigma_{2n},\sigma_{2n+1},t)$ is a non-decreasing sequence, so there exists $r > 0$ such that

$$\lim_{n\to\infty} M(\sigma_{2n},\sigma_{2n+1},t) = r. \tag{19}$$

By (7) and (13), it follows that $\kappa_2(\rho_{2n},\rho_{2n+1},t) < 1$.

Taking limit n tends to ∞ in (18) and using (19), we get

$$\lim_{n\to\infty}\Omega(M(\sigma_{2n+1},\sigma_{2n+2},t)) \geq \lim_{n\to\infty}\Omega(M(\sigma_{2n},\sigma_{2n+1},t) \\ +\lim_{n\to\infty}\Delta(\kappa_2(\rho_{2n},\rho_{2n+1},t)),$$

which gives,

$$\Omega(r) \geq \Omega(r) + \lim_{n\to\infty}\Delta(\kappa_2(\rho_{2n},\rho_{2n+1},t)).$$

This is impossible with Δ function, therefore

$$\lim_{n\to\infty} M(\sigma_{2n},\sigma_{2n+1},t) = 1.$$

Thus, $\forall\ n \in N\cup\{0\}$, we have

$$\lim_{n\to\infty} M(\sigma_{2n+1},\sigma_{2n+2},t) = 1,$$

that is,

$$\lim_{n\to\infty} M(\sigma_n,\sigma_{n+1},t) = 1. \tag{20}$$

Next, we claim that the sequence $\{\sigma_n\}$ is Cauchy.

For this, it is sufficient to prove that the sub-sequence $\{\sigma_{2n}\}$ of the sequence $\{\sigma_n\}$ is Cauchy. Let us assume in a contrary manner that $\{\sigma_{2n}\}$ is not a Cauchy sequence. Consider the sequences $\{2n(k)\}$ and $\{2m(k)\}$ such that $2n(k) > 2m(k) > 2k$ for $k \in N$ and

$$M(\sigma_{2m(k)},\sigma_{2n(k)},t) \leq 1-\varepsilon. \tag{21}$$

Choose $2n(k)$ to be the smallest index in such a way that (21) holds true.

Then,

$$M(\sigma_{2m(k)-1},\sigma_{2n(k)-1},t) > 1-\varepsilon \quad \text{for all } k \in N. \tag{22}$$

Putting $\rho = \rho_{2m(k)-1}$ and $\sigma = \rho_{2n(k)-1}$ in (2),

$$\Omega(M(\sigma_{2m(k)},\sigma_{2n(k)},t)) \geq \Omega(\kappa_1(\rho_{2m(k)-1},\rho_{2n(k)-1},t)) \\ +\Delta(\kappa_2(\rho_{2m(k)-1},\rho_{2n(k)-1},t)), \tag{23}$$

where

$$M(\rho_{2m(k)-1},\rho_{2n(k)-1},t) = \min\{M(\sigma_{2m(k)-1},\sigma_{2n(k)-1},t), M(\sigma_{2m(k)-1},\sigma_{2m(k)},t) \\ *M(\sigma_{2n(k)-1},\sigma_{2n(k)},t), M(\sigma_{2m(k)-1},\sigma_{2n(k)},t) \\ *M(\sigma_{2n(k)-1},\sigma_{2m(k)},t)\}.$$

By triangle inequality, we obtain,

$$M(\sigma_{2m(k)},\sigma_{2n(k)},t) \geq M\left(\sigma_{2m(k)},\sigma_{2n(k)-1},\frac{t}{2}\right) * M\left(\sigma_{2n(k)-1},\sigma_{2n(k)},\frac{t}{2}\right),$$

taking $k \to \infty$, we get

$$\lim_{k\to\infty} M(\sigma_{2m(k)},\sigma_{2n(k)},t) = 1-\varepsilon. \tag{24}$$

Now, for every k, we have

$$\begin{aligned} M(\sigma_{2m(k)-1},\sigma_{2n(k)-1},t) &\geq M\left(\sigma_{2m(k)},\sigma_{2m(k)-1},\frac{t}{3}\right) * M\left(\sigma_{2m(k)},\sigma_{2n(k)},\frac{t}{3}\right) \\ &\quad * M\left(\sigma_{2n(k)-1},\sigma_{2n(k)},\frac{t}{3}\right), \\ M(\sigma_{2m(k)},\sigma_{2n(k)},t) &\geq M\left(\sigma_{2m(k)},\sigma_{2m(k)-1},\frac{t}{3}\right) * M\left(\sigma_{2m(k)-1},\sigma_{2n(k)-1},\frac{t}{3}\right) \\ &\quad * M\left(\sigma_{2n(k)-1},\sigma_{2n(k)},\frac{t}{3}\right). \end{aligned}$$

Letting limit $k \to \infty$ and using (20)-(24), we get

$$\lim_{k\to\infty} M(\sigma_{2m(k)-1},\sigma_{2n(k)-1},t) = 1-\varepsilon. \tag{25}$$

Also, for each positive value of k, we get

$$\begin{aligned} M(\sigma_{2m(k)-1},\sigma_{2n(k)},t) &\geq M\left(\sigma_{2m(k)-1},\sigma_{2m(k)},\frac{t}{2}\right) * M\left(\sigma_{2m(k)},\sigma_{2n(k)},\frac{t}{2}\right), \\ M(\sigma_{2m(k)},\sigma_{2n(k)},t) &\geq M\left(\sigma_{2m(k)},\sigma_{2m(k)-1},\frac{t}{2}\right) * M\left(\sigma_{2m(k)-1},\sigma_{2n(k)},\frac{t}{2}\right). \end{aligned}$$

Taking $k \to \infty$ and using (20)-(25), we get

$$\lim_{k\to\infty} M(\sigma_{2m(k)-1},\sigma_{2n(k)},t) = 1-\varepsilon. \tag{26}$$

Again, for each positive value of k, we get

$$\begin{aligned} M(\sigma_{2n(k)-1},\sigma_{2m(k)},t) &\geq M\left(\sigma_{2n(k)-1},\sigma_{2n(k)},\frac{t}{2}\right) * M\left(\sigma_{2n(k)},\sigma_{2m(k)},\frac{t}{2}\right), \\ M(\sigma_{2n(k)},\sigma_{2m(k)},t) &\geq M\left(\sigma_{2n(k)},\sigma_{2n(k)-1},\frac{t}{2}\right) * M\left(\sigma_{2n(k)-1},\sigma_{2m(k)},\frac{t}{2}\right). \end{aligned}$$

Taking limit $k \to \infty$ and using (20)-(26), we get

$$\lim_{k\to\infty} M(\sigma_{2n(k)-1},\sigma_{2m(k)},t) = 1-\varepsilon. \tag{27}$$

From (23)-(27), one obtains

$$\lim_{k\to\infty} M(\rho_{2m(k)-1},\rho_{2n(k)-1},t) = 1-\varepsilon \tag{28}$$

and

$$\lim_{k\to\infty} \kappa_2(\rho_{2m(k)-1},\rho_{2n(k)-1},t) = 1.$$

Taking $k \to \infty$ in (23), we get

$$\Omega(1-\varepsilon) \geq \Omega(1-\varepsilon) + \lim_{k\to\infty} \Delta(\kappa_2(\rho_{2m(k)-1}, \rho_{2n(k)-1}, t)). \tag{29}$$

As Δ is discontinuous at $t = 1$ where $\Delta(t) = 0$ and $\Delta(t) < 1 \ \ \forall \ \ t < 1$, the last term in (29) vanishes, which eventually lead to a contradiction.

Thus, $\{\sigma_n\}$ is a Cauchy sequence. By the property of completeness, this sequence converges to some point ζ(say) in U. Consequently, its sub-sequences also converges to ζ in U *i.e.*

$$\Theta\rho_{2n} \to \zeta, \ \ T\rho_{2n+1} \to \zeta, D\rho_{2n+1} \to \zeta, E\rho_{2n} \to \zeta.$$

Since $D(U) \subset E(U)$, there exists $\hbar \in V$ such that $\zeta = E\hbar$.

Let $M(\zeta, \Theta\hbar, t) \neq 1$ putting $\rho = \hbar$ and $y = \rho_{2n+1}$ in (2), we get

$$\Omega(M(\Theta\hbar, D\rho_{2n+1}, t)) \geq \Omega(\kappa_1(\hbar, \rho_{2n+1}, t)) + \Delta(\kappa_2(\hbar, \rho_{2n+1}, t)), \tag{30}$$

where

$$\begin{aligned}\kappa_1(\hbar, \rho_{2n+1}, t) = \min\{&M(E\hbar, T\rho_{2n+1}, t), M(E\hbar, \Theta\hbar, t) * M(T\rho_{2n+1}, D\rho_{2n+1}, t), \\ &M(E\hbar, D\rho_{2n+1}, t) * M(T\rho_{2n+1}, \Theta\hbar, t)\}\end{aligned}$$

and

$$\begin{aligned}\kappa_2(\hbar, \rho_{2n+1}, t) = \max\{&M(E\hbar, T\rho_{2n+1}, t), M(E\hbar, \Theta\hbar, t) * M(T\rho_{2n+1}, D\rho_{2n+1}, t), \\ &M(E\hbar, D\rho_{2n+1}, t) * M(T\rho_{2n+1}, \Theta\hbar, t)\}.\end{aligned}$$

Taking $n \to \infty$ and using $\zeta = E\hbar$, we have

$$\begin{aligned}M(\hbar, \zeta, t) &= \max\{M(E\hbar, \zeta, t), M(E\hbar, \Theta\hbar, t) * M(\zeta, \zeta, t), M(E\hbar, \zeta, t) * M(\zeta, \Theta\hbar, t)\} \\ &= M(\zeta, \Theta\hbar, t).\end{aligned}$$

Also, we have

$$\Omega(M(\Theta\hbar, \zeta, t)) \geq \Omega(M(\zeta, \Theta\hbar, t)) + \lim_{n\to\infty} \Delta(\kappa_2(\hbar, \rho_{2n+1}, t)).$$

As Δ is discontinuous at $t = 1$ and $\Delta(t) = 0$, we notice that

$$\Omega(M(\Theta\hbar, \zeta, t)) \geq \Omega(M(\zeta, \Theta\hbar, t)).$$

Consequently, we reach a contradiction with Ω function. Thus,

$$\begin{aligned}M(\zeta, \Theta\hbar, t) &= 1 \\ &\Rightarrow \Theta\hbar = \zeta \\ &\Rightarrow \Theta\hbar = \zeta = E\hbar.\end{aligned}$$

As (Θ,E) is a weakly compatible pair, it commutes at its coincidence point $\hbar$, *i.e.* $\Theta E\hbar = E\Theta\hbar \Rightarrow \Theta\zeta = E\zeta$.

Next, we claim that $\Theta\zeta = E\zeta = \zeta$.

For this, putting $\rho = \zeta$ and $\sigma = \rho_{2n+1}$ in (2), we get

$$\Omega(M(\Theta\zeta, D\rho_{2n+1}, t)) \geq \Omega(\kappa_1(\zeta, \rho_{2n+1}, t)) + \Delta(\kappa_2(\zeta, \rho_{2n+1}, t)), \tag{31}$$

where

$$\begin{aligned}\kappa_1(\zeta, \rho_{2n+1}, t) = \min\{&M(E\zeta, T\rho_{2n+1}, t), M(E\zeta, \Theta\zeta, t) * M(T\rho_{2n+1}, D\rho_{2n+1}, t),\\ &M(E\zeta, D\rho_{2n+1}, t) * M(T\rho_{2n+1}, \Theta\zeta, t)\}\end{aligned}$$

and

$$\begin{aligned}\kappa_2(\zeta, \rho_{2n+1}, t) = \max\{&M(E\zeta, T\rho_{2n+1}, t), M(E\zeta, \Theta\zeta, t) * M(T\rho_{2n+1}, D\rho_{2n+1}, t)\\ &* M(E\zeta, D\rho_{2n+1}, t) * M(T\rho_{2n+1}, \Theta\zeta, t).\end{aligned}$$

Taking $n \to \infty$ and using $\Theta\zeta = E\zeta$, we get

$$\kappa_1(\zeta, \zeta, t) = M(E\zeta, \zeta, t).$$

Now, (30) implies that

$$\Omega(M(E\zeta, \zeta, t)) \geq \Omega(M(E\zeta, \zeta, t)) + \lim_{n\to\infty} \Delta(\kappa(\zeta, \rho_{2n+1}, t)).$$

As Δ is discontinuous at $t = 1$, we get $\Delta(t) = 0$, which implies

$$\Omega(M(E\zeta, \zeta, t)) > \Omega(M(E\zeta, \zeta, t)),$$

but it is a contradiction. Therefore $M(E\zeta, \zeta, t) = 1$, implies

$$E\zeta = \zeta \Rightarrow E\zeta = \Theta\zeta = \zeta.$$

Likewise, we can demonstrate that $T\zeta = D\zeta = \zeta$.

Hence, $E\zeta = \Theta\zeta = T\zeta = D\zeta = \zeta$.

We now assert that ζ is the unique common fixed point of Θ, D, E and T. To show this, let w be another fixed point of Θ, D, E and T.

Now put $\rho = \zeta$ and $\sigma = w$ in (2), we obtain

$$\Omega(M(\zeta, w, t)) \geq \Omega(M(\zeta, w, t)) + \Delta(M(\zeta, w, t)),$$

which contradicts itself. Thus, $M(\zeta, w, t) = 1 \Rightarrow \zeta = w$.

Hence Θ, D, E and T possess an unrepeated common fixed point in U. $\square$

Assuming $E = T = I$ (identity map), we deduce the following result:

Theorem 3. *Let $(U,M,*)$ be a fuzzy metric space equipped with completeness. Let Θ, $D : U \to U$ be two self-mappings which satisfy the following inequality:*

$$\Omega(M(\Theta\rho,D\sigma,t) \geq \Omega(\kappa_1(\rho,\sigma,t)) + \Delta(\kappa_2(\rho,\sigma,t)), \tag{32}$$

where $\rho,\sigma \in U$, $\rho \neq \sigma$,

$$\kappa_1(\rho,\sigma,t) = \min\{M(\rho,\sigma,t), M(\rho,\Theta\rho,t) * M(\sigma,D\sigma,t), M(\rho,D\sigma,t) * M(\sigma,\Theta\rho,t)\}$$

and

$$\kappa_2(\rho,\sigma,t) = \max\{M(\rho,\sigma,t), M(\rho,\Theta\rho,t) * M(\sigma,D\sigma,t), M(\rho,D\sigma,t) * M(\sigma,\Theta\rho,t)\};$$

and

1. *$\Delta : [0,1] \to [0,1]$ with $\Delta(t) < 1$ is upper semi-continuous for each $t \in (0,1)$ and Δ is discontinuous at the point $t = 1$ with $\Delta(t) = 0$.*
2. *$\Omega : [0,1] \to [0,1]$ is an altering distance function.*

Then Θ and D possess a unique fixed point in U.

The result below is obtained by taking $\Omega = I$(identity function):

Corollary 1. *Let $(U,M,*)$ be a fuzzy metric space equipped with completeness property. Let Θ, D, E and $T : U \to U$ be self-mappings holding following inequality:*

$$M(\Theta\rho,D\sigma,t) \geq \kappa_1(\rho,\sigma,t) + \Delta(\kappa_2(\rho,\sigma,t)), \tag{33}$$

where $\rho,\sigma \in U$, $\rho \neq \sigma$,

$$\kappa_1(\rho,\sigma,t) = \min\{M(E\rho,T\sigma,t), M(E\rho,\Theta\rho,t) * M(T\sigma,D\sigma,t), M(E\rho,D\sigma,t) * M(T\sigma,\Theta\rho,t)\}$$

and

$$\kappa_2(\rho,\sigma,t) = \max\{M(E\rho,T\sigma,t), M(E\rho,\Theta\rho,t) * M(T\sigma,D\sigma,t), M(E\rho,D\sigma,t) * M(T\sigma,\Theta\rho,t)\};$$

1. *$\Theta(U) \subset T(U)$ and $D(U) \subset E(U)$,*
2. *(Θ,E) and (D,T) are weakly compatible pairs,*
3. *$\Delta : [0,1] \to [0,1]$ with $\Delta(t) < 1$ is upper semi-continuous for each $t \in (0,1)$ and Δ is discontinuous at the point $t = 1$ with $\Delta(t) = 0$.*

Then Θ, D, E and T possess a unique common fixed point in U.

Corollary 2. *Let $(U,M,*)$ be a fuzzy metric space equipped with completeness property. Let Θ and $D : U \to U$ be self-mappings satisfying the following inequality:*

$$M(\Theta\rho,D\sigma,t) \geq \kappa_1(\rho,\sigma,t) + \Delta(\kappa_2(\rho,\sigma,t)), \tag{34}$$

where $\rho,\sigma \in U$, $\rho \neq \sigma$,

$$\kappa_1(\rho,\sigma,t) = \min\{M(\rho,\sigma,t), M(\rho,\Theta\rho,t) * M(\sigma,D\sigma,t), M(\rho,D\sigma,t) * M(\sigma,\Theta\rho,t)\}$$

and

$$\kappa_2(\rho,\sigma,t) = \max\{M(\rho,\sigma,t), M(\rho,\Theta\rho,t) * M(\sigma,D\sigma,t), M(\rho,D\sigma,t) * M(\sigma,\Theta\rho,t)\};$$

$\Delta : [0,1] \to [0,1]$ *with* $\Delta(t) < 1$ *is upper semi-continuous for each* $t \in (0,1)$ *and* Δ *is discontinuous at the point* $t = 1$ *with* $\Delta(t) = 0$.
Then Θ *and* D *possess a unique fixed point in* U.

If the aforementioned condition

$$\kappa_2(\rho,\sigma,t) = \max\{M(\rho,\sigma,t), M(\rho,\Theta\rho,t) * M(\sigma,D\sigma,t), M(\rho,D\sigma,t) * M(\sigma,\Theta\rho,t)\}$$

is changed to

$$\kappa_2(\rho,\sigma,t) = \max\{M(\rho,\sigma,t), M(\rho,\Theta\rho,t) * M(\sigma,D\sigma,t)\};$$

another result will be deduced as follows:

Theorem 4. *Let* $(U,M,*)$ *be a fuzzy metric space equipped with completeness. Let* Θ, D, E *and* T *be self-mappings defined on* U *such that they satisfy the following inequality:*

$$\Omega(M(\Theta\rho,D\sigma,t) \geq \Omega(\kappa_1(\rho,\sigma,t)) + \Delta(\kappa_2(\rho,\sigma,t)), \tag{35}$$

where $\rho,\sigma \in U$, $\rho \neq \sigma$,

$$\kappa_1(\rho,\sigma,t) = \min\{M(E\rho,T\sigma,t), M(E\rho,\Theta\rho,t) * M(T\sigma,D\sigma,t), M(E\rho,D\sigma,t) * M(T\sigma,\Theta\rho,t)\}$$

and

$$\kappa_2(\rho,\sigma,t) = \max\{M(E\rho,T\sigma,t), M(E\rho,\Theta\rho,t) * M(T\sigma,D\sigma,t)\};$$

1. $\Theta(U) \subset T(U)$ *and* $D(U) \subset E(U)$,
2. (Θ,E) *and* (D,T) *are weakly compatible pairs,*
3. $\Delta : [0,1] \to [0,1]$ *with* $\Delta(t) < 1$ *is upper semi-continuous for each* $t \in (0,1)$ *and* Δ *is discontinuous at the point* $t = 1$ *with* $\Delta(t) = 0$.
4. $\Omega : [0,1] \to [0,1]$ *is a strictly monotonically increasing altering distance function.*

Then Θ, D, E *and* T *possess a unique common fixed point in* U.

On the same lines, the above theorem is easily demonstrable as Theorem 3.1.

Theorem 5. *Let* $(U,M,*)$ *be a fuzzy metric space equipped with completeness. Let* Θ *and* D *be self-mappings defined on* U *such that they satisfy the following inequality:*

$$\Omega(M(\Theta\rho,D\sigma,t) \geq \Omega(\kappa_1(\rho,\sigma,t)) + \Delta(\kappa_2(\rho,\sigma,t)), \tag{36}$$

where $\rho,\sigma \in U$, $\rho \neq \sigma$,

$$\kappa_1(\rho,\sigma,t) = \min\{M(\rho,\sigma,t), M(\rho,\Theta\rho,t) * M(\sigma,D\sigma,t), M(\rho,D\sigma,t) * M(\sigma,\Theta\rho,t)\}$$

and

$$\kappa_2(\rho,\sigma,t) = \max\{M(\rho,\sigma,t), M(\rho,\Theta\rho,t) * M(\sigma,D\sigma,t)\};$$

1. $\Delta : [0,1] \to [0,1]$ *with* $\Delta(t) < 1$ *is upper semi-continuous for each* $t \in (0,1)$ *and* Δ *is discontinuous at the point* $t = 1$ *with* $\Delta(t) = 0$,
2. $\Omega : [0,1] \to [0,1]$ *is a strictly monotonically increasing altering distance function.*

Then Θ, D *possess a unique common fixed point in* U.

Following are a few corollaries that arise from the results stated above:

Corollary 3. *Let* $(U,M,*)$ *be a fuzzy metric space equipped with completeness. Let* Θ, D, E *and* T *be self-mappings defined on* U *such that they satisfy the following inequality:*

$$M(\Theta\rho, D\sigma, t) \geq \kappa_1(\rho,\sigma,t) + \Delta(\kappa_2(\rho,\sigma,t)), \tag{37}$$

where $\rho,\sigma \in U$, $\rho \neq \sigma$,

$$\kappa_1(\rho,\sigma,t) = \min\{M(E\rho,T\sigma,t), M(E\rho,\Theta\rho,t) * M(T\sigma,D\sigma,t), M(E\rho,D\sigma,t) * M(T\sigma,\Theta\rho,t)\}$$

and

$$\kappa_2(\rho,\sigma,t) = \max\{M(E\rho,T\sigma,t), M(E\rho,\Theta\rho,t) * M(T\sigma,D\sigma,t)\};$$

1. $\Theta(U) \subset T(U)$ *and* $D(U) \subset E(U)$,
2. (Θ,E) *and* (D,T) *are weakly compatible pairs,*
3. $\Delta : [0,1] \to [0,1]$ *with* $\Delta(t) < 1$ *is upper semi-continuous for each* $t \in (0,1)$ *and* Δ *is discontinuous at the point* $t = 1$ *with* $\Delta(t) = 0$.

Then Θ, D, E *and* T *possess a unique common fixed point in* U.

Corollary 4. *Let* $(U,M,*)$ *be a fuzzy metric space equipped with completeness. Let* Θ *and* D *be self-mappings defined on* U *such that they satisfy the following inequality:*

$$M(\Theta\rho, D\sigma, t) \geq \kappa_1(\rho,\sigma,t) + \Delta(\kappa_2(\rho,\sigma,t)), \tag{38}$$

where $\rho,\sigma \in U$, $\rho \neq \sigma$,

$$\kappa_1(\rho,\sigma,t) = \min\{M(E\rho,T\sigma,t), M(E\rho,\Theta\rho,t) * M(T\sigma,D\sigma,t), M(E\rho,D\sigma,t) * M(T\sigma,\Theta\rho,t)\}$$

and

$$\kappa_2(\rho,\sigma,t) = \max\{M(E\rho,T\sigma,t), M(E\rho,\Theta\rho,t) * M(T\sigma,D\sigma,t)\};$$

Here, $\Delta : [0,1] \to [0,1]$ with $\Delta(t) < 1$ is upper semi-continuous for each $t \in (0,1)$ and Δ is discontinuous at the point $t = 1$ with $\Delta(t) = 0$.

Then Θ, D possess a unique common fixed point in U.

Example 2. *Let $U = [0,2]$ be equipped with the (usual) metric $\hat{d}(\rho,\sigma) = |\rho - \sigma|$ and $(U,M,*)$ be a fuzzy metric space. Let Θ, D, E and T be self mappings defined on U as*

$$\Theta(\rho) = \begin{cases} 0 & \text{if } \rho = 0 \\ \frac{\rho}{7}+1 & \text{otherwise} \end{cases} \qquad E(\rho) = \begin{cases} 0 & \text{if } \rho = 0 \\ \frac{3\rho}{7}+1 & \text{otherwise} \end{cases}$$

$$D(\rho) = \begin{cases} 0 & \text{if } \rho = 0 \\ \frac{2\rho}{7}+1 & \text{otherwise} \end{cases} \qquad T(\rho) = \begin{cases} 0 & \text{if } \rho = 0 \\ \frac{4\rho}{7}+1 & \text{otherwise} \end{cases},$$

where $\rho,\sigma \in U$, $\Theta(U) = \left[0,\frac{8}{7}\right]$, $E(U) = \left[0,\frac{9}{7}\right]$, $D(U) = \left[0,\frac{10}{7}\right]$, $T(U) = \left[0,\frac{11}{7}\right]$.

Here, $\Theta(U) \subset T(U)$ and $D(U) \subset E(U)$, and (Θ,E) and (D,T) are weakly compatible maps at $\rho = 0$.

Let $\Omega(t) = t$ and $\Delta(t) = \begin{cases} \frac{t}{2}, & \text{if } t \neq 1 \\ 0, & t = 1 \end{cases}$.

Now, we examine Theorem 3.1's inequality in several cases.

Case I. If $\rho = 0$ and $\sigma = 0$

$$\Omega(M(\Theta\rho,D\sigma,t)) = \Omega\left(\frac{t}{t+|\Theta\rho - D\sigma|}\right) = \Omega(1) = 1$$
$$\kappa_1(\rho,\sigma,t) = 1 \Rightarrow \Omega(\kappa_1(\rho,\sigma,t)) = 1$$
$$\kappa_2(\rho,\sigma,t) = 1 \Rightarrow \Delta(\kappa_2(\rho,\sigma,t)) = 0.$$

Hence,

$$\Omega(M(\Theta\rho,D\sigma,t)) = \Omega(\kappa_1(\rho,\sigma,t)) + \Delta(\kappa_2(\rho,\sigma,t)).$$

Case II. If $\rho = 0$ and $\sigma \neq 0$,

$$\Omega(M(\Theta\rho,D\sigma,t)) = \Omega\left(\frac{t}{t+|\Theta\rho - D\sigma|}\right) = \Omega\left(\frac{t}{t+|0-(\frac{2\sigma}{7}+1)|}\right) = \frac{t}{t+|1+\frac{2\sigma}{7}|}.$$

Now,

$$
\begin{aligned}
&\{M(E\rho,T\sigma,t),M(E\rho,\Theta\rho,t)*M(T\sigma,D\sigma,t),M(E\rho,D\sigma,t)*M(T\sigma,\Theta\rho,t)\}\\
&=\left\{\frac{t}{t+|E\rho-T\sigma|},\frac{t}{t+|E\rho-\Theta\rho|}*\frac{t}{t+|T\sigma-D\sigma|},\frac{t}{t+|E\rho-D\sigma|}*\frac{t}{t+|T\sigma-\Theta\rho|}\right\}\\
&=\left\{\frac{t}{t+|\frac{4\sigma}{7}+1|},1*\frac{t}{t+|\frac{4\sigma}{7}+1-1-\frac{2\sigma}{7}|},\frac{t}{t+|0-(\frac{2\sigma}{7}+1)|}*\frac{t}{t+|\frac{4\sigma}{7}+1-0|}\right\}\\
&=\left\{\frac{t}{t+|\frac{4\sigma}{7}+1|},\frac{t}{t+|\frac{2\sigma}{7}|},\frac{t}{t+|1+\frac{2\sigma}{7}|}*\frac{t}{t+|\frac{4\sigma}{7}+1|}\right\},
\end{aligned}
$$

implies,

$$
\begin{aligned}
\kappa_1(\rho,\sigma,t)&=\frac{t}{t+|\frac{4\sigma}{7}+1|}\\
\kappa_2(\rho,\sigma,t)&=\frac{t}{t+|\frac{2\sigma}{7}|}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\Omega(\kappa_1(\rho,\sigma,t))+\Delta(\kappa_2(\rho,\sigma,t))&=\Omega\left(\frac{t}{t+|\frac{4\sigma}{7}+1|}\right)+\Delta\left(\frac{t}{t+|\frac{2\sigma}{7}|}\right)\\
&=\frac{t}{t+|\frac{4\sigma}{7}+1|}+\frac{1}{20}\left(\frac{t}{t+|\frac{2\sigma}{7}|}\right)\\
&\leq\frac{t}{t+|\frac{2\sigma}{7}+1|}=\Omega(M(\Theta\rho,D\sigma,t)).
\end{aligned}
$$

Hence,

$$
\Omega(M(\Theta\rho,D\sigma,t))\geq\Omega(\kappa_1(\rho,\sigma,t))+\Delta(\kappa_2(\rho,\sigma,t)).
$$

Case III. If $\rho\neq 0$ and $\sigma=0$.

$$
\Omega(M(\Theta\rho,D\sigma,t))=\Omega\left(\frac{t}{t+|\Theta\rho-D\sigma|}\right)=\Omega\left(\frac{t}{t+|\frac{\rho}{7}+1|}\right)=\frac{t}{t+|\frac{\rho}{7}+1|}.
$$

Now,

$$
\begin{aligned}
&\{M(E\rho,T\sigma,t),M(E\rho,\Theta\rho,t)*M(T\sigma,D\sigma,t),M(E\rho,D\sigma,t)*M(T\sigma,\Theta\rho,t)\}\\
&=\left\{\frac{t}{t+|\frac{3\rho}{7}+1|},\frac{t}{t+|\frac{3\rho}{7}+1-\frac{\rho}{7}-1|}*1,\frac{t}{t+|\frac{3\rho}{7}+1|}*\frac{t}{t+|\frac{\rho}{7}+1|}\right\}\\
&=\left\{\frac{t}{t+|\frac{3\rho}{7}+1|},\frac{t}{t+|\frac{2\rho}{7}|},\frac{t}{t+|\frac{3\rho}{7}+1|}*\frac{t}{t+|\frac{\rho}{7}+1|}\right\},
\end{aligned}
$$

By definition of κ_1 and κ_2 in Theorem 2, we get

$$\begin{aligned}
\kappa_1(\rho,\sigma,t) &= \frac{t}{t+|\frac{3\rho}{7}+1|},\\
\kappa_2(\rho,\sigma,t) &= \frac{t}{t+|\frac{2\rho}{7}|},\\
\Omega(\kappa_1(\rho,\sigma,t))+\Omega(\kappa_2(\rho,\sigma,t)) &= \frac{t}{t+|\frac{3\rho}{7}+1|}+\frac{1}{20}\left(\frac{t}{t+|\frac{2\rho}{7}|}\right) \leq \frac{t}{t+|\frac{\rho}{7}+1|}.
\end{aligned}$$

Hence,

$$\Omega(M(\Theta\rho,D\sigma,t)) \geq \Omega(\kappa_1(\rho,\sigma,t))+\Delta(\kappa_2(\rho,\sigma,t)).$$

Case IV. If $\rho \neq 0$ and $\sigma \neq 0$.

$$\begin{aligned}
\Omega(M(\Theta\rho,D\sigma,t)) &= \Omega\left(\frac{t}{t+|\Theta\rho-D\sigma|}\right)\\
&= \Omega\left(\frac{t}{t+|\frac{\rho}{7}+1-\frac{2u}{7}-1|}\right)\\
&= \Omega\left(\frac{t}{t+\frac{\rho}{7}}\right).
\end{aligned}$$

Now,

$$\begin{aligned}
&\left\{\frac{t}{t+|E\rho-T\sigma|},\frac{t}{t+|E\rho-\Theta\rho|}*\frac{t}{t+|T\sigma-D\sigma|},\frac{t}{t+|E\rho-D\sigma|}*\frac{t}{t+|T\sigma-\Theta\rho|}\right\}\\
&=\left\{\frac{t}{t+\frac{\rho}{7}},\frac{t}{(t+\frac{2\rho}{7})}*\frac{t}{(t+\frac{2\sigma}{7})},\frac{t}{t+|\frac{3\rho}{7}-\frac{2\sigma}{7}|}*\frac{t}{t+|\frac{4\sigma}{7}-\frac{\rho}{7}|}\right\},
\end{aligned}$$

and

$$\begin{aligned}
\kappa_1(\rho,\sigma,t) &= \begin{cases} \frac{t}{t+\frac{2\rho}{7}}, & \sigma<\frac{2\rho}{7}\\ \frac{t}{t+\frac{2\sigma}{7}}, & \sigma>\frac{2\rho}{7}\end{cases}\\
\kappa_2(\rho,t,\sigma) &= \frac{t}{t+|\frac{3\rho}{7}-\frac{2\sigma}{7}|},
\end{aligned}$$

this implies,

$$\Omega(\Theta\rho,D\sigma,t) \geq \Omega(\kappa_1(\rho,\sigma,t))+\Delta(\kappa_2(\rho,\sigma,t)).$$

The inequality is therefore true in each instance. As a result, Theorem 3.1's criteria are all fulfilled and therefore Θ, D, E and T possess a unique common fixed point. Here, $\rho=0$ is the unique common fixed point in U.

4 CONCLUSION

In this work, control functions are well used to locate fixed point for pairs of discontinuous maps in the setting of fuzzy environment. This is a fruitful strategy to broaden and generalize the literature's findings in the direction of fuzzy metric space.

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