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ARTICLES

A NOVEL APPROACH BASED ON INVERSE NON-RADIAL DEA PROCESS FOR DEALING WITH UNDESIRABLE OUTPUTS

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ABSTRACT. We present a novel approach to the inverse data envelopment analysis (DEA) process, based on the slacks-based measure (SBM) model in the presence of undesirable outputs. In the first step, we determine the minimum undesirable outputs levels that a decision-making unit (DMU) can produce, according to the level of inputs and desired outputs, based on weak disposability. We then introduce counterpart (hypothetical) units corresponding to each DMU. In the second step, we determine the optimal level of inputs and outputs for the newly created units, based on the SBM model in the presence of undesirable outputs. We present two new criteria models based on the SBM model, to compare the efficiency of the newly created unit in the inverse DEA process with the original unit. We demonstrate that the efficiency scores obtained from these two models are equal. Using the SBM-based inverse non-radial DEA process, we can use all inefficiency slacks related to all input and undesirable output components in the inverse non-radial DEA process to estimate inputs and outputs simultaneously.

Keywords: Data Envelopment Analysis, inverse DEA, undesirable output, efficiency, non-radial DEA.

1 INTRODUCTION

According to the World Environment Organization, adverse waste products and social activities, such as air pollutants and hazardous waste, are increasingly being produced in the environment and pose a serious threat to it. For example, the number of defective goods, amount of pollution and waste, or release of CO2 in the production process are all undesirable and should be reduced.

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Therefore, evaluating the performance of organizations that produce undesirable outputs without considering these outputs is not a logical or correct evaluation, and these outputs should be included in the performance evaluation (Scheel, 2001; Sueyoshi and Goto, 2010, 2012). In this regard, the DEA technique, which was first introduced by Charnes et al. (1978), is a suitable technique. When evaluating the performance of a DMU based on DEA models, we can use input or output-oriented models. In the input-oriented models, by decreasing the inputs (or increasing the outputs), we depict the inefficient unit on the efficiency frontier. In these models, the amount of outputs (or inputs) does not change. However, if we have an undesirable output among the output components, increasing these outputs will reduce the efficiency of the evaluated DMU. The DEA models are important for dealing with undesirable outputs. The DEA technique uses the Pareto optimality property in measuring efficiency. The efficient unit is denoted as non-dominated. In this method, if the DMU can maximize the production of one desirable output, minimize one of the undesirable outputs, or minimize one of the inputs, the DMUs can be regarded as efficient without taking into account the number of other parameters linked with it. Only if a DMU does not produce any undesirable outputs can it be considered efficient. Any amount of undesirable output generated should decrease the efficiency measures. However, since the production of desirable outputs normally comes with the generation of undesirable outputs, a certain quantity of undesirable outputs should be permitted because they will be generated until the desirable outputs are achieved. In this method, if a DMU generates a moderate amount of undesirable output, it still has a chance to be judged as efficient. The DMU is defined as inefficient if an excessive level of undesirable outputs is produced (Liu et al., 2010; Kao and Hwang, 2021).

DEA models with undesirable inputs and outputs have been frequently discussed in DEA literature. For instance, Shi et al. (2010) considered undesirable outputs for assessing energy efficiency in the Chinese manufacturing industry, and Yeh et al. (2010) compared the total factors of energy efficiency in China. Sueyoshi and Goto (2010) proposed a new DEA model to calculate the efficiency of electric fossil fuels, taking into account the CO2 produced by each production unit. Chen et al. (2017) proposed a stochastic network DEA model for evaluating Chinese airline efficiency under CO2 emissions and flight delays. Izadikhah and Saen (2018) developed a chance-constrained two-stage DEA model to present undesirable factors for evaluating the sustainability of supply chains. Ren et al. (2020) proposed a new chance-constrained DEA model to measure the energy and carbon emission efficiency of regional transportation systems in China.

One of the convenient methods proposed for dealing with undesirable outputs is the approach introduced by Kao and Hwang (2021). They provided a suitable model for measuring the effect of undesirable outputs on efficiency measurement by addressing the minimum level of such outputs that can be allowed for each DMU. They used the results to create a new efficiency frontier of production by introducing the concept of a counterpart (hypothetical) unit corresponding to each of the DMUs. The efficiency of a production unit was presented based on the minimum level of the undesirable output, and the new frontier was used to calculate the efficiency scores of a manufacturing unit and compare with other units with the minimum level of undesirable outputs. (For more detail see: Anderson, Daim, and Kim, 2008; Inman, Anderson, and Harmon, 2006;

Lim, Anderson, and Shott, 2015). The difference between these two efficiencies was then used to calculate the impact of creating too many undesirable outputs on a specific manufacturing unit's efficiency. Data transformation technologies can produce skewed efficiency measurements (Färe and Grosskopf, 2004, & Kao, 2020). The approaches that use the source data to determine efficiency are provided by Kao and Hwang (2021) for this purpose. The SBM model, one of Kao's (2017) three types of direct approaches, can distinguish between the observed amount and the optimal one, making it suitable for analyzing the effect of creating a large level of undesirable outputs in measuring efficiency. Therefore, the SBM idea is employed in this study to create models in inverse DEA in the presence of undesirable outputs.

DEA models can be divided into radial and non-radial categories for evaluating efficiency (see: Gerami et al. 2022). Radial models do not consider slacks in inputs and outputs, which are important components for evaluating efficiency. Radial approaches may not produce robust results in efficiency evaluation based on radial reduction of inputs and outputs. All inverse DEA models are radial models, and as a result, they may help DMs estimate the level of outputs (inputs) by ignoring the effects of slack variables corresponding to input and output components. However, radial models suffer from the limitation of being incapable of addressing the slacks in the model. Limited studies have used non-radial inverse DEA models. For instance, Jahanshahloo et al. (2014) proposed an inverse DEA model based on the Enhanced Russell Model, assuming constant efficiency scores among dimensions. Zhang and Cui (2020) proposed a general inverse DEA model for non-radial DEA, assuming unchanged overall efficiency scores but not for every dimension. They presented a basic form of non-radial inverse DEA models, which was nonlinear in a special case for the SBM model. They used nonlinear optimization methods to solve the proposed model and one-dimensional search as a suitable method to solve it.

We have developed a non-radial SBM model for estimating optimal inputs and outputs in the presence of undesirable outputs. The efficiency scores of each DMU are first obtained using the non-radial SBM model, and then a non-radial SBM model is presented under the inverse DEA process for simultaneous estimation of optimal inputs and outputs. The proposed approach can deal with both non-radial measurements and undesirable outputs. In the first step, the optimal level of undesirable outputs is obtained for each DMU using the approach proposed by Kao and Hwang (2021). In the second step, the inverse DEA process is performed using the non-radial SBM model to obtain the optimal levels of inputs and outputs for the created new units that have the same efficiency scores as their corresponding original units. We present two new criteria models for evaluating the efficiency of the created new units in the presence of counterpart units and show that the efficiency of the created new units in the inverse DEA process is equal to that of the original units.

2 LITERATURE REVIEW

2.1 The undesirable outputs in DEA

When additional input and output parameters are included in the evaluation, the DEA approach can yield higher efficiency scores for the DMUs being evaluated. This can lead to unreasonable results. Moreover, when considering undesirable outputs in the efficiency measurement of a DMU, the scores will either be equal to or higher than those that do not consider the undesirable outputs. To arrive at a more robust efficiency metric, we should develop traditional DEA models and provide suitable models. In many manufacturing processes, undesirable outputs may be generated unexpectedly along with the planned outputs. For instance, in the pulping process, wastewater or sewage is created, and in the combustion of fossil fuels, CO2 and NOx are formed. Although these undesirable outputs may not have a direct impact on the DMU generating them, they have a negative impact on the environment. The production of such outputs should be minimized to protect the environment. However, due to the costs of preventing the production of undesirable outputs, they are usually kept at the minimum standard level allowed by law. If the costs of preventing their production outweigh the costs of breaking the rules, the production units may be forced to control the creation of undesirable outputs. Undesirable outputs are often ignored in efficiency measurements, but understanding their impact on efficiency measurements is crucial to increase environmental awareness. There are numerous ways to determine the efficiency of a production unit, with the DEA framework being the most commonly used. This technique was first developed by Charnes et al. (1978), who created efficiency evaluation models based on the constant returns to scale (CRS) technology. Banker et al. (1984) further developed this method for the variable returns to scale (VRS) technology. DEA measures the relative efficiency of a set of DMUs that use multiple inputs to generate multiple outputs.

Scheel (2001) categorized methods of measuring the efficiency of DMUs with the presence of undesirable outputs into two categories: direct and indirect. Direct methods use original data, while indirect methods involve data transformations. The concept of weak disposability is directly linked to direct approaches (Färe et al., 1989; Liu et al., 2010). Inverse input methods, additive inverse, translated inverse, and multiplicative inverse methods are examples of indirect methods. The direct approaches are classified by Zhao et al. (2008) into the directional distance function (DDF), hyperbolic models (Kao, 2017), and the slacks-based measure (SBM). Vazquez-Rowe et al. (2010) used the SBM-DEA model in the presence of undesirable outputs to assess fisheries in Spain.

Undesirable outputs are considered in the evaluation of environmental efficiency, and Song et al. (2012) classified the methods into three categories including disposability-related methods, data transformation, and input reversal. In an extensive review of papers on performance benchmarking in the presence of undesirable outputs, five approaches are classified by Dakpo et al. (2016), including data transformation, materials balance principles, free disposability of the inputs, weak disposability of the undesirable outputs, as well as the two sub-technologies from the biproportional production model of Murty et al. (2012), and the natural mammalian performance benchmarking model. Wu et al. (2013) considered wasting water, emission of toxic gases, and the production of useless solid material as undesirable stochastic outputs in evaluating several provinces in China.

Dakpo et al. (2016) proposed the drawbacks of each method. They evaluated the efficiency of 136 winter wheat farms in Poland under six DEA models with the life cycle assessment approach. The methods included the ones that disregard the undesirable outputs, treat the undesirable outputs as inputs, impact rate, data transformation, You and Yan's (2011) ratio model, as well as the slacks-based model. The comparison among the models showed that the slacks-based model provided better results in differentiating the performance among the firms. However, they did not discuss weak disposability. Pishgar-Komleh et al. (2020) classified DEA models in the face of undesirable outputs as follows: a) ignoring undesirable outputs, b) treating undesirables as inputs to the DEA model, c) data transformation, d) impact rate, e) ratio model. They discussed several papers that used each method.

Cecchini et al. (2018) proposed a slack-based measure (SBM)-DEA model for analyzing the environmental efficiency of dairy cattle farms with undesirable outputs in Italy. Adenuga et al. (2018) used the directional output distance function DEA-based model to optimize both desirable and undesirable outputs on dairy farms in Ireland. Dong et al. (2018) considered the SBM-DEA model in the presence of undesirable outputs to evaluate the resource use in crop production in a province in China. Angulo-Meza et al. (2019) proposed a multi-objective DEA model to reduce the CF of organic blueberry orchards while maintaining high production levels.

Pishgar-Komleh et al. (2020) defined efficiency under different methods for incorporating undesirable outputs in an LCA+DEA framework. They applied the proposed approach to winter wheat production in Poland. Kao and Hwang (2021) proposed an approach for measuring the effects of undesirable outputs on the efficiency of production units. They developed a concept for determining the minimum amount of undesirable outputs that a DMU is allowed to generate based on the assertion of weak disposability and proposed a suitable approach for measuring the effect of undesirable outputs on the efficiency measurement.

2.2 Inverse DEA

The inverse DEA models, compared to the traditional DEA models, achieve the rate of output variations by increasing inputs while maintaining the efficiency level of the unit under evaluation. The inverse optimization problem is useful for the decision-maker (DM) to obtain information regarding resource allocation. The traditional DEA models focus only on efficiency evaluation. The idea of the inverse DEA was first introduced by Zhang and Cui (1999) and then formulated by Wei et al. (2000) based on multi-objective programming techniques. The inverse DEA models were then studied from various theoretical and practical perspectives. Hadi-Vencheh et al. (2008) proposed an inverse DEA model for estimating inputs. They initially increased the outputs and assumed that the efficiency of the unit under evaluation would not change. Zhang and Cui (2016) integrated twelve different scenarios into the inverse DEA models. Other applications of the inverse DEA include input estimation for resource allocation (Xiaoya & Jinchuan, 2008) and inter-temporal application (Jahanshahloo et al., 2015). Ghobadi (2018) presented inverse DEA models with fuzzy data. Ghiyasi (2017) developed inverse DEA models based on the concepts of cost and revenue efficiency. Other applications of the inverse DEA models include recent studies on sustainability (Hassanzadeh et al., 2018), enterprise merger (Amin et al., 2017a, b), revenue target management (Lin, 2010), and pricing strategy (Frija et al., 2011). Lertworasirikul et al. (2011) proposed inverse DEA models in the VRS technology. Amin et al. (2017b) developed a generalized inverse DEA model for restructuring, considering the number of original DMUs as pre-restrictions for producing new units. Lim (2016) proposed an inverse DEA model based on the frontier changes for new product target setting. He discusses an inverse DEA problem that considered expected changes in the production frontier in the future by adding the inverse optimization problem with a time series application of DEA to provide a tool for new product target setting practices. Wegener and Amin (2019) applied an inverse DEA model in the oil and gas industry. Emrouznejad et al. (2019) proposed an inverse DEA model to allocate undesirable outputs. Ghiyasi (2019) proposed a new criterion model, reducing the computational complications in the inverse DEA. Ghiyasi and Zhu (2020) proposed an inverse semi-oriented radial DEA in the presence of negative data. Amin and Ibn Boamah (2021) developed a two-stage inverse DEA approach for estimating potential merger gains in the US banking sector. Zeinodin and Ghobadi (2020) examined the merger issue under the inter-temporal dependence structure. Gerami et al. (2023) proposed a generalized inverse DEA model for firm restructuring based on value efficiency.

3 MEASURING THE EFFICIENCY IN THE PRESENCE OF UNDESIRABLE OUTPUTS BASED ON THE SBM NON-RADIAL MODEL

Suppose we have n decision units as $DMU_j = (x_j, y_j)$, $j = 1, \ldots, n$. The input and output vectors corresponding to DMU_j, $j = 1, \ldots, n$ are denoted as $x_j = (x_{1j}, \ldots, x_{mj})$ and $y_j = (y_{1j}, \ldots, y_{sj})$. Assume that s_i^x, $i = 1, \ldots, m$ and s_r^y, $r = 1, \ldots, s$ are the slacks of the inputs and outputs, respectively. λ_j is the intensity variable corresponding to DMU_j.

The SBM model for measuring the efficiency of DMU_k as a unit under evaluation in the variable returns to scale technology was presented by Tone (2001) as follows:

$$\begin{aligned}
&\min \frac{1 - \frac{1}{m}\sum_{i=1}^{m} \frac{s_i^x}{x_{ik}}}{1 + \frac{1}{s}\sum_{r=1}^{s} \frac{s_r^y}{y_{rk}}} \\
&s.t.\ \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^x = x_{ik},\ i = 1, \ldots, m, \\
&\qquad \sum_{j=1}^{n} \lambda_j y_{rj} - s_r^y = y_{rk},\ r = 1, \ldots, s, \\
&\qquad \sum_{j=1}^{n} \lambda_j = 1, \lambda_j \geq 0,\ s_i^x \geq 0, s_r^y \geq 0, i = 1, \ldots, m, r = 1, \ldots, s,\ \ j = 1, \ldots, n.
\end{aligned} \tag{1}$$

Suppose, $(\lambda^*, s^{x*}, s^{y*})$ where $s^{x*} = (s_1^{x*}, \ldots, s_m^{x*})$ and $s^{y*} = \left(s_1^{y*}, \ldots, s_s^{y*}\right)$ is an optimal solution of model (1).

Definition 1. *DMU_k is efficient in evaluation with model (1) if and only if the objective function score of model (1) is equal to one or equivalent to $s_i^{x*} = 0$, $i = 1, \ldots, m, s_r^{y*} = 0, r = 1, \ldots, s$.*

When there are undesirable outputs, such as $u_j = \left(u_{1j}, \ldots, u_{fj}\right), j = 1, \ldots, n$, model (1) cannot be used to assess the DMUs' efficiency. In this regard, Kao and Hwang (2021) proposed a new approach for efficiency evaluation in the presence of undesirable outputs. They first introduced the concept of a counterpart (hypothetic) unit and presented the efficiency based on the minimum level of undesirable output.

To obtain the amount of efficiency corresponding to each of the DMUs on the basis of the projection of the unit under evaluation on the new frontier created based on the counterpart units (for more detail see: Entani et al. 2002), they used this new frontier to calculate the efficiency scores that produce the least undesirable output. The difference between these two efficiencies is then used to calculate the impact of creating too many undesirable outputs on a production unit's efficiency. The counterpart units have the same input and output levels as the original units, but their undesirable output level is lower than that of the original units. To determine the level of undesirable outputs from these DMUs, assume that s_f^u, $f = 1, \ldots, h$ are slacks of the undesirable outputs, we can solve the following model (Kao and Hwang, 2021):

$$
\begin{aligned}
&\max \sum_{f=1}^{h} s_f^h \\
&s.t. \sum_{j=1}^{n} \lambda_j u_{fj} + s_f^u = u_{fk},\ f = 1, \ldots, h, \\
&\sum_{j=1}^{n} \lambda_j y_{rj} = y_{rk}, r = 1, \ldots, s, \\
&\lambda_j \geq 0, j = 1, \ldots, n, s_f^u \geq 0,\ f = 1, \ldots, h.
\end{aligned}
\tag{2}
$$

Suppose (λ^*, s^{u*}) where $s^{u*} = \left(s_1^{u*}, \ldots, s_f^{u*}\right)$ is an optimal solution of model (2). In model (2), the level of desirable outputs remains constant, and DMU_k is allowed to generate the lowest undesirable outputs as below:

$$u_{fk}^{cp} = u_{fk} - s_f^{u*}, f = 1, \ldots, h.$$

On the basis of the frontier constructed from the counterpart DMUs, namely $DMU_j^{cp} = \left(x_j, y_j, u_j^{cp}\right)$, $j = 1,\dots,n$, *the efficiencies can be calculated through the following model (Kao and Hwang, 2021):*

$$
\begin{aligned}
&\min \frac{1-\frac{1}{m+h}\left(\sum_{i=1}^{m}\frac{s_i^x}{x_{ik}}+\sum_{f=1}^{h}\frac{s_f^u}{u_{fk}}\right)}{1+\frac{1}{s}\sum_{r=1}^{s}\frac{s_r^y}{y_{rk}}}\\
&s.t.\ \sum_{j=1}^{n}\lambda_j x_{ij}+s_i^x = x_{ik},\ i=1,\dots,m,\\
&\sum_{j=1}^{n}\lambda_j u_{fj}^{cp}+s_f^u = u_{fk},\ f=1,\dots,h,\\
&\sum_{j=1}^{n}\lambda_j y_{rj}-s_r^y = y_{rk},\ r=1,\dots,s,\\
&\sum_{j=1}^{n}\lambda_j = 1, \lambda_j \geq 0,\ j=1,\dots,n,\\
&s_i^x \geq 0, s_f^u \geq 0, s_r^y \geq 0, i=1,\dots,m, f=1,\dots,h, r=1,\dots,s.
\end{aligned}
\tag{3}
$$

In model (3), $s_i^x,\ \ i=1,\dots,m,\ s_r^y,\ \ r=1,\dots,s,$ *and* $s_f^u, f=1,\dots,h,$ *are slacks of the inputs, desirable and undesirable outputs, respectively.* λ_j *is the intensity variable corresponding to* DMU_j*. In model (3), the undesirable outputs are reduced relative to the desirable output under weak disposability. Suppose,* $(\lambda^*, s^{x*}, s^{u*}, s^{y*})$ *where* $s^{x*} = (s_1^{x*},\dots,s_m^{x*})$, $s^{u*} = \left(s_1^{u*},\dots,s_h^{u*}\right)$ *and* $s^{y*} = \left(s_1^{y*},\dots,s_s^{y*}\right)$ *is an optimal solution of model (3).*

Definition 2. DMU_k *is called efficient in evaluation with model (3) if and only if the objective function score of model (3) is equal to one or equivalent* $s_i^{x*} = 0$, $i=1,\dots,m, s_r^{y*} = 0, r=1,\dots,s,$ $s_f^{u*} = 0, f=1,\dots,h.$

Now, based on model (3), we present the input-oriented non-radial SBM model as follows:

$$
\begin{aligned}
&\psi_k = \min 1-\frac{1}{m+h}\left(\sum_{i=1}^{m}\frac{s_i^x}{x_{ik}}+\sum_{f=1}^{h}\frac{s_f^u}{u_{fk}}\right)\\
&s.t.\ \sum_{j=1}^{n}\lambda_j x_{ij}+s_i^x = x_{ik},\ i=1,\dots,m,\\
&\sum_{j=1}^{n}\lambda_j u_{fj}^{cp}+s_f^u = u_{fk},\ f=1,\dots,h,\\
&\sum_{j=1}^{n}\lambda_j y_{rj} \geq y_{rk}, r=1,\dots,s\\
&\sum_{j=1}^{n}\lambda_j = 1, \lambda_j \geq 0,\ j=1,\dots,n,\\
&s_i^x \geq 0, s_f^u \geq 0, i=1,\dots,m,\ f=1,\dots,h.
\end{aligned}
\tag{4}
$$

In model (3), s_i^x, $i = 1,\ldots,m$, and s_f^u, $f = 1,\ldots,h$, represent the slacks of the inputs and undesirable outputs, respectively. λ_j is the intensity variable corresponding to DMU_j. Model (4) obtains the efficiency scores of the DMU under evaluation. Suppose, $(\lambda^*, s^{x*}, s^{u*})$ is an optimal solution of model (4) where $s^{x*} = (s_1^{x*},\ldots,s_m^{x*})$ and $s^{u*} = (s_1^{u*},\ldots,s_h^{u*})$

Definition 3. *DMU_k is efficient in evaluation with model (4) if and only if the objective function score of model (4) is equal to one, i.e., $\psi_k = 1$ or equivalently, $s_i^{x*} = 0, i = 1,\ldots,m$ and $s_f^{u*} = 0$, $f = 1,\ldots,h$.*

4 SBM-BASED INVERSE NON-RADIAL DEA

We present a new approach based on the non-radial SBM model for dealing with the presence of undesirable outputs. We use the input-oriented SBM model, which calculates the efficiency of DMUs based only on the slack values of the inputs and undesirable outputs. Therefore, we first solve model (4) to obtain the efficiency and slacks of the inputs and undesirable outputs. Suppose $(\lambda^*, s^{x*}, s^{u*})$ is an optimal solution of model (4) where $s^{x*} = (s_1^{x*},\ldots,s_m^{x*})$ and $s^{u*} = (s_1^{u*},\ldots,s_h^{u*})$, To estimate the input, desirable output, and output components in the inverse DEA process, we created a new corresponding DMU_k as follows, taking into account the level of inputs and outputs:

$$DMU_k^{new} = (x_k + \Delta x_k, y_k + \Delta y_k, u_k + \Delta u_k).$$

where $\Delta x_k, \Delta y_k$, Δu_k, show the deviations of the inputs, desirable and undesirable outputs of DMU_k, respectively. In the inverse DEA process based on the SBM model, the optimal inputs and outputs of the newly created DMU, i.e. DMU_k^{new}, are determined in such a way that this unit's efficiency is the same as that of the corresponding unit, i.e. DMU_k. To obtain the optimal inputs and outputs of the newly created unit, we solve the following model in the inverse non-radial DEA process:

$$\begin{aligned}
\varphi_k^{inv} = \min\ & 1 - \frac{1}{m+h}\left(\sum_{i=1}^{m} \frac{s_i^{x-inv}}{x_{ik}+\Delta x_{ik}} + \sum_{f=1}^{h} \frac{s_f^{u-inv}}{u_{fk}+\Delta u_{fk}}\right) \\
s.t.\ & \sum_{j=1}^{n} \lambda_j x_{ij} + \lambda_k^{new}(x_{ik}+\Delta x_{ik}) + s_i^{x-inv} = x_{ik}+\Delta x_{ik},\ i = 1,\ldots,m, \\
& \sum_{j=1}^{n} \lambda_j u_{fj}^{cp} + \lambda_k^{new}\left(u_{fk}+\Delta u_{fk}\right) + s_f^{u-inv} = u_{fk}+\Delta u_{fk},\ f = 1,\ldots,h, \\
& \sum_{j=1}^{n} \lambda_j y_{rj} + \lambda_k^{new}(y_{rk}+\Delta y_{rk}) \geq y_{rk}+\Delta y_{rk},\ r = 1,\ldots,s, \\
& \sum_{j=1}^{n} \lambda_j + \lambda_k^{new} = 1, \lambda_k^{new} \geq 0,\ \lambda_j \geq 0, j = 1,\ldots,n, \\
& x_{ik}+\Delta x_{ik} \geq 0,\ u_{fk}+\Delta u_{fk} \geq 0,\ i = 1,\ldots,m, f = 1,\ldots,h, \\
& y_{rk}+\Delta y_{rk} \geq 0,\ r = 1,\ldots,s, \\
& s_i^{x-inv} \geq 0, s_f^{u-inv} \geq 0,\ i = 1,\ldots,m,\ f = 1,\ldots,h.
\end{aligned} \tag{5}$$

In model (5), $s_i^{x-inv},\ \ i=1,\ldots,m$, and $s_f^{u-inv}, f=1,\ldots,h$, are the slacks of the inputs and undesirable outputs, respectively. λ_j is the intensity variable corresponding to DMU_j, and λ_k^{new} is the intensity variable corresponding to DMU_k^{new}. Additionally, $\Delta x_{ik},\ i=1,\ldots,m, \Delta y_{rk}, r=1,\ldots,s$, $\Delta u_{fk},\ f=1,\ldots,h$, represent the deviations of the inputs, desirable and undesirable outputs of DMU_k, respectively. The constraint $x_{ik}+\Delta x_{ik}\geq 0,\ u_{fk}+\Delta u_{fk}\geq 0, i=1,\ldots,m, f=1,\ldots,h$, $y_{rk}+\Delta y_{rk}\geq 0, r=1,\ldots,s$, ensures that the optimal level the newly created unit's inputs and outputs, i.e. DMU_k^{new}, is non-negative.

If we present model (5) with the slack values corresponding to the desirable output components, the linearization process of is not easy. Therefore, we only present the input-oriented model (5) with the slack values corresponding to the input and undesirable output components. Model (5) is a nonlinear model used to determine the efficiency of the created new unit, i.e., DMU_k^{new}. However, the optimal level of the new unit's inputs and outputs, i.e., DMU_k^{new}, is unclear, making it difficult to solve model (5) and determine its efficiency. Additionally, the optimal solution of this model may be zero. To determine the efficiency score of the created new unit, i.e., DMU_k^{new}, we need to obtain the optimal level of inputs and outputs corresponding to it. However, due to the non-linear shape of model (5), we need to present a new linear model. This model assumes that the created new unit's efficiency, i.e., DMU_k^{new} , is the same as that of the corresponding original unit, i.e., DMU_k, and that we obtain the efficiency score corresponding to DMU_k from model (4). To do this, we need to present the model in a way that includes the optimal values of the slack variables regarding the input and undesirable output components resulting from model (4) corresponding to the unit under evaluation, i.e., DMU_k. This ensures that the optimal values of the slack variables regarding the input and undesirable output components corresponding to the unit under evaluation, i.e., DMU_k, from model (4) are equal to the ones of the created new unit, i.e., DMU_k^{new}, from model (5) in a special case:

$$\frac{s_i^{x-inv}}{x_{ik}+\Delta x_{ik}}=\frac{s_i^{x*}}{x_{ik}},\ i=1,\ldots,m, \frac{s_f^{u-inv}}{u_{fk}+\Delta u_{fk}}=\frac{s_f^{u*}}{u_{fk}},\ i=1,\ldots,m, f=1,\ldots,h. \tag{6}$$

In this case, we can guarantee that the created new unit's efficiency, i.e. DMU_k^{new} is the same as the one of the corresponding original unit, i.e. DMU_k, and with this assumption, the created new unit's optimal inputs and outputs are obtained. Equation (6) is equivalent to the following equation:

$$s_i^{x-inv}=s_i^{x*}\left(1+\frac{\Delta x_{ik}}{x_{ik}}\right),\ i=1,\ldots,m, s_f^{u-inv}=s_u^{u*}\left(1+\frac{\Delta u_{fk}}{u_{fk}}\right),\ f=1,\ldots,h. \tag{7}$$

We can now use the slacks values of the input and undesirable output components corresponding to the newly created unit, i.e., DMU_k^{new} , in the SBM-based inverse non-radial DEA process to determine the optimal level of this unit's inputs and outputs. We propose model (8) as follows:

$$
\begin{aligned}
\psi_k^{inv} = \min\ & \left(\sum_{i=1}^{m} \Delta x_{ik} + \sum_{f=1}^{h} \Delta u_{fk}\right) \\
s.t.\ & \sum_{j=1}^{n} \mu_j x_{ij} + s_i^{x*}\left(1 + \frac{\Delta x_{ik}}{x_{ik}}\right) = x_{ik} + \Delta x_{ik},\ i = 1,\dots,m, \\
& \sum_{j=1}^{n} \mu_j u_{fj}^{cp} + s_f^{u*}\left(1 + \frac{\Delta u_{fk}}{u_{fk}}\right) = u_{fk} + \Delta u_{fk},\ f = 1,\dots,h, \\
& \sum_{j=1}^{n} \mu_j y_{rj} \geq y_{rk} + \Delta y_{rk},\ r = 1,\dots,s, \\
& \sum_{j=1}^{n} \mu_j = 1, \mu_j \geq 0,\ j = 1,\dots,n, \\
& x_{ik} + \Delta x_{ik} \geq 0, u_{fk} + \Delta u_{fk} \geq 0, i = 1,\dots,m, f = 1,\dots,h, \\
& y_{rk} + \Delta y_{rk} \geq 0, r = 1,\dots,s.
\end{aligned}
\tag{8}
$$

Model (8) obtains the minimum level of inputs and undesirable outputs from the created new unit, i.e. DMU_k^{new}, in a manner that the created new unit's efficiency, i.e. DMU_k^{new}, is equal to that of the corresponding unit, i.e. DMU_k. In this model, we use the optimal values of the slack variables regarding the input and undesirable output components corresponding to the created new unit., i.e. $DMU_k^{new}$from model (5), which we obtain in equation (7), according to the optimal values of the slack variables regarding the input and undesirable output components corresponding to the unit under evaluation, i.e. DMU_k from model (4). We apply model (8) to determine the minimum level of inputs and undesirable outputs from the created new unit, i.e., DMU_k^{new}. In model (8), the optimal level of inputs, desirable outputs, and undesirable outputs of the created new unit, i.e. DMU_k^{new}, is determined based on the counterpart DMUs, namely, $DMU_j^{cp} = \left(x_j, y_j, u_j^{cp}\right)$, $j = 1,\dots,n$.

Suppose $(\mu^*, \Delta x_k{}^*, \Delta y_k{}^*,\ \Delta u_k{}^*)$ such that $\Delta x_k{}^* = (\Delta x_{1k}{}^*,\dots,\Delta x_{mk}{}^*), \Delta y_k{}^* = (\Delta y_{1k}{}^*,\dots,\Delta y_{sk}{}^*)$ and $\Delta u_k{}^* = (\Delta u_{1k}{}^*,\dots,\Delta u_{hk}{}^*)$ is an optimal solution of model (8). In this case, the optimal levels of the created new unit's inputs and outputs corresponding to i.e. DMU_k based on model (8) are considered as follows:

$$DMU_k^{new} = (x_k + \Delta x_k{}^*, y_k + \Delta y_k{}^*,\ u_k + \Delta u_k{}^*)\,.$$

$\Delta x_k{}^*, \Delta y_k{}^*,\ \Delta u_k{}^*$show the optimal deviations of the inputs, desired outputs, and undesirable outputs of DMU_k, respectively. In the following, to show that the created new unit's efficiency, i.e. DMU_k^{new}, in the inverse non-radial DEA process is the same as the one of the original unit corresponding to it, first we provide the following criterion model:

$$
\begin{aligned}
\psi_k^{cr} = \min\ & 1-\frac{1}{m+h}\left(\sum_{i=1}^{m}\frac{s_i^{x-cr}}{x_{ik}+\Delta x_{ik}{}^{*}}+\sum_{f=1}^{h}\frac{s_f^{u-cr}}{u_{fk}+\Delta u_{fk}{}^{*}}\right)\\
s.t.\ & \sum_{j=1}^{n}\gamma_j x_{ij}+s_i^{x-cr}=x_{ik}+\Delta x_{ik}{}^{*}, i=1,\ldots,m,\\
& \sum_{j=1}^{n}\gamma_j u_{fj}^{cp}+s_f^{u-cr}=u_{fk}+\Delta u_{fk}{}^{*}, f=1,\ldots,h,\\
& \sum_{j=1}^{n}\gamma_j y_{rj}\geq y_{rk}+\Delta y_{rk}{}^{*}, r=1,\ldots,s,\\
& \sum_{j=1}^{n}\gamma_j=1, \gamma_j\geq 0, j=1,\ldots,n,\\
& s_i^{x-cr}\geq 0, s_f^{u-cr}\geq 0, i=1,\ldots,m, f=1,\ldots,h.
\end{aligned}
\tag{9}
$$

Model (9) obtains the created new unit's efficiency score, i.e. DMU_k^{new} , based on the slack variables of the input and undesirable output components in the structure of the SBM model. In Theorem (1), we show that the score of the optimal objective function obtained from model (9) in the evaluation DMU_k^{new} and the optimal score obtained from model (4) in the evaluation DMU_k are equal. In other words, the efficiency scores of the unit under evaluation, i.e. DMU_k, and its corresponding unit in the inverse non-radial DEA process, i.e. DMU_k^{new}, are equal.

Theorem 1. *The optimal objective function scores of models (9) is less than or equal to the optimal objective function scores of models (4).*

Proof. As stated earlier, to solve model (9), we need to first solve model (8) and use the optimal solution obtained from model (8) to solve model (9). Similarly, to solve model (8), we need to solve model (4) and use the optimal solution obtained from model (4) to solve model (8). Let $(\psi_k, \lambda^*, s^{x*}, s^{u*})$ be a desired optimal solution of model (4). Based on this solution, we solve model (8) and let $(\mu^*, \Delta x_k{}^*, \Delta y_k{}^*,\ \Delta u_k{}^*)$ be the desired optimal solution of model (8). Next, based on this solution, we solve model (9) and let $\left(\psi_k^{cr}, \gamma^*, s^{x-cr*}, s^{u-cr*}\right)$ be the desired optimal solution of model (9).

To prove the theorem, we need to show that $\psi_k^{cr} = \psi_k$. Given that $(\mu^*, \Delta x_k{}^*, \Delta y_k{}^*, \Delta u_k{}^*)$ is a desired optimal solution of model (8), it satisfies in constraints of model (8). Therefore, we have

$$\begin{aligned}
&\sum_{j=1}^{n} \mu_j^* x_{ij} + s_i^{x*}\left(1+\frac{\Delta x_{ik}{}^*}{x_{ik}}\right) = x_{ik} + \Delta x_{ik}{}^*,\ i=1,\ldots,m,\\
&\sum_{j=1}^{n} \mu_j^* u_{fj}^{cp} + s_f^{u*}\left(1+\frac{\Delta u_{fk}{}^*}{u_{fk}}\right) = u_{fk} + \Delta u_{fk}{}^*,\ f=1,\ldots,h,\\
&\sum_{j=1}^{n} \mu_j^* y_{rj} \geq y_{rk} + \Delta y_{rk}{}^*,\ r=1,\ldots,s,\\
&\sum_{j=1}^{n} \mu_j^* = 1, \mu_j^* \geq 0,\ j=1,\ldots,n.
\end{aligned} \tag{10}$$

$(\lambda^*, s^{x*}, s^{u*})$ is a desired optimal solution of model (4) which was used to solve model (8).

Now we get a feasible solution for model (9). For this purpose, we put

$$\begin{aligned}
&s_i^{x-cr} = s_i^{x*}\left(1+\frac{\Delta x_{ik}{}^*}{x_{ik}}\right),\ i=1,\ldots,m,\\
&s_f^{u-cr} = s_f^{u*}\left(1+\frac{\Delta u_{fk}{}^*}{u_{fk}}\right),\ f=1,\ldots,h,\\
&\gamma_j = \mu_j^*, j=1,\ldots,n,
\end{aligned}$$

According to the relations (10) and the constraints of model (9), it is clear that $\left(s_i^{x-cr} : i=1,\ldots m, s_f^{u-cr} : f=1,..h,\ \gamma_j : j=1,\ldots,n\right)$ satisfies the constraints of model (9), and therefore is a feasible solution for model (9). The objective function score of model (9) for this feasible solution is as follows:

$$1-\frac{1}{m+h}\left(\sum_{i=1}^{m}\frac{s_i^{x*}\left(1+\frac{\Delta x_{ik}{}^*}{x_{ik}}\right)}{x_{ik}+\Delta x_{ik}{}^*}+\sum_{f=1}^{h}\frac{s_f^{u*}\left(1+\frac{\Delta u_{fk}{}^*}{u_{fk}}\right)}{u_{fk}+\Delta u_{fk}{}^*}\right) = 1-\frac{1}{m+h}\left(\sum_{i=1}^{m}\frac{s_i^{x*}}{x_{ik}}+\sum_{f=1}^{h}\frac{s_f^{u*}}{u_{fk}}\right) = \psi_k \tag{11}$$

According to relations (11), the optimal objective function scores of models (9) is less than or equal to the optimal objective function scores of models (4), this is attributed to the fact that the objective function score of model (9) for a feasible solution is equal to the optimal objective function score of model (4), and both models are of the minimization type. Then $\psi_k^{cr} \leq \psi_k$.

Conversely, suppose $\left(\psi_k^{cr}, \gamma^*, s^{x-cr*}, s^{u-cr*}\right)$ is a desired optimal solution of model (9). Then it satisfies the constraints of model (9).

$$\begin{aligned}
&\sum_{j=1}^{n} \gamma_j{}^* x_{ij} + s_i^{x-cr*} = x_{ik} + \Delta x_{ik}{}^*, i = 1, \ldots, m, \\
&\sum_{j=1}^{n} \gamma_j{}^* u_{fj}^{cp} + s_f^{u-cr*} = u_{fk} + \Delta u_{fk}{}^*, f = 1, \ldots, h, \\
&\sum_{j=1}^{n} \gamma_j{}^* y_{rj} \geq y_{rk} + \Delta y_{rk}{}^*, r = 1, \ldots, s, \\
&\sum_{j=1}^{n} \gamma_j{}^* = 1, \gamma_j{}^* \geq 0, j = 1, \ldots, n,
\end{aligned}$$

To solve model (9), we must first solve model (8) and use the optimal solution obtained from model (8) to solve model (9). In model (8), we replace the variables $x_{ik} + \Delta x_{ik}{}^*$, $i = 1, \ldots, m, u_{fk} + \Delta u_{fk}{}^*$, $f = 1, \ldots, h$, with these scores, then we will have

$$\sum_{j=1}^{n} \mu_j x_{ij} + s_i^{x*} \left(1 + \frac{\Delta x_{ik}{}^*}{x_{ik}}\right) = \sum_{j=1}^{n} \gamma_j{}^* x_{ij} + s_i^{x-cr*}, i = 1, \ldots, m,$$

$$\sum_{j=1}^{n} \mu_j u_{fj}^{cp} + s_f^{u*} \left(1 + \frac{\Delta u_{fk}{}^*}{u_{fk}}\right) = \sum_{j=1}^{n} \gamma_j{}^* u_{fj}^{cp} + s_f^{u-cr*}, f = 1, \ldots, h,$$

Based on the relations above, we can derive a special case where we can establish a relation between the scores s_i^{x-cr*} and $s_i^{x*}, i = 1, \ldots, m$, as well as a relation between the scores s_f^{u-cr*}, s_f^{u*}, $f = 1, \ldots, h$, we put

$$s_i^{x*} = s_i^{x-cr*} \left(\frac{x_{ik}}{x_{ik} + \Delta x_{ik}{}^*}\right), \quad i = 1, \ldots, m, s_f^{u*} = s_f^{u-cr*} \left(\frac{u_{fk}}{u_{fk} + \Delta u_{fk}{}^*}\right), f = 1, \ldots, h, \mu_j = \gamma_j{}^*, j = 1, \ldots, n,$$

Given that $\left(s_i^{x*} \ : i = 1, \ldots m, s_f^{u*} \ : f = 1, ,..h\right)$ is an optimal solution of model (4) which we used to solve model (8). Then we have

$$\begin{aligned}
&1 - \frac{1}{m+h} \left(\sum_{i=1}^{m} \frac{s_i^{x*}}{x_{ik}} + \sum_{f=1}^{h} \frac{s_f^{u*}}{u_{fk}}\right) = \\
&1 - \frac{1}{m+h} \left(\sum_{i=1}^{m} \frac{s_i^{x-cr*}\left(\frac{x_{ik}}{x_{ik}+\Delta x_{ik}{}^*}\right)}{x_{ik}} + \sum_{f=1}^{h} \frac{s_f^{u-cr*}\left(\frac{u_{fk}}{u_{fk}+\Delta u_{fk}{}^*}\right)}{u_{fk}}\right) = \\
&1 - \frac{1}{m+h} \left(\sum_{i=1}^{m} \frac{s_i^{x-cr*}}{x_{ik} + \Delta x_{ik}{}^*} + \sum_{f=1}^{h} \frac{s_f^{u-cr*}}{u_{fk} + \Delta u_{fk}{}^*}\right) = \psi_k^{cr},
\end{aligned}$$

Based on the relations above, we can conclude that the optimal objective function scores of models (4) is less than or equal to the optimal objective function scores of models (9) because the objective function score of the model (4) for a feasible solution is equal to the optimal objective function score of model (9), and both models are of the minimization type. Therefore, we have $\psi_k \leq \psi_k^{cr}$.

From these results, we can deduce that $\psi_k^{cr} = \psi_k$. The proof of the theorem is complete. □

We will now calculate the efficiency score of the newly created unit, denoted as $DMU_k^{new} = \left(x_k + \Delta x_k{}^*, y_k + \Delta y_k{}^*,\ u_k + \Delta u_k{}^*\right)$, based on model (5). Assume that $\left(\mu^*, \Delta x_k{}^*, \Delta y_k{}^*,\ \Delta u_k{}^*\right)$ is an optimal solution of model (8). We put the following values in model (5):

$$\Delta x_{ik} = \Delta x_{ik}{}^*,\ i = 1,\ldots,m, \Delta y_{rk} = \Delta y_{rk}{}^*,\ r = 1,\ldots,s, \Delta u_{fk} = \Delta u_{fk}{}^*,\ f = 1,\ldots,h.$$

Therefore, the new criteria model for evaluating the efficiency of DMU_k^{new} based on model (5) can be presented as follows:

$$\begin{aligned}
\psi_k^{inv} = \min\ & 1 - \frac{1}{m+h}\left(\sum_{i=1}^{m}\frac{s_i^{x-inv}}{x_{ik}+\Delta x_{ik}{}^*} + \sum_{f=1}^{h}\frac{s_f^{u-inv}}{u_{fk}+\Delta u_{fk}{}^*}\right)\\
s.t.\ & \sum_{j=1}^{n}\lambda_j x_{ij} + \lambda_k^{new}\left(x_{ik}+\Delta x_{ik}{}^*\right) + s_i^{x-inv} = x_{ik}+\Delta x_{ik}{}^*, i = 1,\ldots,m,\\
& \sum_{j=1}^{n}\lambda_j u_{fj}^{cp} + \lambda_k^{new}\left(u_{fk}+\Delta u_{fk}{}^*\right) + s_f^{u-inv} = u_{fk}+\Delta u_{fk}{}^*, f = 1,\ldots,h,\\
& \sum_{j=1}^{n}\lambda_j y_{rj} + \lambda_k^{new}\left(y_{rk}+\Delta y_{rk}{}^*\right) \geq y_{rk}+\Delta y_{rk}{}^*, r = 1,\ldots,s,\\
& \sum_{j=1}^{n}\lambda_j + \lambda_k^{new} = 1, \lambda_k^{new} \geq 0, \lambda_j \geq 0, j = 1,\ldots,n,\\
& s_i^{x-inv} \geq 0, s_f^{u-inv} \geq 0, i = 1,\ldots,m, f = 1,\ldots,h.
\end{aligned} \tag{12}$$

Theorem 2. Suppose that $\left(\psi_k^{inv}, \lambda^*,\ \lambda_{n+1}^*, s^{x-inv*}, s^{u-inv*}\right)$ and $\left(\psi_k^{cr}, \gamma^*, s^{x-cr*}, s^{u-cr*}\right)$ are the optimal solutions of models (12) and (9), respectively. Then, we have $\psi_k^{cr} = \psi_k$.

Proof. Let $\left(\lambda^*, \lambda_k^{new*}, s^{x-inv*}, s^{u-inv*}\right)$ be an optimal solution of model (12), and assume that DMU_k^{new} is not efficient based on model (12). In this case, we will have $\lambda_k^{new*} = 0$.

we can present the optimal solution by considering the constrains of model (12) as follows:

$$\begin{aligned}
& \sum_{j=1}^{n}\lambda_j^* x_{ij} + s_i^{x-inv*} = x_{ik}+\Delta x_{ik}{}^*, i = 1,\ldots,m,\\
& \sum_{j=1}^{n}\lambda_j^* u_{fj}^{cp} + s_f^{u-inv*} = u_{fk}+\Delta u_{fk}{}^*, f = 1,\ldots,h,\\
& \sum_{j=1}^{n}\lambda_j^* y_{rj} \geq y_{rk}+\Delta y_{rk}{}^*, r = 1,\ldots,s,\\
& \sum_{j=1}^{n}\lambda_j^* = 1.
\end{aligned} \tag{13}$$

By placing $s_i^{x-cr} = s_i^{x-inv*}, i = 1,\dots,m, s_f^{u-cr} = s_f^{u-inv*}, f = 1,\dots,h, \gamma_j = \lambda_j^*, j = 1,\dots,n,$

By considering the constraints in model (9) and relations (13), it is possible to obtain a feasible solution for model (9). The objective function score of model (9) for this feasible solution will be equal to ψ_k^{inv}. From the optimization point of view, since model (9) is a minimization type, its optimal objective function score will be ψ_k^{cr}, we will therefore have:

$$\psi_k^{cr} \leq \psi_k^{inv}. \tag{14}$$

On the other hand, suppose that $\left(\gamma^*, s^{x-cr*}, s^{u-cr*}\right)$ is an optimal solution of model (9). Now consider the constraints of model (9) for this optimal solution as follows:

$$\begin{aligned}
&\sum_{j=1}^{n} \gamma_j^* x_{ij} + s_i^{x-cr*} = x_{ik} + \Delta x_{ik}{}^*, i = 1,\dots,m,\\
&\sum_{j=1}^{n} \gamma_j^* u_{fj}^{cp} + s_f^{u-cr*} = u_{fk} + \Delta u_{fk}{}^*, f = 1,\dots,h,\\
&\sum_{j=1}^{n} \gamma_j^* y_{rj} \geq y_{rk} + \Delta y_{rk}{}^*, r = 1,\dots,s,\\
&\sum_{j=1}^{n} \gamma_j^* = 1, \gamma_j^* \geq 0, j = 1,\dots,n.
\end{aligned} \tag{15}$$

By placing

$$s_i^{x-inv} = s_i^{x-cr*}, i = 1,\dots,m, s_f^{u-inv} = s_f^{u-cr*}, f = 1,\dots,h, \lambda_j = \gamma_j^*, j = 1,\dots,n, \lambda_k^{new} = 0.$$

Given the constraints in model (12) and the relationships in (15), we can obtain a feasible solution for model (12). The objective function score of model (12) for this feasible solution will be equal to ψ_k^{cr}. From the optimization point of view, considering that model (12) is a model of minimization type, and its optimal objective function score is equal to ψ_k^{inv}, we will therefore have:

$$\psi_k^{inv} \leq \psi_k^{cr}. \tag{16}$$

According to the relations (14) and (16), we have:

$$\psi_k^{inv} = \psi_k^{cr}.$$

Therefore, the optimal objective function scores of models (9) and (12) are equal.

If this unit is efficient in evaluating DMU_k^{new} based on model (12), the objective function score of model (12) will be equal to one. In this case, model (12) is equivalent to the input-oriented SBM model or model (4), which we use to evaluate $n+1$ DMUs including the counterpart DMUs, namely $DMU_j^{cp} = \left(x_j, y_j, u_j^{cp}\right)$, $j = 1,\dots,n$, and $DMU_k^{new} = (x_k + \Delta x_k{}^*, y_k + \Delta y_k{}^*,\ u_k + \Delta u_k{}^*)$. Therefore, the objective function score of model (4) in the evaluation of this $n+1$ DMU will be equal to one. According to Theorem (1), the optimal objective function score obtained from models (4) and (9) are equal. Therefore, the optimal objective function score of model (9) is also equal to one, meaning DMU_k^{new} is also efficient in evaluation with model (9). Therefore, if $DMU_k^{new}$is efficient in evaluation with model (12), it also will be an efficient unit in evaluation with model (9). The optimal objective function scores of models (9) and (12) are then equal, and the proof of the theorem is complete. □

If the efficiency frontier remains unchanged in the presence of newly created DMUs, or in other words, the new units can be expressed as a combination of the original observed DMUs, then $\lambda_k^{new} = 0$ in model (12). Model (12) calculates the efficiency score of the created new unit, namely $DMU_k^{new} = (x_k + \Delta x_k{}^*, y_k + \Delta y_k{}^*,\ u_k + \Delta u_k{}^*)$, based on model (5). We can say that model (12) is a criteria model for evaluating the efficiency of DMU_k^{new} based on model (5).

Now, according to Theorems (1) and (2), the optimal objective function scores of models (4) and (12) are equal. This is because, according to Theorem (1), the optimal objective function scores of models (4) and (9) are equal, and according to Theorem (2), the optimal objective function scores of models (9) and (12) are equal. Therefore, the optimal objective function scores of models (4) and (12) are also equal. This means we can use either of the criteria models (9) and (12) to show that the created new unit's efficiency in the inverse non-radial DEA process is equal to that of the original unit. Given that model (9) has a smaller number of variables compared to model (12), model (9) is proposed as a suitable criteria model. We now present the following algorithm as a three-step approach to estimate the optimal level of the newly created unit's inputs and outputs in the SBM-based inverse non-radial DEA process as follows (Table 1):

Table 1 – An algorithm three-step approach for SBM-based inverse non-radial DEA in presence of undesirable output.

Step One: Evaluation of DMUs.
Solve model (4) to obtain the efficiency scores and optimal levels of slacks of the inputs and undesirable outputs of DMU_k, which is the unit under evaluation in the inverse non-radial DEA. Go to Step 2.
Step Two: Inverse non-radial DEA.
Solve model (8) to obtain the optimal levels of input, desirable output, and undesirable output components of DMU_k^{new}, which is the new unit corresponding to DMU_k in the inverse non-radial DEA.

5 NUMERICAL EXAMPLE

This example presented below is intended to help the reader better understand the proposed approach. We consider a case of 5 DMUs, each having one input, one desirable output and one undesirable output, as shown in Table 2. The input, desirable output, and undesirable output are indicated by the symbols X, Y, and U, respectively.

Table 2 – Data for five DMUs as an example and the efficiency score of models (1), (3), and (4).

DMU	Original DMUs			Counterpart DMUs			Efficiency score		
	X	Y	U	X	Y	U	Model (1)	Model (3)	Model (4)
A	1	0.5	0.5	1	0.5	0.1	1	0.6	0.6
B	2	2	0.6	2	2	0.4	0.8	0.7333	0.7708
C	3	4.5	1	3	4.5	0.9	1	0.95	0.95
D	4	4	0.8	4	4	0.8	0.6667	0.8438	0.8438
E	5	5	1.5	5	5	1	1	0.8333	0.8333

First, using model (1), we obtain efficiency score for each DMU, as reported in Table 2. It can be seen that all units are inefficient except for units A, C and E. The SBM model divides the DMUs into two categories: strong efficient and inefficient. Model (1) calculates the efficiency score of each DMU without considering the undesirable output. To calculate the efficiency of each DMU in the presence of undesirable outputs, we solve model (2) and obtain the lowest undesirable output based on the input and desirable output values. To do this, we solve model (2) and introduce the counterpart (hypothetic) units corresponding to each of the DMUs, and we obtain the true efficiency scores and slacks of the inputs and undesirable outputs by the non-radial projection of these units on the frontier of efficiency that is created on the basis of the counterpart (hypothetic) units. These units have the same levels of inputs and desirable outputs as the original units, but their undesirable output level is less than the original units. These units have a minimum level of undesirable outputs based on the levels of inputs and desirable outputs. The results further report the input, desirable output, and undesirable output values corresponding to each of the counterpart DMUs. Model (3) is solved to obtain the corresponding efficiency score for each of the DMUs. This model obtains the efficiency score based on the slack variable of the inputs and undesirable outputs based on the counterpart (hypothetic) units. According to this model, none of the DMUs are efficient. Unit C has the highest efficiency score among the DMUs.

To perform the SBM-based inverse non-radial DEA process, we solve model (4) to obtain the efficiency scores and slacks of the inputs and undesirable outputs. This model achieves the true efficiency scores and slacks of each DMU by representing the units in the non-radial form on the frontier of efficiency that is created based on the counterpart (hypothetic) units. The results are given in the second to fourth columns of Table 3.

We can see that the results of models (3) and (4) are the same, and none of the DMUs are efficient. It is shown in Tables (2) and (3) that the efficiency scores of each DMU based on model (1) are greater than the ones obtained from model (3); however, this is not true from an optimization perspective. This is because we expect the efficiency of each DMU to deteriorate and decrease in the presence of undesirable outputs. Therefore, it is recommended that instead of solving model (1), we use models (3) and (4) to calculate the efficiency of DMUs.

As stated in the third section, in the inverse non-radial DEA process, we must solve model (8), through which the lowest inputs and undesirable outputs can be determined. Before solv-

Table 3 – The result of models (4), (9), (12) in the inverse non-radial DEA process for data set in numerical example.

DMU	Model (4)			Model (9)			Model (12)		
	Efficiency score	Slack of input	Slack of undesirable output	Efficiency score	Slack of input	Slack of undesirable output	Efficiency score	Slack of input	Slack of undesirable output
A	0.6	0	0.4	0.6	0	0.4	0.6	0	0.4
B	0.7708	0.25	0.2	0.7708	0.1429	0.05	0.7708	0.1429	0.05
C	0.95	0	0.1	0.95	0	0.0111	0.95	0	0.0111
D	0.8438	1.25	0	0.8438	0.4545	0	0.8438	0.4545	0
E	0.8333	0	0.5	0.8333	0	0.05	0.8333	0	0.05

ing model (8), we must first solve model (4) and obtain the efficiency scores and optimal slacks of the inputs and undesirable outputs of the unit under evaluation. As previously stated, model (8) obtains the minimum level of inputs and undesirable outputs from the created new unit in such a way that the created new unit's efficiency is the same as the original unit. The third to fifth columns of Table 4 show the deviations of the input, desirable outputs, and undesirable output components of the DMUs, respectively. Also, the sixth to eighth columns of Table 4 show the respective new levels.

Table 4 – The result of model (8).

DMU	The objective function score	Variations of input	Variations of undesirable output	Variations of desirable output	The new input	The new undesirable output	The new desirable output
A	0	0	0	0	1	0.5	0.5
B	-1.3071	-0.8571	-0.45	-1.5	1.1429	0.15	0.5
C	-2.8889	-2	-0.8889	-4	1	0.1111	0.5
D	-3.2455	-2.5455	-0.7	-4	1.4545	0.1	0
E	-5.35	-4	-1.35	-4.5	1	0.15	0.5

To show that the efficiency scores corresponding to the original DMUs and the ones of the created new units are equal, we can solve two criterion models (9) and (12). We observe that the efficiency scores of the created new unit are the same as those of the original units. Model (9) with the SBM model structure obtains the efficiency scores of the created new unit, and we obtain their input and undesirable output levels from model (8). Additionally, model (9) non-radially depicts the created new units on the frontier of efficiency that is created on the basis of the counterpart units. Next, we use model (12) as a criterion model to obtain the efficiency scores of the created new unit. The level of inputs and outputs of this new unit was determined based on model (8). The results are presented in the last column of Table 4, showing that the efficiency scores of the created new units based on model (12) are the same as those of the corresponding original units.

6 CASE STUDY

In this section, we apply our method to the dataset of the energy sector in the Iranian oil industry. The energy sector in Iran is of particular importance, which is why it is necessary to have accurate, comprehensive, and reliable statistics about it. Energy statistics are important tools for making correct and appropriate decisions and policies in the energy sector. These statistics must have the necessary coordination with the decision-making process of energy policies in the country. In recent years, the planning office and macroeconomics of electricity and energy of the Ministry of Energy have tried to provide comprehensive information and publish a set of basic information chronologically from 2005 to meet the need of officials, managers, researchers, and experts of the country for reliable statistical sources and also to create appropriate, quick, and easy access to energy information. In this paper, we evaluate the performance of this industry over 14 years during the period 2005-2018 using the DEA technique.

To evaluate the performance of this industry, we consider each year as a Decision Making Unit (DMU). We must determine the inputs and outputs corresponding to each DMU for performance evaluation. The correct selection of inputs and outputs can determine the efficiency values accurately. After determining the efficiency of each of the DMUs, one of the managers' questions is how to develop DEA models to obtain the amount of changes in the inputs and outputs from the DMUs that do not change the efficiency of these units. Therefore, inverse DEA models are suggested. It is important to use models that correctly calculate the efficiency score and consider all inefficiency factors in all input and output components. Therefore, we can develop inverse DEA models in terms of input and output slacks based on the SBM model. One of the important issues in the evaluation of energy sectors is the presence of unwanted data that has a negative effect on the performance of these sectors. According to the above explanations, the models presented in this paper are among the most appropriate models in dealing with unfavorable outcomes. These models use the inverse DEA technique to evaluate energy sectors during different years based on non-radial models.

We used the opinions of senior managers of this industry to select inputs and outputs. Inputs and outputs were selected according to the indicators that have had the greatest impact on the performance of this industry over many years. These inputs and outputs are as follows. Each DMU has three inputs, two desirable outputs, and two undesirable outputs. The first input is the amount of drilled oil wells during that year, including the fields of exploration, development, repair, and supplementary wells, measured in meters. The second input shows the amount of gasoline imports during different years, measured in one thousand cubic meters. Due to the inability of the country's refineries to produce the gasoline needed by the country, gasoline is exported, and the required gasoline is imported from other producing countries. The third input includes the amount of different petroleum products transported by various means, including pipelines, railway tanks, road tankers, petroleum products trucks, refueling vessels, and refueling vessels, which are measured in millions of tons per kilometer.

Outputs include desirable and undesirable outputs. Desirable outputs include two outputs, which are factors whose larger value increases the rate of efficiency and performance. Total crude oil extracted from oil wells is used in three ways. The first part is the total amount of crude oil provided to the country's refineries in the form of petroleum products, from which other petroleum products and derivatives are produced and used for domestic consumption and in various industries, and the rest of the refinery products are also exported. The second part is the amount of crude oil that is exported directly as furnace oil. The third part is exported as crude oil to other countries.

The first desirable output is the total production of the country's petroleum products, which are produced in ten different refineries. These refineries produce petroleum products, including gasoline, liquefied petroleum gas, heavy furnace oil, sulfur, gas condensate, solvents, jet fuel, industrial oil, gas oil, crude oil, engine oils, kerosene, and other products. Most of the petrochemical products in the country are used in various industries and households, and a limited part is exported to other countries. The total amount of crude oil supplied to different refineries is measured in cubic meters per day. The second desirable output is the amount of crude oil exported to other countries during the 14 years studied. The amount of crude oil exported is measured in million barrels per year.

In this study, the undesirable outputs include the emission of polluting and greenhouse gases resulting from the country's energy production and consumption. Emission coefficients of pollutants and greenhouse gases used in the power plant sector from 2005 onwards, based on the studies conducted under the title of "Compilation of Pollution Atlas of Power Plants in 2005," have been revised. In this study, nitrogen oxides and sulfur, which are the most significant pollutants and greenhouse gases emitted from the country's energy production and consumption, are considered as undesirable outputs. The emissions of these gases (NOx, SO2) resulting from the country's energy production and consumption in each of the studied years are considered as undesirable outputs. Therefore, the first and second undesirable outputs are the emissions of NOx and SO2 gases caused by the country's energy production and consumption in each of the studied years, respectively. The dataset is provided in Table 5.

We first solve model (1) to obtain the efficiency score of each DMU. Model (1) calculates the efficiency score of each DMU without considering the undesirable output. We can see that Units 1, 3, 7, 8, 9, 11, 12, and 14 are efficient, while the others are inefficient. The optimal scores of the inefficient slacks related to the two input components and the two desirable output components are provided in the other columns of Table 6.

Now, to calculate the efficiency of each DMU in the presence of undesirable outputs, we solve model (2) and obtain the minimum level of undesirable output based on the levels of input and desirable output. As stated in the third section, we first solve model (2) and introduce a counterpart (hypothetical) unit for each of the DMUs. Table 7 displays the slack of undesirable outputs and the corresponding values of undesirable outputs for each of the counterpart DMUs.

Table 5 – The energy data related to the energy sector in Iran during the years 2005-2018.

DMUs	Input			Desirable output		Undesirable output	
	Input1	Input2	Input3	Desirable output1	Desirable output2	Undesirable output1	Undesirable output2
DMU1	345004	9055.2	35405	250987	866.9	1256222	768793
DMU2	326139	10037.5	39183	257580	881.6	1345241	837761
DMU3	346213	6952.6	38001	256893	908.7	1677948	1435620
DMU4	391630	7542.5	41802	260947	862.7	1807615	1597779
DMU5	334375	7664.7	40926	266526	800	1770498	1500437
DMU6	356752	5449.1	36290	265371	820	1791802	1352650
DMU7	454372	1803	34212	273221	810.5	1860248	1427839
DMU8	417493	554.9	38618	283427	413.5	1867075	1540481
DMU9	339639	1339.6	38354	290811	372.8	1945104	1609006
DMU10	382784	1668.1	37192	280248	391	1943018	1302639
DMU11	356290	3646	36429	272754	411.7	1842891	997169
DMU12	293606	4428.6	38024	272146	764.9	1896902	881844
DMU13	254269	4608.1	42749	283403	772.3	1952789	800831
DMU14	193738	4934.7	44013	300939	831.6	2056315	819572

Model (3) is used to obtain the corresponding efficiency for each DMU. We can observe that Units 1, 3, 9, 11, and 14 are efficient, while the others are inefficient. Table 8 illustrates the efficiency scores, input, desirable output, and undesirable output slacks corresponding to all DMUs based on model (3).

In order to perform the SBM-based inverse non-radial DEA process, we solve model (4) and obtain the efficiency scores and slacks. Table 6 shows the results of model (4). We can see that units 1, 3, 9, 11, and 14 are efficient and others are inefficient. The model (4) introduces the oil industry as efficient in 2005, 2007, 2013, 2015, and 2018. Over these years, due to the low amount of costs incurred in this sector, including the cost of operating wells and oil rigs, the cost of transporting crude oil to the country's refineries, and the cost of imported gasoline, we observe favorable performance for the oil industry in the energy sector. However, during the years 2006, 2008, 2009, 2010, 2011, 2012, 2014, and 2016, the oil industry did not perform well in the energy sector due to rising costs. The model (4) introduces the oil industry as inefficient in these years.

The difference between models (3) and (4) is that the former measures the efficiency scores based on the counterpart (hypothetic) units. Model (3) is based on the SBM model in terms of the slack variables of the input, desirable output, and undesirable output components. Model (4) is similar to model (3), but model (4) does not consider the slacks of the desirable output components. We can say that model (3) is the input-oriented SBM model in the presence of undesirable outputs. In models (3) and (4), we consider the undesirable outputs, but in model (1), we do not consider the undesirable outputs. As can be seen in Tables (6), (7), and (8), the efficiency scores of each DMU based on model (1) are greater than their corresponding ones from models (3) and (4), which is not correct because we expect the efficiency score of each DMU to deteriorate in the presence of undesirable outputs. Therefore, it is recommended that instead of solving model (1), we use models (3) and (4) to calculate the efficiency of the DMUs.

Table 6 – The result of models (1), (4) for the data set in the case study.

DMUs	Model (1)					
	Efficiency score	Slack of input1	Slack of input2	Slack of input3	Slack of desirable output1	Slack of desirable output2
DMU1	1	0	0	0	0	0
DMU2	0.8391	8922.265	4004.016	0	11559.3	0
DMU3	1	0	0	0	0	0
DMU4	0.7701	136387.8	1793.835	214.0739	22225.07	0
DMU5	0.7332	58545.67	3716.385	0	25682.72	24.9542
DMU6	0.9833	0	219.7463	0	1737.571	0
DMU7	1	0	0	0	0	0
DMU8	1	0	0	0	0	0
DMU9	1	0	0	0	0	0
DMU10	0.8028	0	0	381.1447	3497.141	183.8244
DMU11	1	0	0	0	0	0
DMU12	1	0	0	0	0	0
DMU13	0.9038	26917.96	77.2842	0	13961.31	56.5788
DMU14	1	0	0	0	0	0
DMUs	Model (4)					
	Efficiency score	Slack of input1	Slack of input2	Slack of input3	Slack of undesirable output1	Slack of undesirable output2
DMU1	1	0	0	0	0	0
DMU2	0.9817	0	0	0	48749.65	46312.15
DMU3	1	0	0	0	0	0
DMU4	0.7657	58889.57	0	4896.477	357445.3	720117.7
DMU5	0.7755	0	4180.553	2971.85	127091.6	635784.1
DMU6	0.8783	0	219.7463	0	334277.5	516074.7
DMU7	0.8828	0	0	0	372771.1	550333.1
DMU8	0.9874	0	0	0	0	97181.84
DMU9	1	0	0	0	0	0
DMU10	0.9697	0	114.4303	0	161058.2	0
DMU11	1	0	0	0	0	0
DMU12	0.9614	0	0	0	365814.8	0
DMU13	0.9936	0	0	0	62874.98	0
DMU14	1	0	0	0	0	0

As stated in the fourth section, in the inverse non-radial DEA process, we must solve model (8), through which the lowest inputs and undesirable outputs can be determined. Before solving model (8), we must first solve model (4) and obtain the efficiency scores and optimal slacks of the inputs and undesirable outputs of the unit under evaluation. As previously stated, in the inverse non-radial DEA process, model (8) obtains the minimum level of inputs and undesirable outputs from the created new unit in a manner that the created new unit's efficiency is the same as that of the original unit. Table 9 shows the optimal objective function score and the deviation of the input, desirable outputs, and undesirable output components of the 14 DMUs based on model (8).

Table 7 – The result of model (2) for the data set in the case study.

DMU	Model (2)				
	The objective function score	Slack of undesirable output1	Slack of undesirable output2	The undesirable output1 of counterpart DMU	The undesirable output2 of counterpart DMU
DMU1	0	0	0	1256222	768793
DMU2	95061.8	48749.65	46312.15	1296491	791448.9
DMU3	0	0	0	1677948	1435620
DMU4	1253482	466773.1	786709	1340842	811070
DMU5	975188.6	327901.3	647287.3	1442597	853149.7
DMU6	886945.7	376593.7	510352	1415208	842298
DMU7	923104.3	372771.1	550333.1	1487477	877505.9
DMU8	97181.84	0	97181.84	1867075	1443299
DMU9	0	0	0	1945104	1609006
DMU10	73027.86	59069.55	13958.32	1883948	1288681
DMU11	0	0	0	1842891	997169
DMU12	365814.8	365814.8	0	1531087	881844
DMU13	62874.98	62874.98	0	1889914	800831
DMU14	0	0	0	2056315	819572

Table 8 – The result of model (3) for the data set in the case study.

DMU	Model (3)							
	Efficiency score	Slack of input1	Slack of input2	Slack of input3	Slack of desirable output1	Slack of desirable output2	Slack of undesirable output1	Slack of undesirable output2
DMU1	1	0	0	0	0	0	0	0
DMU2	0.9817	0	0	0	0	0	48749.65	46312.15
DMU3	1	0	0	0	0	0	0	0
DMU4	0.7969	130127.8	1085.245	387.9763	19462.03	0	0	609574.4
DMU5	0.7475	9679.484	4303.546	1837.586	20485.88	20.9981	0	651755.7
DMU6	0.8755	0	219.7463	0	1737.571	0	334277.5	516074.7
DMU7	0.8828	0	0	0	0	0	372771.1	550333.1
DMU8	0.9874	0	0	0	0	0	0	97181.84
DMU9	1	0	0	0	0	0	0	0
DMU10	0.7842	0	0	0	3657.476	185.0738	174298.4	46619.1
DMU11	1	0	0	0	0	0	0	0
DMU12	0.9614	0	0	0	0	0	365814.8	0
DMU13	0.9936	0	0	0	0	0	62874.98	0
DMU14	1	0	0	0	0	0	0	0

Table 10 shows the new levels of the input, desirable outputs, and undesirable output components corresponding to each of the DMUs derived from model (8). As described in the fourth section, the vector of the inputs, desirable outputs, and undesirable output of the created new unit in the inverse non-radial DEA process is as follows:

$$DMU_k^{new} = \left(x_k + \Delta x_k{}^*, y_k + \Delta y_k{}^*,\ u_k + \Delta u_k{}^*\right).$$

Table 9 – The input and output variations based on model (8).

DMUs	Model (8)			
	The objective function score	Variations of input1	Variations of input2	Variations of input3
DMU1	0	0	0	0
DMU2	-51660.4	18865	-982.3	-3778
DMU3	-1090255	-1209	2102.6	-2596
DMU4	-425700	14434.02	1512.7	-1699.61
DMU5	-563337	10629	12255.64	-2748.75
DMU6	-365713	-11748	3986.615	-885
DMU7	-567024	-109368	7252.2	1193
DMU8	-1397977	-72489	8500.3	-3213
DMU9	-1518963	5365	7715.6	-2949
DMU10	-1138614	-37780	8054.03	-1787
DMU11	-821946	-11286	5409.2	-1024
DMU12	-400183	51398	4626.6	-2619
DMU13	-598974	90735	4447.1	-7344
DMU14	-704094	151266	4120.5	-8608
DMU	Model (8)			
	Variations of desirable output1	Variations of desirable output2	Variations of undesirable output1	Variations of undesirable output2
DMU1	0	0	0	0
DMU2	-6593	-14.7	-41783.5	-23981.6
DMU3	-6593	-41.8	-421726	-666827
DMU4	-9960	-41.8	-241753	-198194
DMU5	-15539	-41.8	-417127	-166346
DMU6	-15539	-41.8	-247470	-109597
DMU7	-22234	-41.8	-289209	-176893
DMU8	-32440	-41.8	-610853	-719923
DMU9	-39824	-41.8	-688882	-840213
DMU10	-39824	-41.8	-573255	-533846
DMU11	-39824	-41.8	-586669	-228376
DMU12	-39824	-41.8	-340537	-113051
DMU13	-39824	-41.8	-654774	-32038
DMU14	-49952	-41.8	-800093	-50779

It should be noted that Table 5 lists the vector of the inputs, desirable outputs, and undesirable output corresponding to each of the original DMUs as $DMU_k = (x_k, y_k,\ u_k)$.

Table 10 – The new levels of inputs and outputs of DMUs based on model (8).

DMUs	Model (8)			
	The objective function score	The new input1	The new input2	The new input3
DMU1	0	345004	9055.2	35405
DMU2	-51660.4	345004	9055.2	35405
DMU3	-1090255	345004	9055.2	35405
DMU4	-425700	406064	9055.2	40102.39
DMU5	-563337	345004	19920.34	38177.25
DMU6	-365713	345004	9435.715	35405
DMU7	-567024	345004	9055.2	35405
DMU8	-1397977	345004	9055.2	35405
DMU9	-1518963	345004	9055.2	35405
DMU10	-1138614	345004	9722.13	35405
DMU11	-821946	345004	9055.2	35405
DMU12	-400183	345004	9055.2	35405
DMU13	-598974	345004	9055.2	35405
DMU14	-704094	345004	9055.2	35405
DMU	Model (8)			
	The new desirable output1	The new desirable output2	The new undesirable output1	The new undesirable output2
DMU1	250987	866.9	1256222	768793
DMU2	250987	866.9	1303457	813779.4
DMU3	250300	866.9	1256222	768793
DMU4	250987	820.9	1565862	1399585
DMU5	250987	758.2	1353371	1334091
DMU6	249832	778.2	1544332	1243053
DMU7	250987	768.7	1571039	1250946
DMU8	250987	371.7	1256222	820558.2
DMU9	250987	331	1256222	768793
DMU10	240424	349.2	1369763	768793
DMU11	232930	369.9	1256222	768793
DMU12	232322	723.1	1556365	768793
DMU13	243579	730.5	1298015	768793
DMU14	250987	789.8	1256222	768793

Now, in order to analyze the results of the models, we will examine the results of DMU_4, which is an inefficient unit. Based on the proposed algorithm, model (4) is solved to obtain the efficiency and the optimal levels of slacks of the inputs and undesirable outputs for DMU_4. The efficiency score is equal to 0.8168, the optimal levels of slacks of the inputs, desirable outputs, and undesirable outputs are equal to 0.8168, 58889.57, 0, 4896.477, 357445.3, and 720117.7 respectively. These values indicate the rate of improvement in the inputs and undesirable output components. They are obtained by a non-radial projection of these units on the frontier of efficiency, which is built on by the counterpart units on the basis of the model (4).

In the following, we solve model (8), based on which we obtain the input and output variations. The input variations for the first, second, and third inputs (the amount of drilled oil wells during 2008 year, the amount of gasoline imports during 2008 year, the amount of different petroleum products transported by various means during 2008 year) are 14434.02, 1512.7 and -1699.61 respectively. This shows that the first and second inputs can be increased and decreased by 14434.02, 1512.7, respectively, and third input can be decreased by 1699.61. In this case, the newly created unit, DMU_4^{new} , has an efficiency score of 0.8168, which is equal to the efficiency score of the original unit, i.e., DMU_4. Similarly, the desirable outputs (the total production of the country's petroleum products during 2008 year, the amount of crude oil exported to other countries during 2008 year) variations for the first and second desirable outputs are -9960 and -41.8, respectively. This shows that the first and second desirable outputs can be decreased by 9960 and 3723, respectively. In this case, the newly created unit, DMU_4^{new} , has an efficiency score of 0.8168, which is the same as the one of the original unit, i.e. DMU_4. The undesirable outputs (the emissions of NO_x gases caused by the country's energy production and consumption during 2008 year, the emissions of SO_2 gases caused by the country's energy production and consumption during 2008 year) variation is -241753 and -198194. This shows that the first and second undesirable outputs can be decreased by 241753 and 198194, respectively. With this amount of changes, the efficiency score of the newly created unit, DMU_4^{new} , is the same as the one of the original unit, i.e. DMU_4. Of course, all the changes related to the inputs, desirable output and undesirable output components are done together. Then, the optimal levels of the newly created unit's inputs and outputs corresponding to DMU_{18} based on model (8) are considered as follows:

$$DMU_4^{new} = (406064, 9055.2, 40102.39, 250987,\ 820.9, 1565862, 1399585).$$

The efficiency scores of the counterpart (hypothetic) units corresponding to DMU_4, and DMU_4^{new} based on models (9) and (12) are equal, which is 0.7657.

To show that the efficiency scores of the original DMU and the newly created unit are equal, we can solve the criterion models (9) and (12). We can observe that the efficiency scores of the new unit based on model (9) are equivalent to those of their original units based on model (4). Model (9) with the SBM model structure obtains the efficiency scores of the new unit, and we acquire their input and undesirable output levels from model (8). Additionally, model (9) displays the new unit non-radially on the frontier of efficiency that is created based on the counterpart units. Subsequently, we use the criterion model (12) to determine the efficiency score of the new unit. The level of inputs and outputs of this new unit was determined based on model (8). We can see that the efficiency scores of the new unit based on model (12) are the same as those of the corresponding unit based on model (4). We find that the efficiency scores obtained from models

(4), (9), and (12) are the same. Therefore, we have demonstrated the validity of the theorems (1) and (2). Table 11 displays the results based on models (9) and (12).

Table 11 – The result of models (9) and (12).

DMUs	Model (9)					
	Efficiency score	Slack of input1	Slack of input2	Slack of input3	Slack of undesirable output1	Slack of undesirable output2
DMU1	1	0	0	0	0	0
DMU2	0.9817	0	0	0	47235.48	44986.43
DMU3	1	0	0	0	0	0
DMU4	0.7657	0	5861.308	4472.288	37092.18	544336.1
DMU5	0.7755	18166.23	12500.37	1846.583	0	525340.8
DMU6	0.8783	0	380.5148	0	288109.6	474260.4
DMU7	0.8828	0	0	0	314817.2	482153.2
DMU8	0.9874	0	0	0	0	51765.24
DMU9	1	0	0	0	0	0
DMU10	0.9697	0	666.9302	0	113540.7	0
DMU11	1	0	0	0	0	0
DMU12	0.9614	0	0	0	300142.6	0
DMU13	0.9936	0	0	0	41792.87	0
DMU14	1	0	0	0	0	0
DMUs	Model (12)					
	Efficiency score	Slack of input1	Slack of input2	Slack of input3	Slack of undesirable output1	Slack of undesirable output2
DMU1	1	0	0	0	0	0
DMU2	0.9817	0	0	0	47235.48	44986.43
DMU3	1	0	0	0	0	0
DMU4	0.7674	12435.75	5837.618	3942.825	0	545313.2
DMU5	0.7755	18166.23	12500.37	1846.583	0	525340.8
DMU6	0.8783	0	380.5148	0	288109.6	474260.4
DMU7	0.8828	0	0	0	314817.2	482153.2
DMU8	0.9874	0	0	0	0	51765.24
DMU9	1	0	0	0	0	0
DMU10	0.9697	0	666.9302	0	113540.7	0
DMU11	1	0	0	0	0	0
DMU12	0.9614	0	0	0	300142.6	0
DMU13	0.9936	0	0	0	41792.87	0
DMU14	1	0	0	0	0	0

Now, we compare the results obtained from our approach with the results of Hadi-Vencheh and Foroughi (2006). In this way, we ignore undesirable output in the inverse DEA process. We also assume that in the radial model it is possible to decrease or increase inputs. At first, we apply input orientation model in the radial form. The results are given in Table 12.

In the following, we compare the results obtained from our approach in this paper with approach provided by Hadi-Vencheh and Foroughi (2006). We consider models in the output orientation

Table 12 – The results of Hadi-Vencheh and Foroughi (2006) in the input oriented.

DMUs	Efficiency	Variations of input1	Variations of input2	Variations of input3
DMU1	1	-176361.4489	-7110.7892	-9571.3551
DMU2	0.9495	-133078.8624	-7905.4754	-10107.3781
DMU3	0.9871	-39213.0972	-5741.7347	-3332.7956
DMU4	0.9114	-21573.894	-6082.9292	-13.1245
DMU5	0.956	-3076.2763	-6357.9956	-3513.8322
DMU6	1	-68178.4144	-3697.9654	-449.4293
DMU7	1	-150363.9522	-78.8542	2905.1804
DMU8	1	-92225.0737	746.8437	-1786.9474
DMU9	1	0	0	0
DMU10	1	-100647.4605	886.1386	2318.8351
DMU11	1	-133395.7527	-271.9629	1983.0306
DMU12	1	-91290.8238	-378.6601	2082.2263
DMU13	0.9636	-58951.2984	161.8896	535.1303
DMU14	1	0	0	0
DMUs	-	The new input1	The new input2	The new input3
DMU1	-	168642.5511	1944.4108	25833.6449
DMU2	-	193060.1376	2132.0246	29075.6219
DMU3	-	306999.9028	1210.8653	34668.2044
DMU4	-	370056.106	1459.5708	41788.8755
DMU5	-	331298.7237	1306.7044	37412.1678
DMU6	-	288573.5856	1751.1346	35840.5707
DMU7	-	304008.0478	1724.1458	37117.1804
DMU8	-	325267.9263	1301.7437	36831.0526
DMU9	-	339639	1339.6	38354
DMU10	-	282136.5395	2554.2386	39510.8351
DMU11	-	222894.2473	3374.0371	38412.0306
DMU12	-	202315.1762	4049.9399	40106.2263
DMU13	-	195317.7016	4769.9896	43284.1303
DMU14	-	193738	4934.7	44013

and radial form. Also, we ignore undesirable output in the inverse DEA process. The results are given in Table 13.

In the comparison made between the approach presented in this paper and the approach presented by Hadi-Vencheh and Foroughi (2006) based on the inverse DEA process, we did not consider the undesirable outputs, then the results are not comparable in any way, and the results are provided only so that the reader can see the difference between approaches based on radial and non-radial models. As can be seen, the input vector based on the approach presented in this paper and the approach of Hadi-Vencheh and Foroughi (2006) are larger in some components and smaller in

Table 13 – The results obtained from Hadi-Vencheh and Foroughi (2006) in the output oriented.

DMUs	Efficiency	Variations of output1	Variations of output2	The new output1	The new output2
DMU1	1	565355.5212	714958.8116	1821578	1483752
DMU2	0.6818	1541905.223	1413053.718	2887146	2250815
DMU3	0.9085	452181.5826	318615.3411	2130130	1754235
DMU4	0.993	345165.5448	167181.0419	2152781	1764960
DMU5	0.9493	389404.8535	153428.575	2159903	1653866
DMU6	1	77891.8802	168037.5982	1869694	1520688
DMU7	1	0	0	1860248	1427839
DMU8	1	0	0	1867075	1540481
DMU9	1	0	0	1945104	1609006
DMU10	1	-12658.5715	255006.4296	1930359	1557645
DMU11	1	32452.3292	529432.6331	1875343	1526602
DMU12	1	-7194.9146	487478.0279	1889707	1369322
DMU13	0.9595	190549.3804	389663.9982	2143338	1190495
DMU14	1	0	0	2056315	819572

others according to the results in tables (10) and (12). We can provide a similar interpretation for the output vector according to the results in tables (10) and (13).

In general, the approaches provided in the field of inverse DEA are divided into two categories. The first category includes approaches that consider two or more DMUs. By merging the levels of inputs and outputs of these DMUs, a new unit is proposed; this unit has a predetermined target efficiency. However, in this paper, the inverse DEA process is done only based on the DMU under evaluation, and the target efficiency is predetermined. This efficiency score is the efficiency of the DMU under evaluation based on the DEA model. Also, by solving only one model, we can determine the optimal level of inputs and outputs from the unit under evaluation in such a way that the newly created unit has the same efficiency score as the corresponding original unit.

The second category includes approaches that have an orientation. In this approach, the decision-maker predetermines the target efficiency score. In this situation, we are faced with two types of models in the inverse DEA process. These models are input- and output-oriented. In the input-oriented approach, we obtain the optimal level of inputs according to the target efficiency score. Also, the output level is constant during the inverse DEA process. Similarly, in the output orientation, we obtain the optimal level of outputs according to the target efficiency score. The input level is constant during the inverse DEA process. However, our approach in this paper is different from these approaches, and the target efficiency score is the same as the efficiency score of the unit under evaluation. The model determines the optimal level of inputs and outputs simultaneously. In other words, the model determines the best level of inputs and outputs (undesirable and desirable) for the unit under evaluation based on the conditions of this unit.

Therefore, this paper introduces a new approach in the field of inverse DEA that is different from previous approaches.

7 CONCLUSION

In this paper, we present the SBM-based inverse non-radial DEA process in the presence of undesirable outputs. Firstly, we presented the counterpart (hypothetical) units corresponding to each of the DMUs. These units have the same input and desirable output levels as the corresponding original units, but they have the lowest level of undesirable outputs. The observed units are used on the basis of weak disposability in DEA. We obtain the true efficiency scores and slacks regarding the input and undesirable output components of each DMU by depicting the units non-radially on the frontier of efficiency created on the basis of the counterpart (hypothetical) units. Next, based on the efficiency score and slacks regarding the input and undesirable output components of each DMU, the SBM-based inverse non-radial DEA process is presented. We demonstrated that this model obtains the optimal level of input, desirable outputs, and undesirable output components for the created new unit in the inverse non-radial DEA process. We presented two new criteria models to compare the efficiency scores between the original unit and the created new unit and showed that these efficiency scores are equal based on the two criteria models presented in this paper. By using the SBM-based inverse non-radial DEA process, we can use all the inefficiency slacks related to all the input and undesirable output components in the inverse non-radial DEA process to estimate the inputs and outputs simultaneously. This paper proposes a new idea of creating a set of hypothetical counterpart DMUs of the original DMUs. These DMUs used the same amount of inputs to produce the same amount of desirable outputs, however generated the smallest amount of undesirable outputs calculated from the observed DMUs based on weak disposability. We used hypothetical DMUs to construct the production frontier in order to do inverse DEA process. We do inverse DEA process using hypothetical DMUs. The new level of inputs, desirable output and undesirable output of the original DMUs obtained by considering the new frontier created by these DMUs.

We can say that the advantages of our approach in this paper over the previous approach are as follows:

1. In our proposed approach in this paper for the inverse DEA process, we presented two criteria models in order to check whether the efficiency score of the new unit created and the initial unit corresponding to it are equal or not. The new unit created in the inverse DEA process has an efficiency score equal to that of the corresponding initial unit; however, this unit has input and undesirable output levels less than or equal to the input and undesirable output levels of the initial unit corresponding to this unit. Also, the undesirable output level of the new unit created is greater than or equal to the undesirable output level of the initial unit corresponding to itself.

2. This paper presents a new approach to the inverse DEA. By solving a model, we can determine the optimal level of inputs and outputs from the unit under evaluation in such

a way that this unit has the same efficiency score as the corresponding initial unit. It can be said that the inverse DEA process presented in this paper is done only based on the unit under evaluation. The new unit created can be considered the target (benchmark) unit corresponding to the initial unit corresponding to this unit.

3. Most of the approaches provided in the inverse DEA are based on radial models. However, the our proposed approach in this paper is based on non-radial models. In non-radial models such as SBM, the target corresponding to the unit under evaluation is determined in the best way. These models obtain the efficiency score of the unit under evaluation based on all the inefficiency slack values corresponding to the input and output components. The image of the inefficient unit on the efficiency frontier is determined non-radially. These models obtain the optimal level of input and output components for the target unit. Also, considering that the inverse DEA process presented in this paper is based on non-radial models, the target unit corresponding to the initial unit is determined in the best way possible.

4. The approach presented in this paper provides the ability to deal with undesirable outputs in the inverse DEA process. In this regard, we first determine the lowest level of undesirable output. In the following, new units are introduced with the title of counterpart units corresponding to each of the original units. The counterpart unit has the same level of inputs and desirable outputs as the corresponding original unit; however, it has a Less amount of undesirable outputs than the original unit. We perform the inverse DEA process for original units based on counterpart units. In other words, the unit under evaluation is projected on the new frontier created by the counterpart units. Next, in the inverse DEA process, we determine the optimal level of input and output (desirable and undesirable) components from the target unit corresponding to the original unit. It can be said that the target unit has the best level of inputs and outputs (desirable and undesirable) compared to the corresponding initial unit, but it has the same efficiency score as this unit.

5. The models presented in this paper are without orientation. These models get the optimal level of inputs and outputs simultaneously. However, the models in the previous approaches were often presented as input- and output-oriented. In input-output-oriented models, we first place the target efficiency score in the inverse DEA model, and the new level of inputs (outputs) is determined by keeping the level of outputs (inputs) constant.

6. The inverse DEA model presented in this paper can be converted into a linear model with a simple transformation, and it can be easily solved with available software for solving linear programming models.

7. The approach presented in this paper determines the target unit corresponding to the unit under evaluation in the presence of undesirable outputs. The target unit can be considered a benchmark for the original unit under evaluation. Although this unit may be an inefficient one, it has optimal input and output levels compared to the original unit. In other words, we showed that, considering the existing conditions, the unit under evaluation can have a

better performance than its initial performance and should improve the level of its inputs and outputs.

8. In the models presented in this paper, the level of inputs and outputs may decrease or increase, but the amount of input and output from the newly created unit is non-negative. And it is possible for the units to adjust the level of their inputs and outputs in order to reach the optimal level of their inputs and outputs.

9. From a managerial point of view, the target unit achieved for the unit under evaluation may not be able to be created in practice. As we know, the target unit corresponding to each of the DMUs based on DEA models may not be able to be created, and the optimal level of inputs and outputs cannot be reached in some cases. However, the DEA offers targets. Only if the target unit is an observed unit does it have input and output levels that can be created and are real. However, if the target unit is a virtual unit, it may not be able to be created. The models presented in this paper are also based on the SBM model in DEA and are not exempt from this rule. In DEA, we can achieve attainable targets only if we use FDH models. However, by changing the conditions of production, it is possible to achieve targets that can be produced in practice.

In future work, we can apply the proposed approach in the industrial sector, as one of the primary issues in the industrial sector is how to deal with the outcome of undesirable outputs among the data and how to determine and control the level of these outputs to protect the environment. One of the limitations of the research is that we may obtain targets based on inverse DEA models for DMUs that cannot be created in the real world. In terms of future studies, we can solve the proposed approach in this paper by considering the target efficiency of DMUs and obtaining the amount of deviation of the input and output components needed to reach the target efficiency level. We can also solve the models presented in this paper for other technologies such as CRS, IRS, DRS, and semi-additivity technologies. Additionally, we can apply the proposed approach to other data structures in DEA, such as the two-stage network structures (Mills et al., 2021), or use them in the presence of inaccurate data, such as fuzzy and probabilistic data (Valami, 2009).

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