Pesquisa Operacional (2023) 43: e267995 p.1-22
doi: 10.1590/0101-7438.2023.043.00267995

Printed version ISSN 0101-7438 / Online version ISSN 1678-5142
www.scielo.br/pope
ARTICLES

REDISTRIBUTING MULTIPLE INPUTS WITH A PARABOLIC DEA MODEL

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Received September 19, 2022 / Accepted February 8, 2023

ABSTRACT. Parametric DEA models presume a functional form for the efficiency frontier and are used for resource redistribution. Parabolic DEA is a particular case of these models used to redistribute a single input in Variable Returns to Scale scenarios. In this study, we extend this model to redistribute multiple inputs simultaneously, ensuring that all DMUs will become extreme efficient after redistribution maintaining their outputs. The proposed model is a single multi-objective Linear Programming Problem (PPL) which provides a single optimal solution. To solve this model, two approaches are used, the Weighted Sum of the Objective Functions and the Separation of Variables. Two numerical examples considering single and multiple outputs are used and the results obtained are identical for the two approaches.

Keywords: parabolic DEA, resources redistribution, variable returns to scale.

1 INTRODUCTION

Classic Data Envelopment Analysis (DEA) models (Charnes et al., 1978; Banker et al., 1984) are tools used for efficiency analysis of a set of productive units, the so-called Decision Making Units (DMUs). Dai et al. (2016) have stated that issues regarding resources fair allocation or redistribution are one of the main uses of DEA.

In classical models, an inefficient DMU would have freedom of production reaching the efficiency frontier by modifying its inputs or outputs without affecting other DMUs. However, this situation is not feasible in realities of competition or cooperation among DMUs and in limited resources environment, where it is undesirable or impossible to change the total sum of some input or output. Thus, one DMU receiving more resources would result in losses for others.

Cook and Kress (1999) have introduced the use of DEA output oriented to allocate shared fixed costs among DMUs in constant returns to scale (CRS) scenarios in an efficiency invariance ap-

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proach. Following this line, Jahanshahloo et al. (2004) have presented an alternative model for resource distribution in constant or variable returns to scale scenarios (CRS and VRS) with no need of Linear Programming Problem resolution. Cook and Zhu (2005) have extended the model of Cook and Kress (1999) for cost allocation in an input-oriented CRS scenario. They have also suggested the initial idea for developing the VRS model.

Beasley (2003) have presented a five-stage model for maximizing DMUs efficiency by resources allocation. However, as pointed by Milioni and Bianca Alves (2013), this model presents a complex formulation in which a phase requires a nonlinear optimization solution. Lins et al. (2003) have developed the Zero Sum Gains DEA model (ZSG-DEA) to set targets for countries at the Olympic Games considering the total sum of medals fixed. Moreover, Fonseca et al (2010) have introduced a ZSG-DEA model with weight restrictions. Recently, Bouzidis and Karagiannis (2022) have extended de ZSG-DEA model to the case of a reverse (undesirable) output. However, most of its uses have been for fixed sum variables redistribution. Similar to the Beasley model, DEA-GSZ does not assure that DMUs will become extreme efficient after redistribution.

Unlike classical DEA models, the so-called Parametric DEA model presumes a functional form for shaping the efficiency frontier and its main use is fixed sum variables redistribution. After reallocation, the efficiency frontier takes the shape of a smooth functional form previously defined by the decision maker. Parametric DEA models provide smoothed efficient frontiers. Other kind of smooth frontiers, although with different goals, are studied in Soares de Mello et al (2002), Nacif et al. (2009), Pereira e Soares de Mello (2015), Brandão and Soares de Mello (2017) and Brandão et al. (2020). Smoothed DEA frontiers do not result in multiple optimal weight for efficient DMUs and also do not present indeterminacy on the scale variation (increasing, decreasing or constant) in VRS models (Benicio and Soares de Mello, 2015; Benicio and Soares de Mello, 2019). An important advantage of the parametric DEA is that after the redistribution all DMUs will become extreme efficient, because the parametric frontier contains no weakly efficient regions.

Avellar et al. (2007) have introduced the spherical parametric frontier to redistribute a new input among DMUs. This redistribution considered that the allocation of inputs would be sufficiently fair to make all DMUs efficient and arranged at the spherical frontier. Avellar et al. (2010) have studied a global target for the input to be redistributed and, unlike the previous paper, this input already exists in the model. In addition, they have showed that, in some cases, some DMUs would have to increase its inputs to become efficient when using the redistribution approach, which is contrary to classic DEA models assumptions. They have additionally presented a theorem for redistribution using the spherical frontier without increasing any DMU's inputs. In this specific case, the redistributed input total sum is not constant.

Guedes et al. (2012) have presented the adjusted spherical frontier model. This new model calculates the spherical frontier using the fraction of the total inputs and outputs of a DMU instead of scalar values. Thus, as they have stated, the model presents results more adherent to the coherence property. Milioni et al. (2011a) and Bianca Alves et al. (2014) have studied the input allocation using parameterized efficiency in elliptical geometry. Milioni et al. (2011b) and

Silva et al. (2017) have introduced the hyperbolic DEA frontier. Milioni and Bianca Alves (2013) have made a brief overview about related studies on Parametric DEA models.

It is noteworthy that all parametric DEA models previously mentioned only deal with CRS scenarios which generate zero-degree homogeneous functions whenever centred at the origin (Coelli et al., 2005). On the other hand, Silveira et al. (2019) have developed the parabolic model that respects VRS conditions by parameterizing the frontier using a paraboloid functional form for redistributing one input. Based on this model, Moreira et al (2021) developed an extension to take into account integer variables. In this paper, we extend the original Parabolic DEA (Silveira et al., 2019) to redistribute multiple fixed sum resources in VRS scenarios.

2 PARABOLIC DEA FOR ONE INPUT REDISTRIBUTION REVIEW

Parabolic DEA (Silveira et al., 2019) is a particular case of parametric models to perform input redistribution in VRS scenarios that may have multiple inputs. However, in order to generate the parabolic frontier, only one of the inputs is allowed to change (redistribute). In such frontier, all DMUs become efficient without changing the total sum of the redistributed input. Thus, all DMUs become extreme efficient without changing neither the outputs nor the other inputs.

The Parabolic DEA model developed by Silveira et al. (2019) describes the new efficiency frontier by the input being a quadratic function of the output, in one input and one output scenarios. Equation (1) describes the frontier in a set of k DMUs. The new input of a DMU k is represented by x_k and the outputs by y_k in which k varies from 1 to n. Also, a, b and c are the parabola coefficients.

$$x_k = ay_k^2 + by_k + c; \forall k \tag{1}$$

The model also ensures some inherent properties of a DEA frontier such as convexity, increasing monotonicity, frontier in non-negative quadrants and non-negative values for inputs. Restrictions (2), (3), (4) and (5) represent these conditions respectively.

$$\frac{d^2 x_k}{dy_k^2} \geq 0; \forall k \tag{2}$$

$$\frac{dx}{dy_k} \geq 0; \forall k \tag{3}$$

$$c \geq 0 \tag{4}$$

$$x_k \geq 0;\ \forall k \tag{5}$$

The objective function, represented by (6), minimizes the difference between the original input of a DMUk (x_{ok}) and the input obtained after redistribution (x_k). Thus, the new frontier ensures that the computed optimal solution is as close as possible to the original configuration.

$$Min \sum_{k=1}^{n} |x_{ok} - x_k| \tag{6}$$

To linearize the objective function's absolute value (or modulus), Silveira et al. (2019) have adopted the auxiliary variable Mk. For this, they have replaced the objective function (6) by (7) and have included restrictions (8) and (9). In addition, restriction (10) ensures the redistributed input fixed total sum.

$$Min \sum_{k=1}^{n} M_k \tag{7}$$

Subject to

$$M_k \geq x_{ok} - x_k;\ \forall k \tag{8}$$

$$M_k \geq -x_{ok} + x_k;\ \forall k \tag{9}$$

$$\sum_{k=1}^{n} (x_{ok} - x_k) = 0 \tag{10}$$

The Linear Programme (11) describes the parabolic frontier for cases using one input and one output. Unlike classic DEA, the redistribution by parabolic DEA requires a solution of a single Linear Programme.

$$Min \sum_{k=1}^{n} M_k \tag{11}$$

Subject to

$$M_k \geq x_{ok} - x_k;\ \forall k$$

$$M_k \geq -x_{ok} + x_k;\ \forall k$$

$$\sum_{k=1}^{n} (x_{ok} - x_k) = 0$$

$$x_k = ay_k^2 + by_k + c;\ \forall k$$

$$\frac{d^2 x_k}{dy_k^2} \geq 0; \forall k$$

$$\frac{dx}{dy_k} \geq 0; \forall k$$

$$c \geq 0$$

$$x_k \geq 0;\ \forall k$$

The bi-dimensional model described can be generalised for multiple outputs scenarios. In this case, the frontier previously named as parabolic becomes a paraboloid and is represented by equation (12). Moreover, restrictions to convexity and increasing monotonicity must be adapted to the new outputs as in (13) and (14).

$$x_k = ay_{1k}^2 + by_{1k} + cy_{2k}^2 + dy_{2k} + \ldots + my_{sk}^2 + ny_{sk} + e; \forall k \tag{12}$$

$$\frac{d^2 x_k}{dy_{jk}^2} \geq 0; \forall k \ and \ \forall j \tag{13}$$

$$\frac{dx_k}{dy_{jk}} \geq 0; \forall k \text{ and } \forall j \tag{14}$$

Silveira et al. (2019) have also generalized the Parabolic DEA model for multiple inputs and outputs scenarios. It is noteworthy that this model only redistributes one input. For this generalization, equation (15) represents the new frontier that will be determined by the redistribution of input x_1. Restrictions (16) and (17) represent the restrictions to frontier's convexity and increasing monotonicity.

$$x_1 + bx_2 + \ldots + mx_r = ny_1^2 + oy_1 + py_2^2 + qy_2 + \ldots + uy_2^2 + vy_2 + w; b \neq 0 \tag{15}$$

$$\frac{\partial^2 x_{ik}}{\partial y_{jk}^2} \geq 0;\ \forall k, \forall i \text{ and } \forall j \tag{16}$$

$$\frac{\partial x_{ik}}{\partial y_{jk}} \geq 0;\ \forall k, \forall i \text{ and } \forall j \tag{17}$$

3 PARABOLIC DEA FOR MULTIPLE INPUTS REDISTRIBUTION

In this paper, we extend the Parabolic DEA model developed by Silveira et al. (2019) to redistribute multiple inputs. For this, as main strategy, we have used the multi-objective approach, that is, one objective function for each input to be redistributed. Equation (18) represents the objective functions.

$$Min \sum_{k=1}^{n} |x_{ok} - x_k|, \forall i \tag{18}$$

Similarly to the original model (Silveira et al., 2019), the auxiliary variable M_{ik} and its restrictions linearize the objective functions.

$$Min \sum_{k=1}^{n} M_{ik};\ \forall i \tag{19}$$

Subject to

$$M_{ik} \geq x_{ok} - x_k\ ; \forall i \text{ and } \forall k \tag{20}$$

$$M_{ik} \geq -x_{ok} + x_k\ ;\ \forall i \text{ and } \forall k \tag{21}$$

In this approach, we have stated that the efficiency frontier is the intersection of the paraboloid functions that will be determined for each input by the model, as represented in (22).

$$x_{1k} = ay_k^2 + by_k + c;\ \forall k \tag{22}$$

$$x_{2k} = dy_k^2 + ey_k + f;\ \forall k$$

$$\ldots$$

$$x_{rk} = vy_k^2 + wy_k + z;\ \forall k$$

Therefore, restrictions (23) and (24) assure frontier's convexity and increased monotonicity. Furthermore, restrictions (25) and (26) state that the independent term of all paraboloids equations and the new inputs must be non-negative.

$$\frac{dx_{ik}}{dy} \geq 0;\ \forall i \ \ and\ \forall k \tag{23}$$

$$\frac{d^2x_{ik}}{dy^2} \geq 0;\ \forall i \ \ and\ \forall k \tag{24}$$

$$c, f, \ldots, z \geq 0 \tag{25}$$

$$x_{ik} \geq 0;\ \forall i \ \ and\ \forall k \tag{26}$$

Combining the objective functions and restrictions presented, the Linear Programme (27) describes the parabolic DEA model for redistributing multiple inputs in one output scenarios.

$$Min\ \sum_{k=1}^{n} M_{ik}; \forall i \tag{27}$$

Subject to

$$\begin{gathered}
M_{ik} \geq x_{oik} - x_{ik};\ \forall i\ and\ \ \forall k \\
M_{ik} \geq - x_{oik} + x_{ik};\ \forall i\ \ and\ \forall k \\
\sum_{k=1}^{n} (x_{oik} - x_{ik}) = 0;\ \forall i \\
x_{1k} = ay_k^2 + by_k + c;\ \forall k \\
x_{2k} = dy_k^2 + ey_k + f;\ \forall k \\
\cdots \\
x_{rk} = vy_k^2 + wy_k + z;\ \forall k \\
\frac{dx_{ik}}{dy} \geq 0;\ \forall i\ \ and\ \forall k \\
\frac{d^2x_{ik}}{dy^2} \geq 0;\ \forall i\ \ and\ \forall k \\
c, f, \ldots, z \geq 0 \\
x_{ik} \geq 0;\ \forall i\ and\ \forall k
\end{gathered}$$

This model can be generalised to redistribute multiple inputs in multiple outputs scenarios by developing the paraboloid surface to consider the new variables, as represented by (28). It is important to stress out that this model do not change outputs.

$$x_{1k} = ay_{1k}^2 + by_{1k} + cy_{2k}^2 + dy_{2k} + \ldots + ny_{sk} + my_{sk}^2 + o;\ \forall k \tag{28}$$
$$\ldots$$
$$x_{rk} = py_{1k}^2 + qy_{1k} + ty_{2k}^2 + uy_{2k} + \ldots + vy_{sk} + wy_{sk}^2 + z;\ \forall k$$

This generalisation also requires the expansion of the restrictions for convexity and monotonicity to limit all outputs as in (29) and (30).

$$\frac{dx_{ik}}{dy_j} \geq 0;\ \forall i\ ,\forall j\ and\ \forall k \tag{29}$$

$$\frac{d^2x_{ik}}{dy_j^2} \geq 0;\ \forall i\ ,\forall j\ and\ \forall k \tag{30}$$

The Linear Programme (31) describes the parabolic DEA model for redistributing multiple inputs in multiple outputs scenarios.

$$Min\ \sum_{k=1}^{n} M_{ik};\forall i \tag{31}$$

Subject to

$$M_{ik} \geq x_{oik} - x_{ik};\ \forall i\ and\ \forall k$$
$$M_{ik} \geq -x_{oik} + x_{ik};\ \forall i\ and\ \forall k$$
$$\sum_{k=1}^{n} (x_{oik} - x_{ik}) = 0;\ \forall i$$
$$x_{1k} = ay_{1k}^2 + by_{1k} + cy_{2k}^2 + dy_{2k} + \ldots + ny_{sk} + my_{sk}^2 + o;\ \forall k$$
$$\ldots$$
$$x_{rk} = py_{1k}^2 + qy_{1k} + ty_{2k}^2 + uy_{2k} + \ldots + vy_{sk} + wy_{sk}^2 + z;\ \forall k$$
$$\frac{dx_{ik}}{dy_j} \geq 0;\ \forall i\ ,\forall j\ and\ \forall k$$
$$\frac{d^2x_{ik}}{dy_j^2} \geq 0;\ \forall i\ ,\forall j\ and\ \forall k$$
$$o \geq 0$$
$$z \geq 0$$
$$x_{ik} \geq 0;\ \forall i\ and\ \forall k$$

The proposed model respects the characteristics of the original parabolic model developed by Silveira et al. (2019). It constructs the efficiency frontier as a smooth and rising convex curve, in a shape corresponding to the intersection of two or more paraboloids, which respects VRS conditions. It redistributes multiple input variables keeping their sum constant, making all DMUs extreme efficient and located at the frontier. It is noteworthy that this method results in the absence of restrictions that simultaneously contain all input variables. This condition assures the solution's uniqueness in multi-objective models.

4 RESOLUTION APPROACHES AND NUMERICAL EXAMPLES

We present two approaches for redistributing multiple inputs using Parabolic DEA. The first one uses scalarizing technique for calculating nondominated solutions (Antunes et al., 2016). That is, the r objective functions are replaced by a single function. In this approach, the single function is represented by a weighted-sum of the objective functions with positive weights λ_i, as in (32).

$$Max\ z_\lambda = \sum_{i=1}^{r} \lambda_i f_i(x) \tag{32}$$

Subject to

$$\sum_{i=1}^{r} \lambda_i = 1$$

$$\lambda_i > 0, i = 1, \ldots, r$$

This approach requires the variable's nondimensionalization since it is not possible to sum different measure units. In addition, the variable's divergence of scale may lead to unfeasible results. This technique consists of dividing each variable by its largest unit with no loss in result since DEA is invariant at scale.

The second approach is the Separation of Variables, which consists in transforming the single Linear Programme in multiple programmes, one for each input. This is possible because there is no restriction that simultaneously consider all input variables, so it is viable to separate each input objective function and its related fixed sum restriction. To illustrate the model, we present a numerical example for a set of five DMUs, one output (y) and two inputs (x_1 and x_2) to be redistributed. Table 1 depicts the data set and the classical input-oriented BCC efficiency indexes.

Table 1 – Numerical example 1 – original data set and BCC efficiencies.

DMU	x_1	x_2	y	BCC eff.
A	0.90	6.00	1.00	1.00
B	4.10	5.00	2.00	1.00
C	8.80	8.00	2.50	0.82
D	16.00	10.00	3.50	0.84
E	25.20	9.00	5.40	1.00
Total	55.00	38.00	14.40	

For nondimensionalization, the variables must be divided by its largest values. Thus, we divided variables x_1, x_2 and y by 25.2, 10 and 5.4, respectively. Table 2 depicts the original data set nondimensionalized.

The objective functions for this example are as shown in (33).

$$Min\ M_{11} + M_{12} + M_{13} + M_{14} + M_{15} \tag{33}$$

$$Min\ M_{21} + M_{22} + M_{23} + M_{24} + M_{25}$$

Table 2 – Numerical example 1 – original data set nondimensionalized and BCC efficiencies.

DMU	x_1	x_2	y	BCC eff.
A	0.04	0.60	0.19	1.00
B	0.16	0.50	0.37	1.00
C	0.35	0.80	0.46	0.82
D	0.63	1.00	0.65	0.84
E	1.00	0.90	1.00	1.00
Total	2.18	3.80	2.67	

They minimize the auxiliary variable M_{ik}, which represents the difference between the original input i value of DMU k and after redistribution. The variable M_{ik} is related to the module restrictions, represented by (34) in this example.

$$\begin{aligned}
M_{11}+x_{11} &\geq 0.04 \\
M_{12}+x_{12} &\geq 0.16 \\
M_{13}+x_{13} &\geq 0.35 \\
M_{14}+x_{14} &\geq 0.63 \\
M_{15}+x_{15} &\geq 1.00 \\
M_{11}-x_{11} &\geq -0.04 \\
M_{12}-x_{12} &\geq -0.16 \\
M_{13}-x_{13} &\geq -0.35 \\
M_{14}-x_{14} &\geq -0.63 \\
M_{15}-x_{15} &\geq -1.00 \\
M_{21}+x_{21} &\geq 0.60 \\
M_{22}+x_{22} &\geq 0.50 \\
M_{23}+x_{23} &\geq 0.80 \\
M_{24}+x_{24} &\geq 1.00 \\
M_{25}+x_{25} &\geq 0.90 \\
M_{21}-x_{21} &\geq -0.60 \\
M_{22}-x_{22} &\geq -0.50 \\
M_{23}-x_{23} &\geq -0.80 \\
M_{24}-x_{24} &\geq -1.00 \\
M_{25}-x_{25} &\geq -0.90
\end{aligned} \tag{34}$$

Equations (35) and (36) represent the restrictions for inputs x_1 and x_2 fixed sum.

$$x_{11}+x_{12}+x_{13}+x_{14}+x_{15}=2.18 \tag{35}$$

$$x_{21} + x_{22} + x_{23} + x_{24} + x_{25} = 3.80 \tag{36}$$

The set of paraboloids restrictions represented by (37) for x_1 and (38) for x_2 will define the new efficiency frontier.

$$\begin{aligned} x_{11} - 0.0361a - 0.19b - c &= 0 \\ x_{12} - 0.1369a - 0.37b - c &= 0 \\ x_{13} - 0.2116a - 0.46b - c &= 0 \\ x_{14} - 0.4225a - 0.65b - c &= 0 \\ x_{15} - 1.0000a - 1.00b - c &= 0 \end{aligned} \tag{37}$$

$$\begin{aligned} x_{21} - 0.0361d - 0.19e - f &= 0 \\ x_{22} - 0.1369d - 0.37e - f &= 0 \\ x_{23} - 0.2116d - 0.46e - f &= 0 \\ x_{24} - 0.4225d - 0.65e - f &= 0 \\ x_{25} - 1.0000d - 1.00e - f &= 0 \end{aligned} \tag{38}$$

Restrictions (39) and (40) ensure the frontier's positive concavity and its increasing monotonicity, developed from the determination that the second derivative and the first derivative are non-negative.

$$\begin{aligned} 0.38a - 1.00b &\geq 0 \\ 0.38d - 1.00e &\geq 0 \end{aligned} \tag{39}$$

$$\begin{aligned} a &\geq 0 \\ d &\geq 0 \\ c &\geq 0 \\ f &\geq 0 \end{aligned} \tag{40}$$

Using the Weighted Sum scalarizing technique and knowing that $Min z_\lambda = Max(-z_\lambda)$, the objective functions in (33) become the mono-objective function represented in (41).

$$Max\ \lambda_1 \left(-\sum_{k=1}^{5} M_{1k} \right) + \lambda_2 \left(-\sum_{k=1}^{5} M_{2k} \right) \tag{41}$$

The Linear Programme (42) represents the proposed solution for redistributing multiple inputs using Parabolic DEA and Weighted Sum of the Objective Functions resolution. Table 3 depicts the obtained results and the variations used for λ_i.

$$Max\ \lambda_1 \left(-\sum_{k=1}^{5} M_{1k} \right) + \lambda_2 \left(-\sum_{k=1}^{5} M_{2k} \right) \tag{42}$$

Subject to

$$M_{11}+x_{11}\geq 0.04$$
$$M_{12}+x_{12}\geq 0.16$$
$$M_{13}+x_{13}\geq 0.35$$
$$M_{14}+x_{14}\geq 0.63$$
$$M_{15}+x_{15}\geq 1.00$$
$$M_{11}-x_{11}\geq -0.04$$
$$M_{12}-x_{12}\geq -0.16$$
$$M_{13}-x_{13}\geq -0.35$$
$$M_{14}-x_{14}\geq -0.63$$
$$M_{15}-x_{15}\geq -1.00$$
$$M_{21}+x_{21}\geq 0.60$$
$$M_{22}+x_{22}\geq 0.50$$
$$M_{23}+x_{23}\geq 0.80$$
$$M_{24}+x_{24}\geq 1.00$$
$$M_{25}+x_{25}\geq 0.90$$
$$M_{21}-x_{21}\geq -0.60$$
$$M_{22}-x_{22}\geq -0.50$$
$$M_{23}-x_{23}\geq -0.80$$
$$M_{24}-x_{24}\geq -1.00$$
$$M_{25}-x_{25}\geq -0.90$$
$$x_{11}+x_{12}+x_{13}+x_{14}+x_{15}=2.18$$
$$x_{21}+x_{22}+x_{23}+x_{24}+x_{25}=3.80$$
$$x_{11}-0.0361a-0.19b-c=0$$
$$x_{12}-0.1369a-0.37b-c=0$$
$$x_{13}-0.2116a-0.46b-c=0$$
$$x_{14}-0.4225a-0.65b-c=0$$
$$x_{15}-1.0000a-1.00b-c=0$$
$$x_{21}-0.0361d-0.19e-f=0$$
$$x_{22}-0.1369d-0.37e-f=0$$
$$x_{23}-0.2116d-0.46e-f=0$$
$$x_{24}-0.4225d-0.65e-f=0$$
$$x_{25}-1.0000d-1.00e-f=0$$
$$0.38a-1.00b\geq 0$$
$$0.38d-1.00e\geq 0$$

Table 3 – Results for the Weighted Sum of the Objective Functions in one output scenario.

DMU	Result 1		Result 2		Result 3	
	λ_1=0.5	λ_2=0.5	λ_1=0.9	λ_2=0.1	λ_1=0.3	λ_2=0.7
	x_1	x_2	x_1	x_2	x_1	x_1
A	0.12	0.68	0.12	0.68	0.12	0.68
B	0.24	0.71	0.24	0.71	0.24	0.71
C	0.31	0.73	0.31	0.73	0.31	0.73
D	0.51	0.78	0.51	0.78	0.51	0.78
E	1.00	0.90	1.00	0.90	1.00	0.90
Total	2.18	3.80	2.18	3.80	2.18	3.80

Table 3 shows that the three situations have achieved the same results. This indicates the independence of the weighting values for obtaining the results and that both variables represent the same importance for the model. We expected these results since there are no simultaneous restrictions for both input variables in the linear programme (42). This condition allows the Separation of Variables resolution approach.

Using the Separation of Variables approach in the numerical example presented, we obtain two linear programmes, one for input x_1, described by (43) and the other one for input x_2, (44). Table 4 depicts the results achieved.

- Linear Programme 1 (input x_1):

$$Min\ M_{11} + M_{12} + M_{13} + M_{14} + M_{15} \tag{43}$$

Subject to

$$M_{11} + x_{11} \geq 0.04$$
$$M_{12} + x_{12} \geq 0.16$$
$$M_{13} + x_{13} \geq 0.35$$
$$M_{14} + x_{14} \geq 0.63$$
$$M_{15} + x_{15} \geq 1.00$$
$$M_{11} - x_{11} \geq -0.04$$
$$M_{12} - x_{12} \geq -0.16$$
$$M_{13} - x_{13} \geq -0.35$$
$$M_{14} - x_{14} \geq -0.63$$
$$M_{15} - x_{15} \geq -1.00$$
$$x_{11} + x_{12} + x_{13} + x_{14} + x_{15} = 2.18$$
$$x_{11} - 0.0361a - 0.19b - c = 0$$
$$x_{12} - 0.1369a - 0.37b - c = 0$$

$$x_{13} - 0.2116a - 0.46b - c = 0$$
$$x_{14} - 0.4225a - 0.65b - c = 0$$
$$x_{15} - 1.0000a - 1.00b - c = 0$$
$$0.38a - 1.00b \geq 0$$
$$a, c \geq 0$$

- Linear Programme 2 (input x_2):

$$Min\ M_{21} + M_{22} + M_{23} + M_{24} + M_{25} \tag{44}$$

Subject to

$$M_{21} + x_{21} \geq 0.60$$
$$M_{22} + x_{22} \geq 0.50$$
$$M_{23} + x_{23} \geq 0.80$$
$$M_{24} + x_{24} \geq 1.00$$
$$M_{25} + x_{25} \geq 0.90$$
$$M_{21} - x_{21} \geq -0.60$$
$$M_{22} - x_{22} \geq -0.50$$
$$M_{23} - x_{23} \geq -0.80$$
$$M_{24} - x_{24} \geq -1.00$$
$$M_{25} - x_{25} \geq -0.90$$
$$x_{21} + x_{22} + x_{23} + x_{24} + x_{25} = 3.80$$
$$x_{21} - 0.0361d - 0.19e - f = 0$$
$$x_{22} - 0.1369d - 0.37e - f = 0$$
$$x_{23} - 0.2116d - 0.46e - f = 0$$
$$x_{24} - 0.4225d - 0.65e - f = 0$$
$$x_{25} - 1.0000d - 1.00e - f = 0$$
$$0.38d - 1.00e \geq 0$$
$$d, f \geq 0$$

Table 5 depicts the results achieved by resolutions 1 and 2, Weighted Sum of the Objective Functions and Separation of Variables, respectively, for multiple inputs and one output scenarios. We can verify that both present equal results, which indicates that the input variables represent the same importance for the model.

Figure 1 shows the DMU's original and redistributed layout, represented respectively by the blue and red dots. We can also visualize the alignment of the DMUs following the parabolic shape by

Table 4 – Resolution 2: Separation of Variables in one output scenario.

DMU	LP 1	LP 2
	x_1	x_2
A	0.12	0.68
B	0.24	0.71
C	0.31	0.73
D	0.51	0.78
E	1.00	0.90
Total	2.18	3.80

Table 5 – Results for the two approaches – multiples inputs and one output.

DMU	Original data		Resolution 1		Resolution 2		Efficiency BCC	
	x_1	x_2	x_1	x_2	x_1	x_2	Original	Final
A	0.04	0.60	0.12	0.68	0.12	0.68	1.00	1.00
B	0.16	0.50	0.24	0.71	0.24	0.71	1.00	1.00
C	0.35	0.80	0.31	0.73	0.31	0.73	0.82	1.00
D	0.63	1.00	0.51	0.78	0.51	0.78	0.84	1.00
E	1.00	0.90	1.00	0.90	1.00	0.90	1.00	1.00
Total	2.18	3.80	2.18	3.80	2.18	3.80	2.18	3.80

redistributing their inputs without changing outputs. It is interesting to note that even DMUs A and B, considered efficient in the original model, have received resources in order to make them efficient from the perspective of the parabolic DEA. There was no change in DMU E variables.

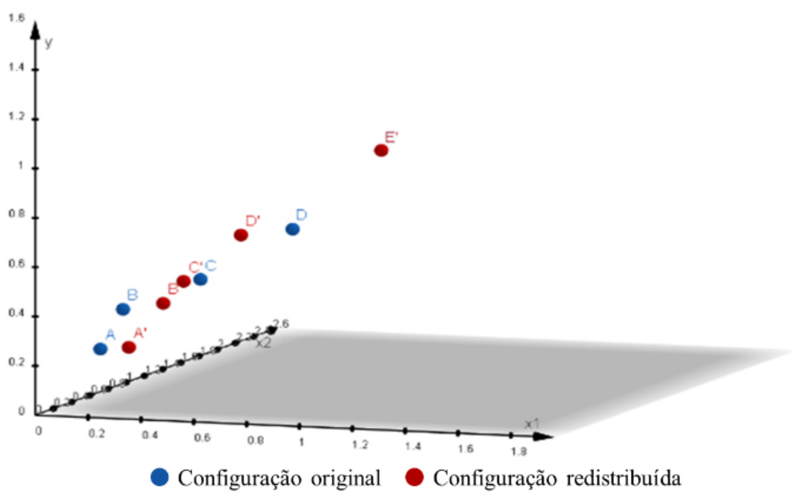

Figure 1 – DMUs – original and redistributed inputs.

The paraboloids defined by the model are represented by (45) and (46) for inputs x_1 and x_2.

$$x_{1k} = 0.69y_k^2 + 0.26y_k + 0.04;\ \forall k \tag{45}$$

$$x_{2k} = 0.17y_k^2 + 0.06y_k + 0.66;\ \forall k \tag{46}$$

Figures 2 and 3 show the paraboloid functions defined for input x_1 and input x_2 on the three-dimensional Cartesian system, respectively.

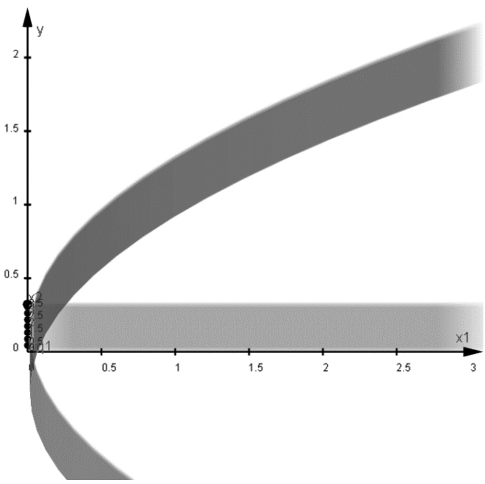

Figure 2 – Paraboloid function x_1.

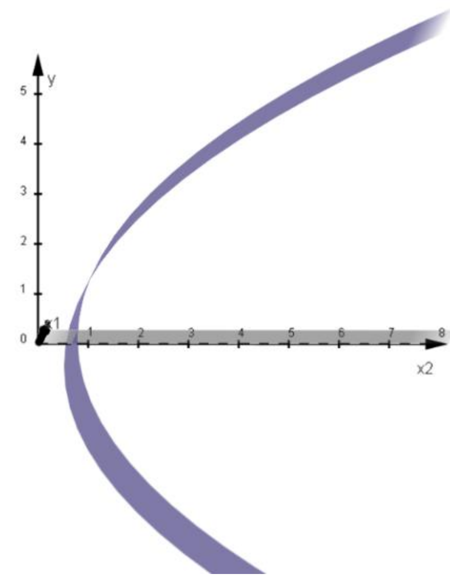

Figure 3 – Paraboloid function x_2.

Figure 4 shows the new DMU's input configuration determined by the model. Note that they are arranged at the intersection of the two paraboloid functions. Figure 5 simultaneously shows the DMUs following the original and the new input configuration determined by the model, respectively represented by the blue and red dots.

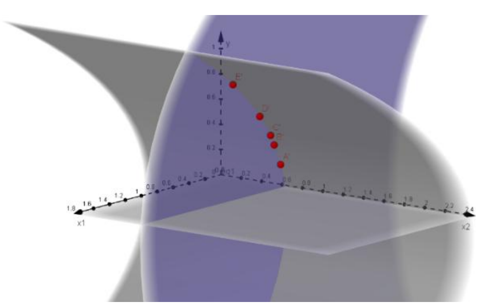

Figure 4 – New DMU's redistribution.

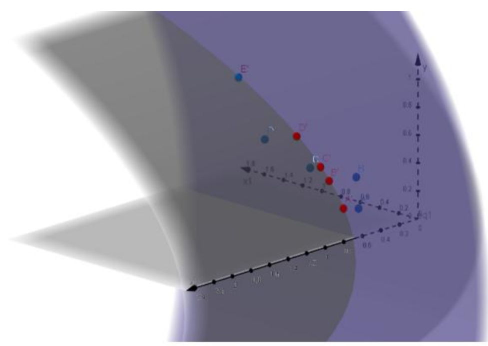

Figure 5 – DMU's original and after redistribution configuration.

To illustrate this approach in multiple outputs scenarios, we will present another numerical example as shown in Table 6. In this example, we will analyse a set of 5 DMUs in which x_1 and x_2 are the inputs to be redistributed. The outputs are represented by y_1 and y_2. The difference between numerical examples 1 and 2 is the inclusion of y_2.

Table 6 – Numerical example 2 – original data-set and BCC efficiencies.

DMU	x_1	x_2	y_1	y_2	BCC ef.
A	0.90	6.00	1.00	4.00	1.00
B	4.10	5.00	2.00	3.00	1.00
C	8.80	8.00	2.50	8.00	1.00
D	16.00	10.00	3.50	6.00	0.84
E	25.20	9.00	5.40	10.00	1.00
Total	55.00	38.00	14.40	31.00	

As in Example 1, the variables must be divided by its largest values for nondimensionalization. Table 7 depicts the original data-set nondimensionalized and equation (47) represents the linear programme for this example.

Table 7 – Numerical example 2 - original data-set nondimensionalized and BCC efficiencies.

DMU	x_1	x_2	y_1	y_2
A	0.04	0.60	0.19	0.40
B	0.16	0.50	0.37	0.30
C	0.35	0.80	0.46	0.80
D	0.63	1.00	0.65	0.60
E	1.00	0.90	1.00	1.00
Total	2.18	3.80	2.67	3.10

$$
\begin{aligned}
&Min\ M_{11}+M_{12}+M_{13}+M_{14}+M_{15} \\
&Min\ M_{21}+M_{22}+M_{23}+M_{24}+M_{25}
\end{aligned}
\tag{47}
$$

Subject to

$$
\begin{aligned}
M_{11}+x_{11} &\geq 0.04 \\
M_{12}+x_{12} &\geq 0.16 \\
M_{13}+x_{13} &\geq 0.35 \\
M_{14}+x_{14} &\geq 0.63 \\
M_{15}+x_{15} &\geq 1.00 \\
M_{11}-x_{11} &\geq -0.04 \\
M_{12}-x_{12} &\geq -0.16
\end{aligned}
$$

$$M_{13} - x_{13} \geq -0.35$$
$$M_{14} - x_{14} \geq -0.63$$
$$M_{15} - x_{15} \geq -1.00$$
$$M_{21} + x_{21} \geq 0.60$$
$$M_{22} + x_{22} \geq 0.50$$
$$M_{23} + x_{23} \geq 0.80$$
$$M_{24} + x_{24} \geq 1.00$$
$$M_{25} + x_{25} \geq 0.90$$
$$M_{21} - x_{21} \geq -0.60$$
$$M_{22} - x_{22} \geq -0.50$$
$$M_{23} - x_{23} \geq -0.80$$
$$M_{24} - x_{24} \geq -1.00$$
$$M_{25} - x_{25} \geq -0.90$$
$$x_{11} + x_{12} + x_{13} + x_{14} + x_{15} = 2.18$$
$$x_{21} + x_{22} + x_{23} + x_{24} + x_{25} = 3.80$$
$$x_{11} - 0.0361a - 0.19b - 0.16c - 0.4d - e = 0$$
$$x_{12} - 0.1369a - 0.37b - 0.09c - 0.3d - e = 0$$
$$x_{13} - 0.2116a - 0.46b - 0.64c - 0.8d - e = 0$$
$$x_{14} - 0.4225a - 0.65b - 0.36c - 0.6d - e = 0$$
$$x_{15} - 1.0000a - 1.00b - 1.00c - 1.0d - e = 0$$
$$x_{21} - 0.0361f - 0.19g - 0.16h - 0.4i - j = 0$$
$$x_{22} - 0.1369f - 0.37g - 0.09h - 0.3i - j = 0$$
$$x_{23} - 0.2116f - 0.46g - 0.64h - 0.8i - j = 0$$
$$x_{24} - 0.4225f - 0.65g - 0.36h - 0.6i - j = 0$$
$$x_{25} - 1.0000f - 1.00g - 1.00h - 1.0i - j = 0$$
$$0.38a - 1.00b \geq 0$$
$$0.60c - 1.00d \geq 0$$
$$0.38f - 1.00g \geq 0$$
$$0.60h - 1.00i \geq 0$$
$$a, c, f, h, e, j \geq 0$$

Using the Weighted Sum of the Objective Functions resolution, Table 8 depicts the obtained results and the variations used for λ_i. As in numerical example 1, the three situations have achieved the same results. This indicates the independence of the weighting values for obtaining the results and that both variables represent the same importance for the model.

Table 8 – Resolution 1: Weighted Sum of the Objective Functions in multiple output scenario.

DMU	Results 1		Results 2		Results 3	
	λ_1=0.5	λ_2=0.5	λ_1=0.1	λ_2=0.9	λ_1=0.7	λ_2=0.3
	x_1	x_2	x_1	x_2	x_1	x_2
A	0.11	0.60	0.11	0.60	0.11	0.60
B	0.20	0.60	0.20	0.60	0.20	0.60
C	0.35	0.80	0.35	0.80	0.35	0.80
D	0.49	0.75	0.49	0.75	0.49	0.75
E	1.03	1.05	1.03	1.05	1.03	1.05

The Separation of Variables approach segregates the objective functions and restrictions of each input variable and calculates the PPLs individually. Thus, using this method in numerical example 2 we have achieved the results shown in Table 9.

Table 9 – Resolution 2: Separation of Variables in multiple outputs scenario.

DMU	PPL 1	PPL 2
	x_1	x_2
A	0.11	0.60
B	0.20	0.60
C	0.35	0.80
D	0.49	0.75
E	1.03	1.05

Equations (48) and (49) represent the paraboloids defined for inputs x_1 and x_2.

$$x_{1k} = 0.64y_{1k}^2 + 0.24y_{1k} + 0.09y_{2k}^2 + 0.06y_{2k} + 0;\ \forall k \tag{48}$$

$$x_{2k} = 0.14y_{1k}^2 + 0.06y_{1k} + 0.22y_{2k}^2 + 0.13y_{2k} + 0.49;\ \forall k \tag{49}$$

Table 10 depicts the results achieved by resolutions 1 and 2, Weighted Sum of the Objective Functions and Separation of Variables respectively, for multiple inputs and outputs scenarios. As in numerical example 2, both approaches have achieved the same results.

Comparing the results obtained for inputs x_1 and x_2 for both numerical examples, Tables 5 and 10, which have as only difference output y_2 in the second one, it is possible to observe that the results obtained are different. This proves that the output variable's choice as well as the number of variables considered directly interfere in the result, although the inputs do not influence the results of redistribution between them.

Table 10 – Resolutions 1 and 2 – multiples inputs and outputs.

DMU	Original Data		Resolution 1		Resolution 2		Efficiency BCC	
	x_1	x_2	x_1	x_2	x_1	x_2	Original	Final
A	0.04	0.60	0.11	0.60	0.11	0.60	1.00	1.00
B	0.16	0.50	0.20	0.60	0.20	0.60	1.00	1.00
C	0.35	0.80	0.35	0.80	0.35	0.80	1.00	1.00
D	0.63	1.00	0.49	0.75	0.49	0.75	0.84	1.00
E	1.00	0.90	1.03	1.05	1.03	1.05	1.00	1.00

5 FINAL COMMENTS

In this paper, we have extended the parabolic DEA model (Silveira et al., 2019) to redistribute multiple inputs simultaneously. For this, we have made an overview about parametric DEA models and other DEA-based models for variable redistribution. In addition, we have detailed the original parabolic DEA model (Silveira et al., 2019), which is the main literature source of our research. The solution for multiple inputs redistribution that we have presented in this study guarantees that all DMUs will become extreme efficient and arranged in the new efficiency frontier without changing the outputs. In addition, the model returns the efficiency frontier as a rising convex curve, of a shape corresponding to the intersection of multiple paraboloids defined for each input variable, which meets the VRS conditions.

To define the parabolic efficiency frontier, the model defines a paraboloid function for each input variable, resulting in the absence of restrictions that simultaneously contemplate all the input variables. In addition, an only multi-objective Linear Programming Problem represents this model.

We have presented two resolutions approaches named Weighted Sum of the Objective Functions and Separation of Variables, which were used in two numerical examples in scenarios of one output and multiple outputs. The results achieved were identical for each scenario, showing the independence between the input variables.

We also have observed that the parabolic model for multiple inputs redistribution assures the solution's uniqueness, unlike classic multi-objective models, which are able to determine only a set of non-dominated solutions. This characteristic is valid regardless the number of inputs analysed, as long as no restrictions simultaneously consider all inputs, that is, as long as the resolution by Separation of Variables is feasible.

In futures studies, we intend to search for a solution that redistributes multiple inputs considering dependence between variables. Moreover, we also intend to study the extension of the model to redistribute multiple outputs. In such case, we believe that the main difficulty will be non-linear nature of the models.

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How to cite

MOREIRA LS, SOARES DE MELLO JCCB & ANGULO MEZA L. 2023. Redistributing multiple inputs with a parabolic DEA model. *Pesquisa Operacional*, **43**: e267995. doi: 10.1590/0101-7438.2023.043.00267995.