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SOME RESULTS ABOUT THE CONNECTIVITY OF TREES

Lilian Markenzon[1], Nair Maria Maia de Abreu[2*] and Luciana Lee[3]

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ABSTRACT. The second smallest Laplacian eigenvalue of a graph G is called algebraic connectivity, denoted $a(G)$. The ordering of trees via this graph invariant is frequently studied in the literature. In this paper, we present a new invariant, the *Internal Degree Sequence* (IDS), that also supports an accurate evaluation of the connectivity of trees. We compare the IDS with $a(G)$ for all elements in six classes of trees known to have the largest algebraic connectivity and we show that the IDS provides a strict total ordering of the elements of these classes. This result is also proved for a subclass of trees of diameter 4.

Keywords: trees; internal degree sequence, algebraic connectivity.

1 PRELIMINARIES

Let $G = (V, E)$ be a simple graph, $n = |V|$ its order and $m = |E|$ its size. The degree of a vertex v is denoted by $d(v)$. A graph is connected if for any two vertices $u, v \in V$ there is a path from u to v. The graph is said to be disconnected if it has two or more connected components. The cut-vertex (or cutpoint) is a vertex whose removal increases the number of components of G.

The Laplacian matrix $L(G) = [\ell_{ij}]$ is defined as follows: $\ell_{ij} = d_i$, for $i = j$; $\ell_{ij} = -1$ if (v_i, v_j) is an edge of G and $\ell_{ij} = 0$ otherwise. The spectrum of the Laplacian of G is the sequence of their eigenvalues $(\mu_1, \ldots, \mu_{n-1}, \mu_n)$, given in non-increasing order, $\mu_1 \geq \cdots \geq \mu_{n-1} \geq \mu_n$. Since $L(G)$ is a semidefinite positive singular matrix, $\mu_n = 0$. It is well known that $\mu_{n-1} = 0$ if and only if G is a disconnected graph. Due to that, Fiedler [3] called μ_{n-1} the algebraic connectivity of G, denoted $a(G)$. For a survey about $a(G)$ see Abreu [1]. The relation between the algebraic connectivity and the cutpoints of a graph has been addressed by Kirkland [6].

In this paper, we introduce a new concept for the connectivity of trees, the Internal Degree Sequence (IDS), taking into account the number of connected components obtained after the

*Corresponding author
[1]NCE – Universidade Federal do Rio de Janeiro, Brazil. E-mail: markenzon@nce.ufrj.br
[2]Programa de Engenharia de Produção/COPPE, Universidade Federal do Rio de Janeiro. E-mail: nair@pep.ufrj.br
[3]Universidade Federal do Mato Grosso. E-mail: lucianalee@ufmt.br

removal of each cut-vertex. An ordering for trees with n vertices according to their IDS is defined. In Section 3, we compare the IDS with the algebraic connectivity $a(G)$ for all elements in six classes of trees known to have the largest algebraic connectivity, presented by Yuan *et al.* [8], and we show that the IDS provides a strict total ordering of the elements of these classes. This result is also proved for a subclass of trees of diamenter 4 in Section 4.

2 THE NEW INVARIANT

In this section, we present a new invariant, the *internal degree sequence* (IDS), taking into account the number of connected components obtained after the removal of each cut-vertex of the tree. The IDS allows us to differentiate the connectivity of trees, providing an extra resource to compare them. Another important feature to highlight is the simplicity of the determination of the invariant.

The *degree sequence* of a graph is the non-increasing sequence of its vertex degrees. Let $T = (V, E)$ be a tree and $[d_1, d_2, \ldots, d_n]$ its degree sequence. Let $V = \{v_1, v_2, \ldots, v_n\}$ be the ordered set of vertices of T, being $d_i = d(v_i)$.

A vertex v of T such that $d(v) > 1$ is called an *internal vertex* of T. It is a *cut-vertex* of T. A vertex v such that $d(v) = 1$ is called a *leaf* or an *external vertex* of T.

Let $T = (V, E)$ be a tree and let $L \subset V$ be the set of leaves of T. The removal of a vertex $v \in V - L$ results in a disconnected tree; observe that there are $d(v)$ components in the resulting tree. It is not difficult to see that the degree of an internal vertex is directly related with its relevance in the analysis of the connectivity. The next definition takes this fact into consideration.

Definition 1. *The non-increasing sequence of degrees of the internal vertices of a tree T is called the* ***internal degree sequence*** *(IDS) of a tree T. It is denoted*

$$IDS(G) = \left[d_1, d_2, \ldots, d_p\right].$$

Definition 2. *Let $s = [d_1, d_2, \ldots, d_p]$ be a non-increasing sequence of integers; s is a* ***valid*** *IDS for a tree T if T has $n = \sum_{i=1,p} d_i - p + 2$ vertices, p internal vertices and $d_i \neq 1$, $1 \leq i \leq p$.*

It is known that a string t is preceded by a string s in lexicographic order if

- s is a prefix of t, or
- if c and d are respectively the first characters of s and t in which s and t differ, then c precedes d in character order.

It is possible to perform a lexicographic comparison of the IDS values of two trees. For instance, Figure 1 shows all trees with 7 vertices with their respective IDS in lexicographic order.

This example shows some coincidences with the algebraic connectivity that must be highlighted. Among all trees, the maximum and the minimum IDS occur with the star and the path, respectively; the same happens with the algebraic connectivity (Grone & Merris [5]). This observation leads us to compare the two approaches, as we will see in the next section.

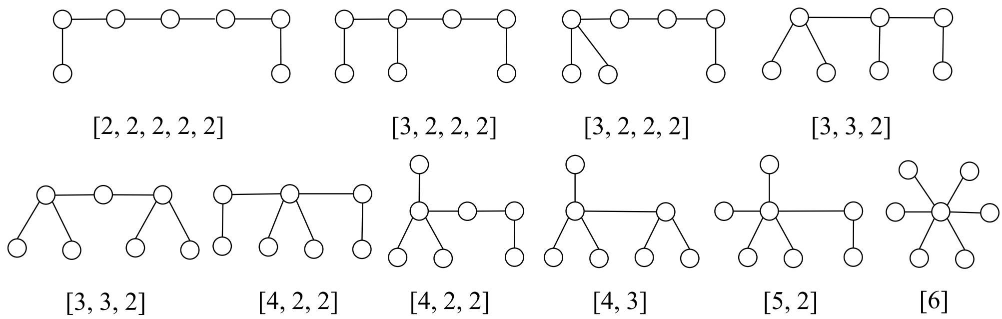


Figure 1 – All trees with $n = 7$ and their respective *IDS*.

It is possible to generate all valid IDS for a tree with n vertices. The sequences can be build recursively, computing each element of the sequence from the first one. Proposition 1 provides some necessary results to build the sequences.

Proposition 1. *Let T be a tree with order n and $IDS(T) = [d_1, \ldots, d_p]$ its internal degree sequence. Let $S = \sum_{i=1}^{p} d_i$. Then*

1. *the maximum value for d_i is*

$$d_i = \begin{cases} n - p, & i = 1 \\ \min\left\{\left(S - \sum_{j=1}^{i-1} d_j\right) - 2(p-i), d_{i-1}\right\}, & 1 < i \leq p; \end{cases}$$

2. *the minimum value for d_i is*

$$d_i = \begin{cases} \left\lceil \frac{S}{p} \right\rceil, & i = 1 \\ \left\lceil \frac{S - \sum_{j=1}^{i-1} d_j}{p-(i-1)} \right\rceil, & 1 < i \leq p. \end{cases}$$

Proof. Let T be a tree of order n and let $[d_1, d_2, \ldots, d_p]$ be the IDS of T. It is known that the sum of the degrees of the vertices of V is $2(n-1)$. The sum of the degrees of the p internal vertices of T is $S = \sum_{i=1}^{p} d_i = n + p - 2$.

1. The maximum degree of the first vertex of the sequence occurs when all the others have the minimum degree for an internal vertex (i.e. 2). Since $S = n + p - 2$, the first element d_1 has value $S - 2(p-1) = n - p$.

   When computing d_i, $1 < i \leq p$, the previous elements $d_1, \ldots, d_{i-1}$ are already determined. It is possible to consider a sequence $[d_i, \ldots, d_p]$ with sum of degrees $S - \sum_{j=1}^{i-1} d_j$. So, applying the previous reasoning and considering that, by definition, $d_i \leq d_{i-1}$, the result is obtained.

2. By Definition 1, $d_1 \geq \ldots \geq d_p$. In order to force d_1 to be minimum, the remaining degrees of S must be scattered in other positions of the sequence, obtaining the value S/p. If the remainder is zero then $d_1 = S/p$. Otherwise, let r be the remainder. The positions d_1, d_2, $\ldots$, d_r will have the value $S/P + 1$. The minimum value of d_1 is $\lceil S/p \rceil$.

   As in item(1), we consider $[d_1, \ldots, d_{i-1}]$ already determined, and the sum of the degrees $S - \sum_{j=1}^{i-1} d_j$. So, the result is obtained. □

The next proposition shows how any sequence realizes a tree that obeys a specific condition.

Proposition 2. *Given a non-increasing sequence of integers* $s = [d_1, d_2, \ldots, d_p]$, $d_i \neq 1$, *there exists always a tree* T *with* $n = \sum_{i=1}^{p} d_i - p + 2$ *vertices such that* $IDS(T) = s$.

Proof. The proof is constructive. By the definition of a valid sequence it is known that T has $n = \sum_{i=1}^{p} d_i - p + 2$ vertices and p internal vertices. The following steps build iteratively the tree T:

- Step 1: Let T be a tree with a central vertex v and d_1 vertices,

  $w_1, \ldots, w_{d_1}$, all adjacent to v.

- Step 2: For $i = 2, \ldots, p$ do

  Add $d_i - 1$ new vertices, all adjacent to a leaf w_j of T.

The graph T, resulting from Step 1, is a tree; as so, it has at least two leaves. Step 2 adds, at each iteration, $d_i - 1$ new vertices, all adjacent to the same vertex w_j. At this point, vertex w_j is no more a leaf, but at least one new leaf is created, maintaining the graph T as a tree. So, tree T has $d_1 + \sum_{i=2,p}(d_i - 1)$ leaves and p internal vertices. □

It is interesting to note that if the leaf w_j chosen in Step 2 is the most recently added leaf, the resulting tree is a caterpillar (a tree such that if all leaves and their incident edges are removed, the remainder of the graph forms a path).

Since the lexicographic order provides a total ordering for all the valid sequences of graphs with a given n, the proof of the following proposition is immediate.

Proposition 3. *The invariant* IDS *provides a total ordering for trees with n vertices.*

3 IDS AND ALGEBRAIC CONNECTIVITY

In 1987, Grone & Merris [4] started an investigation about connectivity of trees through $a(G)$, proving in [5] that it is possible to present a strict total ordering of trees with diameter 3. After that, a large number of results comparing trees via $a(G)$ have appeared in the literature as, for instance, Fallat & Kirkland [2] that extend the results of [5]. In this section, we compare some of the known results with those obtained using IDS.

Proposition 4. **[2]** *Let P_{d-1} be a path with length $d-2$ labeled from 1 to $d-1$. Let $T_d(k,\ell)$ be a tree with n vertices and diameter d, obtained from P_{d-1} by adding k pendant vertices adjacent to vertex 1 and ℓ pendant vertices adjacent to vertex $d-1$. Then*

$$a(T_d(k,\ell)) < a(T_d(k-1,\ell+1)), 1 \leq k \leq \left\lfloor \frac{n-d+1}{2} \right\rfloor.$$

The internal vertices of $T_d(k,\ell)$ are the vertices of the path. So, the IDS of $T_d(k,\ell)$ has $d-1$ elements and can be easily determined:

$$IDS(T_d(k,\ell)) = \big[\ell+1, k+1, \underbrace{2, 2, \ldots, 2}_{d-3}\big].$$

For this class, it is easy to prove that the ordering provided by IDS is a strict total ordering. Moreover, the orders given by $a(G)$ and IDS are isomorphic. However, this behavior of the two measures does not occur in general.

Yuan *et al.* [8] introduced six classes of trees and show that, for $n \geq 15$, $a(T_i) > a(T_j)$ if $1 \leq i < j \leq 6$ and T_i is any tree in the class C_i and T_j is any tree in the class C_j. These classes are defined below and we show the IDS of their elements.

Definition 3. *Let n, k, p and q be nonnegative integers with $3k+2p+q = n-1$. Let $T(k,p,q)$ be the tree of order n which contains a vertex v such that $T(k,p,q) - v = kK_{1,2} \cup pK_{1,1} \cup qK_1$.*

Let $\mathcal{T}$ be the class of all trees $T(k,p,d)$. The following subclasses of $\mathcal{T}$ were defined in [8]:

- $C_1 = \{T(0,0,n-1)\}$;
- $C_2 = \{T(0,1,n-3)\}$;
- $C_3 = \{T(0,p,q) \mid p \geq 2\}$;
- $C_4 = \{T(1,0,n-4)\}$;
- $C_5 = \{T(1,p,q) \mid p \geq 1\}$;
- $C_6 = \{T(k,p,q) \mid k \geq 2\}$.

Notice that $\{T(k,p,q) \mid n = 3k+2p+q+1\} = \cup_{i=1,6} C_i$.

For each n, classes C_1, C_2 and C_4 consist of only one tree. The trees belonging to C_3 and C_6 have the same algebraic connectivity, proved by Zhang [9] and Yuan *et al.* [8], respectively. Two important results about the ordering of all these trees are also presented by Yuan *et al.* [8] in Theorem 1.

Proposition 5. *If $T(k,p,q)$ is any tree in class C_3, then $a(T) = (3-\sqrt{5})/2$.*

Proposition 6. *If $T(k,p,q)$, $k \geq 2$, is any tree in class C_6, then $a(T) = 2-\sqrt{3}$.*

Theorem 1. *Let T_i be a tree in class C_i for $i = 1,\ldots,6$. Then:*

1. *For $n \geq 8$, $a(T_1) > a(T_2) > a(T_3) > a(T_4) > a(T_5) > a(T_6)$;*
2. *For $n \geq 15$ and $T \notin \cup_{i=1}^{6} C_i$, $a(T) < a(T_6) = 2 - \sqrt{3}$.*

For the six classes it is possible to establish the IDS of their elements. And more, since several trees have the same algebraic connectivity, the IDS provides a good insight to distinguish accurately between them.

Let T_i be a tree such that $T_i \in C_i$, $n \geq 8$. So, we have,

- $IDS(T_1) = [n-1]$;
- $IDS(T_2) = [n-2, 2]$;
- $IDS(T_3) = [n-p-1, \underbrace{2, 2, \ldots, 2}_{p}]$, $p \geq 2$;
- $IDS(T_4) = [n-3, 3]$;
- $IDS(T_5) = [n-p-3, 3, \underbrace{2, 2, \ldots, 2}_{p}]$, $p \geq 1$;
- $IDS(T_6) = [n-2k-p-1, \underbrace{3, 3, \ldots, 3}_{k}, \underbrace{2, 2, \ldots, 2}_{p}]$, $k \geq 2$.

The next proposition presents the comparison of the trees belonging to the six classes taking into account their IDS.

Proposition 7. *Let T_i be any tree of order n, $n \geq 8$, such that $T_i \in C_i$. Then*

1. $IDS(T_1) > IDS(T_2) > IDS(T_4)$*;*
2. $IDS(T_4) > IDS(T_3)$*;*
3. $IDS(T_4) > IDS(T_5)$*;*
4. $IDS(T_4) > IDS(T_6)$.

Proof.

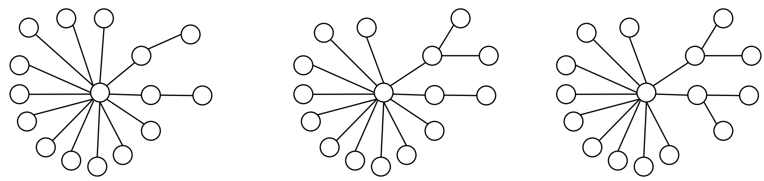
1. Each class C_1, C_2 and C_4 has just one element. It is immediate to state that
$$IDS(T_1) > IDS(T_2) > IDS(T_4).$$
2. The tree $T \in C_3$ with the greatest IDS has $p = 2$ and $IDS(T) = [n-4, 2, 2]$. As $IDS(T_4) = [n-3, 3]$,
$$IDS(T_4) > IDS(T_3).$$

3. Similar reasoning can be made for classes C_5 and C_6.

   The tree $T' \in C_5$ with the greatest IDS has $p = 1$. The tree $T'' \in C_6$ with the greatest IDS has $k = 2$ and $p = 0$. So, $IDS(T') = [n-4, 3, 2]$ and $IDS(T'') = [n-5, 3, 3]$. Comparing with the IDS of T_4

$$IDS(T_4) > IDS(T_5) \text{ and } IDS(T_4) > IDS(T_6). \qquad \square$$

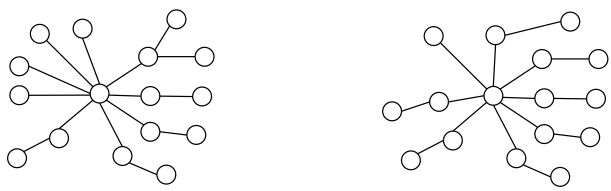
It is impossible to compare the trees belonging to classes C_3, C_5 and C_6 as a whole. Figure 2 shows how the IDS of the elements of these classes depend only on the types of their subtrees.



$T_a\,(0,2,11) \in C_3$ $\quad IDS(T_a) = [13,2,2]$

$T_b\,(1,1,10) \in C_5$ $\quad IDS(T_b) = [12,3,2]$

$T_c\,(2,0,9) \in C_6$ $\quad IDS(T_c) = [11,3,3]$



$T_d\,(1,4,4) \in C_5$ $\quad IDS(T_d) = [9,3,2,2,2,2]$

$T_e\,(0,7,1) \in C_3$ $\quad IDS(T_e) = [8,2,2,2,2,2,2,2]$

Figure 2 – Examples of trees belonging to classes C_3, C_5 and C_6.

However a very interesting result can be stated, showing an advantage of the invariant when the trees have the same algebraic connectivity.

Theorem 2. *Given n, the IDS of the elements of the classes C_i, $i = 1, \ldots, 6$, determine a strict total ordering of the trees $T(k, p, d)$.*

Proof. We must prove that the trees belonging to C_3, C_5 and C_6 have different IDS values, both within and among classes.

Let $T_i = T(k_i, p_i, q_i)$, $T_j = T(k_j, p_j, q_j) \in C_3$. We know that, for $T_3 \in C_3$, $IDS(T_3) = [n-p-1, \underbrace{2, 2, \ldots, 2}_{p}]$, $p \geq 2$. So,

$$IDS(T_i) = IDS(T_j) \Longrightarrow p_i = p_j.$$

The order n of T_i and T_j is given and $p_i = p_j$. As $n = 2p + q + 1$, $q_i = q_j$ and the trees are the same and all trees in C_3 have different IDS's.

The proof is analogous when $T_i, T_j \in C_5$ and $T_i, T_j \in C_6$.

We must now show that trees belonging to different classes have different IDS. It is known that

$$IDS(T_3) = [n - p_i - 1, \underbrace{2, \ldots, 2}_{p}],$$

$$IDS(T_5) = [n - p_j - 3, 3, \underbrace{2, \ldots, 2}_{p}] \text{ and}$$

$$IDS(T_6) = [n - 2k - p - 1, \underbrace{3, 3, \ldots, 3}_{k}, \underbrace{2, 2, \ldots, 2}_{p}],\ \ k \geq 2.$$

Let $T \in C_3$. The tree T cannot have the same IDS as a tree belonging to C_5 or C_6 because the tree T has no internal vertex of degree 3 which is mandatory in the two other classes.

Let $T' \in C_5$. The tree T' cannot have the same IDS as a tree belonging to C_6 because T' has exactly one internal vertex of degree 3 while the trees of C_6 have at least two vertices with such condition.

So, the invariant IDS provides a strict total ordering for trees belonging to the classes C_i, $i = 1, \ldots, 6$. $\square$

4 TREES WITH DIAMETER 4

An important observation about the trees belonging to $\mathcal{T}$, presented in the previous section, is that all of them have diameter less or equal 4. The approach of looking at trees with small diameter was taken up again by Wang & Tan [7]. They give continuity to the work of Yuan *et al.* [8], showing all trees of order $n \geq 45$ with algebraic connectivity in the interval $\left[(5 - \sqrt{21})/2\,,\ 2 - \sqrt{3}\right)$.

The subclass of trees of diameter 4 is defined as follows by Zhang [10].

Definition 4. *Let $p_1, p_2, \ldots, p_k$ be integers such that*

$$p_1 \geq p_2 \geq \ldots \geq p_k \geq 0,\ \ p_1 \geq p_2 > 0,\ \ k \geq 2,\ \ k + 1 + p_1 + \ldots + p_k = n.$$

Let $T(n, k, p_1, p_2, \ldots, p_k)$ be a tree with diameter 4, $N(c) = \{v_1, \ldots, v_k\}$, being c the unique center of the tree, $d(v_1) = p_1 + 1, \ldots, d(v_k) = p_k + 1$. Let **T** *be the class of all trees $T(n, k, p_1, p_2, \ldots, p_k)$.*

Wang & Tan[7] defined new subclasses of trees as follows:

- $C_1' = \{S(3, n-5)\}$, the tree of order n obtained by joining the center of $K_{1,3}$ to the center of $K_{1,n-5}$ with an edge;
- $C_2' = \{T(n, k, 3, 1, p_3, \ldots, p_k)\}$;

– $C'_3 = \{T(n, k, 3, 2, p_3, \ldots, p_k)\}$;

– $C'_4 = \{T(n, k, 3, 3, p_3, \ldots, p_k)\}$;

Notice that C'_1 is not a subclass of **T**.

Wang & Tan [7] proved interesting results about these classes:

Proposition 8. *Let T be a tree of order $n \geq 45$. If $T \notin \mathcal{T} \cup (\cup_{i=1}^{4} C'_i)$, then $a(T) < (5-\sqrt{21})/2$.*

Theorem 3. *If T is a tree of order $n \geq 45$, then $(5 - \sqrt{21})/2 \leq a(T) < 2 - \sqrt{3}$ if and only if $T \in \cup_{i=1}^{4} C'_i$.*

The authors were able to provide the three trees with the first, second and third largest algebraic connectivity in $T_n \backslash \mathcal{T}$. However few results are presented about the ordering of all trees with diameter 4.

Using the IDS, an interesting result can be shown. We define a subclass of **T**. Let $\mathbf{T}'$ be the class of all $T(n, k, p_1, p_2, \ldots, p_k)$ such that $k > p_1$, i.e., the set of all trees of diameter 4 such that the center of the tree has the greatest degree. It is possible to determine the IDS of a tree $T \in \mathbf{T}'$ directly from its notation.

Definition 5. *Let $T(n, k, p_1, \ldots, p_z, \ldots, p_k)$ with $k > p_1$. Let p_z, $2 \leq z \leq k$, be the last non-zero element of the notation. Then*

$$IDS(T) = [k, p_1 + 1, p_2 + 1, \ldots, p_z + 1].$$

Theorem 4. *Given n, the IDS of the elements of the class $\mathbf{T}'$ determine a strict total ordering of the trees $T(n, k, p_1, p_2, \ldots, p_k)$ with $k > p_1$.*

Proof. Let $T(n, k, p_1, p_2, \ldots, p_k) \in \mathbf{T}'$. Let S be the sequence $[n, k, p_1, p_2, \ldots, p_z, \ldots, p_k]$. It is immediate to conclude that S is a non-increasing sequence. The sequence $IDS(T)$ is build from S and its elements obey exactly the original ordering. So, given n, there is a unique tree with this sequence as IDS. □

Analogous result can be shown for all trees of diameter 4 such that the central vertex has the smallest degree.

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