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A NOTE ON THE NP-HARDNESS OF THE SEPARATION PROBLEM ON SOME VALID INEQUALITIES FOR THE ELEMENTARY SHORTEST PATH PROBLEM

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ABSTRACT. In this paper, we investigate the separation problem on some valid inequalities for the $s - t$ elementary shortest path problem in digraphs containing negative directed cycles. As we will see, these inequalities depend to a given parameter $k \in \mathbb{N}$. To show the NP-hardness of the separation problem of these valid inequalities, considering the parameter $k \in \mathbb{N}$, we establish a polynomial reduction from the problem of the existence of $k+2$ vertex-disjoint paths between $k+2$ pairs of vertices $(s_1, t_1), (s_2, t_2) \ldots (s_{k+2}, t_{k+2})$ in a digraph to the decision problem associated to the separation of these valid inequalities. Through some illustrative instances, we exhibit the evoked polynomial reduction in the cases $k = 0$ and $k = 1$.

Keywords: polytope, digraphs, shortest path, valid inequality, separation.

1 INTRODUCTION

Let $G = (V, E)$, be a connected directed graph and $s \in V$ and $t \in V$ two vertices of G. We suppose that G contains q elementary paths from s to t, and we denote $p_1, p_2, \ldots, p_q$, these $s - t$ elementary paths. $V(p_i)$ and $E(p_i)$ are the set of vertices and the set of arcs corresponding to the s-t elementary path p_i respectively.

Given a parameter $k \in \mathbb{N}$, let (S_k, A_k) be a pair of sets $S_k \subset V$ and $A_k \subset E$ in the digraph $G = (V, E)$ such that:

- No arc in A_k has an endpoint in S_k;
- $\forall\ p_i\ (i = 1, \ldots, q),\ |(S_k \cap V(p_i))| + |(A_k \cap E(p_i))| \leq k.$

In Ibrahim (2008) and Ibrahim et al. (2014), we call a pair $(S_k,\ A_k)$ as defined above a k-$subset$ pair with respect to vertices s and t. An element of S_k is called a k-$vertex$ and an element of A_k is called a k-arc.

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For any pair (S_k, A_k) which is a k-*subset* pair, the inequality

$$\sum_{p \in S_k} x_p + \sum_{(q,r) \in A_k} y_{qr} \leq k \tag{1}$$

is shown to be valid for $\mathcal{P}$, the polytope induced by the $s - t$ elementary paths in G. x_p and y_{qr} are binary variables associated to the vertex p and the arc (q, r) respectively, (see Ibrahim (2008), Ibrahim et al. (2014)). Such valid inequality is called a *valid inequality of order k*.

In this paper, we investigate the separation problem of the so-called *valid inequalities of order k*, first presented and exploited in cutting plane framework in view to solve the shortest path problem in digraphs possibly having negative cycles (see Ibrahim (2008), Ibrahim et al. (2014)). Considering a mixed integer linear model of the shortest elementary path problem, we use these valid inequalities in a cutting plane algorithm to build strong linear relaxations. For the mixed integer linear model of the shortest elementary path problem, one could refer to Maculan et al. (2003) and Ibrahim et al. (2009). We prove the NP-hardness of the separation problem of *valid inequalities of order k* by establishing a polynomial reduction from the problem of the existence of $k + 2$ vertex-disjoint paths between $k + 2$ pairs of vertices $(s_1, t_1), (s_2, t_2), \ldots, (s_{k+2}, t_{k+2})$ in a digraph to the decision problem associated to the separation of these valid inequalities.

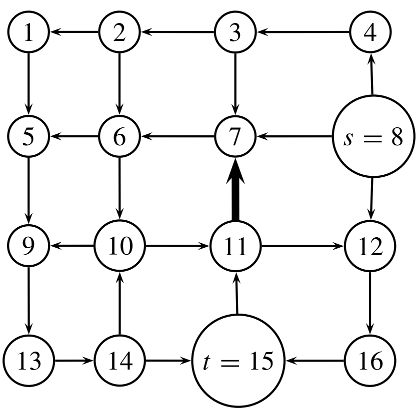

Figure 1 – (11, 7) is a 0-arc w.r.t. the vertices $s = 8$ and $t = 15$.

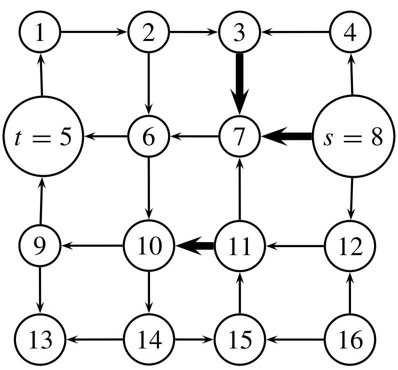

Figure 2 – (11, 10) is a 1-arc w.r.t. the vertices $s = 8$ and $t = 5$.

Figure 1 and Figure 2 illustrate the concept of $k - arc$ and $k - vertex$. In Figure 1, the arc $(11, 7)$ is a $0 - arc$ w.r.t. vertices $s = 8$ and $t = 15$ as it is not belonging to any elementary path between vertices 8 and 15. In Figure 2, $(\emptyset, A_1)$ constitutes a 1-subset pair, with $A_1 = \{(3, 7), (8, 7), (11, 10)\}$ and induces the valid inequality $y_{3,7} + y_{8,7} + y_{11,10} \leq 1$. That is, all $s - t$ elementary paths in Figure 2 passe by at most one of the $1 - arcs$ $(3, 7)$, $(8, 7)$ and $(11, 10)$. One can also observe in Figure 2 that the vertices 14 and 15 are $0 - vertices$. $k - arcs$ and $k - vertices$ induce what we call valid inequalities of *order* k and we will see that in general the problem consisting to detect such vertices and arcs is a difficult task.

The paper is organized as follows. In section 2, we address some cases for which the separation problem of *valid inequalities of order k* can be solved in polynomial time. Then, we show the NP-hardness of the problem of separation of these valid inequalities in digraphs for a given general k. In section 3, considering the cases $k = 0$ and $k = 1$, we present some instances illustrating the evoked polynomial reduction between the problem of the existence of $k + 2$ vertex-disjoint paths between $k + 2$ pairs of vertices $(s_1, t_1), (s_2, t_2), \ldots, (s_{k+2}, t_{k+2})$ and the decision problem associated to the separation of these valid inequalities in a digraph.

2 SEPARATION PROBLEM FOR VALID INEQUALITIES OF *ORDER k*

Given $(\bar{x}, \bar{y})$ an optimal (fractional) solution of a shortest path linear model, the separation problem w.r.t. valid inequalities of *order k* consists in finding in G, a *k-subset* pair (S_k, A_k) such that:

$$\sum_{p \in S_k} \bar{x}_p + \sum_{(q,r) \in A_k} \bar{y}_{qr} > k.$$

The problem of separating valid inequalities of *order k* corresponds in looking for a *k-subset* pair $(S_k,\ A_k)$ in G such that the valid inequality of *order k* is violated.

2.1 Some polynomial cases

We have polynomial algorithms for some special cases:

- if $\Gamma^+(\alpha) = \emptyset$ or $\Gamma^-(\alpha) = \emptyset$, then $\alpha \in S_k,\ k = 0$;
- if $t \in \Gamma^+(\alpha)$ and there is no elementary path from s to α, then $\alpha \in S_k,\ k = 0$;
- if $s \in \Gamma^-(\alpha)$ and there is no elementary path from α to t, then $\alpha_i \in S_k,\ k = 0$.

Where $\Gamma^-(\alpha)$ and $\Gamma^+(\alpha)$ denote the sets of arcs coming into and going out of the vertex α, respectively. That is, considering a digraph $G = (V, E)$, $\Gamma^-(\alpha) = \{(\beta, \alpha) : (\beta, \alpha) \in E\}$ and $\Gamma^+(\alpha) = \{(\alpha, \beta) : (\alpha, \beta) \in E\}$.

In the case of undirected graphs, for $k = 0$, the first polynomial algorithms solving the problem of the existence of $k + 2$ vertex-disjoint elementary paths between $k + 2$ pairs of vertices $(s_1, t_1), (s_2, t_2), \ldots, (s_{k+2}, t_{k+2})$ are due to Ohtsuzi (1981), Seymour (1980), Shiloach (1980) and Thomassen (1985). Robertson & Seymour (1995) treat the general case of k.

In digraphs, there exist particular cases for which the problem of the existence of $k + 2$ vertex-disjoint elementary paths between $k + 2$ pairs of vertices $(s_1, t_1), (s_2, t_2), \ldots, (s_{k+2}, t_{k+2})$ is solvable in polynomial time. Perl & Shiloach (1978) present a polynomial algorithm that solves such problem, with $k = 0$, in three connected directed planar and directed acyclic graphs. The latter result concerning directed acyclic graphs is extended for a given k by Fortune et al. (1980). One can remark that in directed acyclic graphs, the separation problem of these inequalities is not interesting, as in such digraph the shortest path problem can be solved easily. On other hand, Schrijver (1994) present a polynomial method for planar digraphs for a given k.

2.2 The general case for the problem

In the general case, for a given $k \in \mathbb{N}$, the separation problem consists in finding $k+1$-uplet $(\theta_1, \theta_2, \ldots, \theta_{k+1})$ in G such that $\theta_i \in S_k$ or $\theta_i \in A_k$ and $i = 1, \ldots, k+1$. If $\theta_i \in S_k$, we set $\theta_i = \alpha_i$, otherwise $\theta_i \in A_k$ and we set $\theta_i = (\alpha_i, \beta_i)$, where α_i and β_i are the endpoints of the arc θ_i.

Let Π_k be the following decision problem associated to the separation problem of valid inequalities of order k:

"Given $k+1$ vertices and/or arcs $\theta_1, \theta_2, \ldots, \theta_{k+1}$ in G, is θ_i a k-vertex or a k-arc?" With $i = 1, \ldots, k+1$.

Consider the problem Π'_k defined as follow:

"Given $2k+4$ distinct vertices $s_1, t_1, s_2, t_2, \ldots, s_{k+2}, t_{k+2}$, are there no $k+2$ elementary paths, $P_{s_1,t_1}, P_{s_2,t_2} \ldots P_{s_{k+1},t_{k+1}}, P_{s_{k+2},t_{k+2}}$ in G such that $V(P_{s_i,t_i}) \cap V(P_{s_j,t_j}) = \emptyset,\ 1 \le i < j \le k+2$?" Where $V(P_{s_i,t_i})$ are vertex sets of the elementary path P_{s_i,t_i} between s_i and t_i. Π'_k is well known to be NP-complete in general digraph even if $k = 0$ (see Fortune, Hopcroft & Wyllie (1980), Garey & Johnson (1979)).

For a given k, we show the NP-completeness of Π_k, by exhibiting the following polynomial reduction from Π'_k to Π_k:

For any instance of Π'_k, considering a $k+1$-uplet $(\theta_1, \theta_2, \ldots, \theta_{k+1})$ such that $\theta_i \in S_k$ or $\theta_i \in A_k$ and $i = 1, \ldots, k+1$, its corresponding instance of Π_k is obtained:

- by adding a vertex α_i and the arcs (t_i, α_i) and (α_i, s_{i+1}), if θ_i is the vertex α_i;
- by adding the arcs (α_i, β_i), (t_i, α_i) and (β_i, s_{i+1}), if θ_i is the arc (α_i, β_i);
- and by setting $s = s_1$ and $t = t_{k+2}$.

Lemma 2.1. *The answer to the instance of Π'_k is YES **iff** the answer to the instance of Π_k is also YES.*

Proof. $i)$ $\Rightarrow$: Let $P_{s,t}$ be a path that visits the nodes $\alpha_1, \alpha_2, \ldots \alpha_{k+1}$ (in this order) in the graph of the instance of Π_k. This path can be decomposed into the sub-paths P_{s_i,t_i}, $P_{t_i,s_{i+1}}$ and $P_{s_{i+1},t_{i+1}}$, $i = 1, \ldots, k+1$ where $s = s_1$, $t = t_{k+2}$ and paths $P_{t_i,s_{i+1}}$, $i = 1, \ldots, k+1$ are the sequences t_i, α_i, s_{i+1}. $P_{s,t}$ cannot be elementary because $V(P_{s_1,t_1}) \cap \ldots \cap V(P_{s_{k+2},t_{k+2}}) \neq \emptyset$, since the answer to the instance of Π'_k is YES.

$ii)$ $\Leftarrow$: Let $P_{s_1,t_1}, P_{s_2,t_2}, \ldots, P_{s_{k+2},t_{k+2}}$ be paths in the graph of the instance of Π'_k. Consider a path $P_{s,t}$ in the graph of the instance of Π_k. $P_{s,t}$ can be decomposed into the sub-paths P_{s_i,t_i}, $P_{t_i,s_{i+1}}$ and $P_{s_{i+1},t_{i+1}}$, $i = 1, \ldots, k+1$ where $s = s_1$, $t = t_{k+2}$ and paths $P_{t_i,s_{i+1}}$, $i = 1, \ldots, k+1$ are the sequences t_i, α_i, s_{i+1}. Since the answer to the instance of Π_k is YES, $P_{s,t}$ is not elementary. This implies that $V(P_{s_1,t_1}) \cap \ldots \cap V(P_{s_{k+2},t_{k+2}}) \neq \emptyset$. $\square$

Theorem. *The decision problem Π_k is NP-complete.*

Proof. As the decision problem Π'_k is known to be NP-complete in general digraph even if $k = 0$ (see Fortune, Hopcroft & Wyllie (1980), Garey & Johnson (1979)), by Lemma 2.1, it's obvious that the problem Π_k is also NP-complete. □

After such polynomial transformation, for a given $k \in \mathbb{N}$, we conclude that the problem of separation of valid inequalities of order k is NP-hard as its associated decision problem Π_k is NP-complete.

3 SEPARATION PROBLEM IN THE CASES $k = 0$ AND $k = 1$

3.1 Separation problem in the case $k = 0$

In the case $k = 0$, the decision problem associated to the separation problem of valid inequalities of order 0, Π_0, is formulated as follow:

"Given a vertex α or an arc (α, β) in G, is α a 0-vertex or is (α, β) a 0-arc w.r.t. vertices s and t?"

To answer the complexity issue, let us consider the problem Π'_0 defined as:

"Given four distinct vertices s_1, t_1, s_2, t_2, are there no two elementary paths, P_{s_1,t_1} and P_{s_2,t_2} in G such that $V(P_{s_1,t_1}) \cap V(P_{s_2,t_2}) \neq \emptyset$?" Π'_0 is well known to be NP-complete in general digraphs, see Fortune, Hopcroft & Wyllie (1980), Garey & Johnson (1979).

The NP-completeness of problem Π_0 is readily obtained by considering the following polynomial reduction from Π'_0 to Π_0:

For any instance of Π'_0, the corresponding instance of Π_0 is obtained by adding a vertex α, two arcs (t_1, α) and (α, s_2), and by setting $s = s_1$ and $t = t_2$, or by adding the arcs (α, β), (t_1, α), (β, s_2) and we set $s = s_1$ and $t = t_2$.

Lemma 3.1. *The answer to the instance of Π'_0 is YES **iff** the answer to the instance of Π_0 is YES.*

Proof. *i)* $\Rightarrow$: Let $P_{s,t}$ be a path that visits node α in the graph of the instance of Π_0. This path can be decomposed into the sub-paths P_{s_1,t_1}, P_{t_1,s_2}, P_{s_2,t_2}, where $s = s_1$, $t = t_2$ and path P_{t_1,s_2} is the sequence t_1, α, s_2. $P_{s,t}$ cannot be elementary because $V(P_{s_1,t_1}) \cap V(P_{s_2,t_2}) \neq \emptyset$, since the answer to the instance of Π'_0 is YES.

ii) $\Leftarrow$: Let P_{s_1,t_1} and P_{s_2,t_2} be paths in the graph of the instance of Π'_0. Consider a path $P_{s,t}$ in the graph of the instance of Π_0. $P_{s,t}$ can be decomposed into the sub-paths P_{s_1,t_1}, P_{t_1,s_2}, P_{s_2,t_2}, where $s = s_1$, $t = t_2$ and path P_{t_1,s_2} is the sequence t_1, α, s_2. Since the answer to the instance of Π_0 is YES, $P_{s,t}$ is not elementary. This implies that

$$V(P_{s_1,t_1}) \cap V(P_{s_2,t_2}) \neq \emptyset. \qquad \square$$

Example. Consider the next instance of Π'_0, such that $s_1 = 6$, $s_2 = 5$, $t_1 = 4$, and $t_2 = 3$:

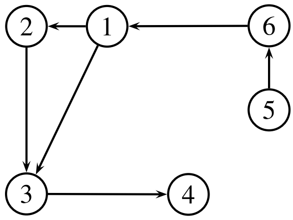

Figure 3 – An instance of Π'_0.

The elementary paths between $s_1 = 6$ and $t_1 = 4$ represented by vertices $[6, 1, 3, 4]$ and $[6, 1, 2, 3, 4]$ are not vertex-disjoints with $[5, 6, 1, 2, 3]$ and $[5, 6, 1, 3]$ the elementary paths between $s_2 = 5$ and $t_2 = 3$, thus $\{6, 1, 3, 4\} \cap \{5, 6, 1, 2, 3\} \neq \emptyset$, $\{6, 1, 2, 3, 4\} \cap \{5, 6, 1, 2, 3\} \neq \emptyset$, $\{6, 1, 3, 4\} \cap \{5, 6, 1, 3\} \neq \emptyset$ and $\{6, 1, 2, 3, 4\} \cap \{5, 6, 1, 3\} \neq \emptyset$.

As explained above, to obtain the following instance of Π_0 from Π'_0, we add the vertex α and the arcs $(4, \alpha)$ and $(\alpha, 5)$ or by adding the arcs (α, β), $(4, \alpha)$ and $(\beta, 5)$

Figure 4 – The corresponding instance Π_0 of Π'_0.

The fact that the answer of the problem Π'_0 is YES, i.e, elementary paths represented by vertices $[6, 1, 3, 4]$, $[6, 1, 2, 3, 4]$ and $[5, 6, 1, 3]$, $[5, 6, 1, 2, 3]$ are not vertex-disjoints, it follows that the answer of the problem Π_0 is also YES. Then, α is a $0 - vertex$. One can observe that α does not belong to any elementary path between vertices $s_1 = 6$ and $t_2 = 3$ (see Fig. 4).

3.2 Separation problem in the case $k = 1$

In the case $k = 1$, the associate decision problem Π_1 is as follow : "Given two vertices α, β in G, are α and β being 1-vertices w.r.t s and t?". Consider the problem Π'_1:

"Given six distinct vertices $s_1, t_1, s_2, t_2, s_3, t_3$, are there no three elementary paths, P_{s_1,t_1}, P_{s_2,t_2} and P_{s_3,t_3} in G such that $V(P_{s_i,t_i}) \cap V(P_{s_j,t_j}) = \emptyset,\ 1 \leq i < j \leq 3$?"

As Π'_0 is a special case of Π'_1. Π'_1 is NP-complete in general digraph, (see Fortune, Hopcroft & Wyllie (1980), Garey & Johnson (1979)). We show the NP-completeness of Π_1, by exhibiting the following polynomial reduction from Π'_1 to Π_1: For any instance of Π'_1, the corresponding instance of Π_1 is obtained by adding the vertices α, β and the four arcs (t_1, α), (α, s_2), (t_2, β) and (β, s_3), and by setting $s = s_1$ and $t = t_3$.

W.r.t arcs (α_1, α_2) and (β_1, β_2) to create the corresponding instance of Π_1 from any instance of Π'_1, we add the arcs (α_1, α_2), (β_1, β_2), (t_1, α_1), (α_2, s_2), (t_2, β_1) and (β_2, s_3).

Lemma 3.2. *The answer to the instance of* Π'_1 *is YES* ***iff*** *the answer to the instance of* Π_1 *is YES.*

Proof. $i)$ $\Rightarrow$: Let $P_{s,t}$ be a path that visits the nodes α and β (in this order) in the graph of the instance of Π_1. This path can be decomposed into the sub-paths P_{s_1,t_1}, P_{t_1,s_2} , P_{s_2,t_2}, P_{t_2,s_3}, P_{s_3,t_3}, where $s = s_1, t = t_3$ and path P_{t_1,s_2} is the sequence t_1, α, s_2 and path P_{t_2,s_3} is the sequence t_2, β, s_3. $P_{s,t}$ cannot be elementary because $V(P_{s_1,t_1}) \cap V(P_{s_2,t_2}) \cap V(P_{s_3,t_3}) \neq \emptyset$, since the answer to the instance of Π'_1 is YES.

$ii)$ $\Leftarrow$: Let P_{s_1,t_1}, P_{s_2,t_2} and P_{s_3,t_3} be paths in the graph of the instance of Π'_1. Consider a path $P_{s,t}$ in the graph of the instance of Π_1. $P_{s,t}$ can be decomposed into the sub-paths P_{s_1,t_1}, P_{t_1,s_2}, P_{s_2,t_2}, P_{t_2,s_3} and P_{s_3,t_3}, where $s = s_1$, $t = t_3$ and path P_{t_1,s_2} is the sequence t_1, α, s_2 and path P_{t_2,s_3} is the sequence t_2, β, s_3. Since the answer to the instance of Π_1 is YES, $P_{s,t}$ is not elementary. This implies that $V(P_{s_1,t_1}) \cap V(P_{s_2,t_2}) \cap V(P_{s_3,t_3}) \neq \emptyset$. □

Example. Consider the below instance of Π'_1:

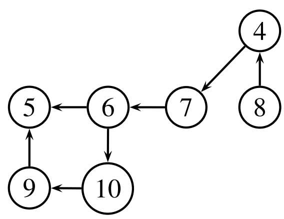

Figure 5 – An instance of Π'_1.

Let $s_1 = 8$, $t_1 = 6$, $s_2 = 7$, $t_2 = 5$, $s_3 = 4$, and $t_3 = 10$, we observe that there is no three vertex-disjoint elementary paths between (s_1, t_1), (s_2, t_2) and (s_3, t_3) in the above instance of digraph. By transformation, we can obtain in polynomial time an instance of Π_1 as follow:

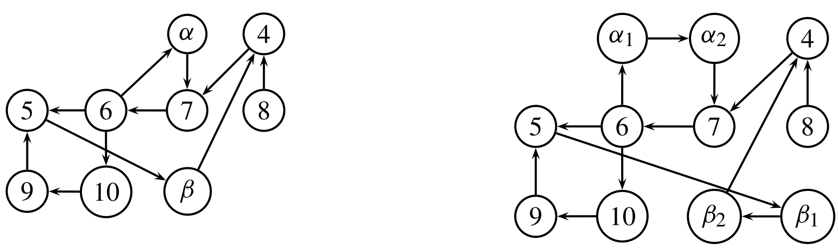

Figure 6 – The corresponding instance Π_1 of Π'_1.

The fact that the answer of the problem Π'_1 is YES, i.e, there is no vertex-disjoint elementary paths between vertices (8, 6), (7, 5) and (4, 10) in the considered instance of Π'_1, it follows that the answer of the problem Π_1 is also YES. Thus, it doesn't exist any elementary path between vertices $s_1 = 8$ and $t_3 = 10$ containing both vertices α and β or arcs (α_1, α_2) and (β_1, β_2).

4 CONCLUSION

In this paper, we prove the NP-hardness of the *separation* problem of the so-called *valid inequalities of order k*. We establish a polynomial reduction from the problem of the existence of $k+2$ vertex-disjoint paths between $k+2$ pairs of vertices $(s_1, t_1), (s_2, t_2) \dots (s_{k+2}, t_{k+2})$ in a digraph to the decision problem associated to the separation of *valid inequalities of order k*. We recall that the problem of the existence of $k+2$ vertex-disjoint paths between $k+2$ pairs of vertices $(s_1, t_1), (s_2, t_2) \dots (s_{k+2}, t_{k+2})$ in a digraph is known to be NP-complete.

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