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ARTICLES

ON THE NP-HARDNESS OF THE MINIMUM DISPERSION ROUTING PROBLEM

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ABSTRACT. In the Minimum Dispersion Routing Problem, a set of vertices must be served by tours with trajectories defined in order to reduce the dispersion of vehicles. Tours must be related to each other and impose a spatial and temporal synchronization among vehicles, quantified by an original dispersion metric used in an objective function to be minimized. In this paper, we demonstrate that Euclidean Traveling Salesman Problem solutions can be found by MDRP solvers. We describe a method to reduce Euclidean Traveling Salesman Problem instances to Minimum Dispersion Routing Problem instances in polynomial time, proving that the last one is NP-Hard.

Keywords: MDRP, NP-Hardness, synchronization, vehicle routing.

1 INTRODUCTION

The Minimum Dispersion Routing Problem (MDRP) is a recently proposed routing problem, first described in Dhein et al. (2019). An MDRP solution is composed of a set of tours that present spatial similarity, but also present a temporal synchronization that keeps vehicles close to each other while traveling their tours. The relative position of vehicles is continuously evaluated to generate a dispersion metric used to compose the MDRP objective function.

Synchronization between vehicles or teams is an important characteristic of routing problems. The most usual synchronization characteristic is probably the one induced by tasks that demand the expertise of more than one team to be solved. In such cases, teams must visit the same vertex concomitantly (Hà et al., 2020; Parragh & Doerner, 2018) and/or subject to precedence constraints (Hojabri et al., 2018; Ait Haddadene et al., 2016).

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Similar problems demand vehicles to meet during their tours to exchange goods as in Anderluh et al. (2021). In those cases, synchronization is not induced by vertices demands but by vehicles characteristics. The use of light vehicles (e.g. drones) to assist heavy vehicles (e.g. trucks) in delivery activities resulted in other similar optimization problems (Murray & Chu, 2015; Sacramento et al., 2019). Light vehicles travel segments of tours attached to heavy vehicles but are launched to serve specific clients. Both vehicles must rejoin later.

Arc routing problems may also present synchronization constraints. In Salazar-Aguilar et al. (2012), more than one vehicle must work together to plow snow from segments of streets with two or more lanes. In Akbari & Salman (2017), normal traversals of edges must be planned to occur after restoration (debris cleaning) traversals.

Problems with multiple synchronization constraints are found in Drexl (2012) and Fink et al. (2019).

Synchronization implies interdependence among tours, which represents a notable difficulty to the development of solution methods. Even small changes involving only one tour cannot be evaluated locally, because all tours must maintain coherence regarding synchronization constraints.

Notice that, as synchronization is usually imposed in routing problems through constraints, the objective functions are related to travel costs, mainly to minimize the longest tour or the total traveled distance. In such cases, the complexity of the problem is straightforward to demonstrate if synchronization constraints are relaxed to produce a classic routing problem.

In MDRP, synchronization is induced by the objective function rather than by constraints. Vehicles must maintain proximity throughout the entire solution to minimize the objective function. Such continuous synchronization imposed by an objective function is somehow similar to the one observed in some formulations for dissimilar path finding problem as described in Liu et al. (2016). This problem has major differences to MDRP, like solutions composed of sets of paths instead of tours and the search for dissimilarities. But evaluation of solutions made according to continuous tracking of vehicles' relative positions is similar.

The minimum dispersion imposed by the objective function in MDRP is relevant for cargo security. To travel as a convoy is a good option if transportation occurs in a hostile region, or is subject to some kind of danger. The disadvantage of such a configuration is an increase in workload for drivers because all vehicles will visit all targets or clients. Our main purpose with MDRP is to define tours that increase security by imposing proximity among vehicles. At the same time, the workload is distributed because only one vehicle visits each client.

To prove that MDRP is NP-Hard is relevant for different reasons. First, it is an obvious theoretical contribution to the field of research. And this theoretical definition is important to guide (and justify) the use of heuristic methods in the search for good solutions. Furthermore, as complexity proofs can be done by reducing one problem to another, each problem whose complexity is already known and established among researchers is a potential source for new proofs.

We would also like to highlight the peculiarity of the objective function of our problem. While the Euclidean Traveling Salesman Problem (ETSP) accumulates traveling costs as its objective function, MDRP presents a constant distance evaluation between each pair of vehicles. This relevant difference makes it impossible to reduce one problem to another by relaxing constraints. Therefore, we also understand that the proof presented in this work is interesting per se. The steps performed to create a carefully designed MDRP instance from an ETSP instance may be interesting and useful for other researchers.

In this work, we demonstrate that instances of ETSP, a known NP-Hard problem (Garey et al., 1976), can be converted into MDRP instances in polynomial time, to be solved by an MDRP solver. MDRP depends on the existence of two vehicles traveling at least two tours to generate a dispersion to be minimized. To fulfill this necessity, the process that results in the MDRP instance essentially creates new vertices and edges to generate a mirror tour to be traveled by a second vehicle while the first one travels the (original) ETSP tour. To create an instance by adding new vertices and edges is straightforward, but the proof is supported by a series of carefully defined values that allow us to demonstrate that the final optimal MDRP solution (with minimum dispersion) is also the optimal solution for the original ETSP instance (with minimum traveled distance).

This paper is organized as follows. In Section 2 we present a detailed description of MDRP. In Section 3 we present an important characteristic of the dispersion metric used in MDRP, fundamental to establishing the correlation between the dispersion objective function and total traveled distance objective function. We present the proof itself in Section 4, which is followed by a brief conclusion in Section 5.

2 MINIMUM DISPERSION ROUTING PROBLEM

An MDRP instance is composed by a graph $G = (V,E)$, where $V = \{0,1,\ldots,m\}$ is a set of $m+1$ vertices and $E = \{e_{ij} \mid i, j \in V\}$ is a set of edges connecting vertices in V. Each vertex $i \in V$ is positioned on a plane according to coordinates (x_i, y_i), has an associated service time s_i and, except for vertex 0 (the depot), is visited only once in a solution. A cost c_{ij} is associated with each edge $e_{ij} \in E$ and represents the time necessary to completely traverse the edge. A set of h tours, all starting and ending at vertex 0, must be planned to be traversed by a set of identical vehicles $K = \{1,\ldots,h\}$.

Solutions for MDRP instances are constructed in order to minimize an objective function that evaluates the dispersion of vehicles while executing tours. This dispersion metric was also used in a different problem (Dhein et al., 2018), composing an objective function designed to scatter vehicles. The MDRP objective function is computed as follows.

The displacement of vehicle $k \in K$ while traversing an edge $e_{ij} \in E$ can be represented by Equation (1), parameterized by time t.

$$X_k(t) = x_i + t\ v_{xk}\ ,\ \ Y_k(t) = y_i + t\ v_{yk},\ \ t \in [0, c_{ij}] \tag{1}$$

Vehicle k has a velocity vector $\vec{v}_k = \langle v_{xk}, v_{yk} \rangle$, and its direction is the same as $\langle x_j - x_i, y_j - y_i \rangle$. We consider, w.l.g., that the velocity vector has unit magnitude.

Every time a vehicle arrives at a vertex or departs from a vertex, its velocity vector and its displacement equation change. Those events mark timestamps illustrated in Figure 1. We call time slice the period between two of such timestamps, i.e., the interval in which displacement equations remain unchanged for all vehicles.

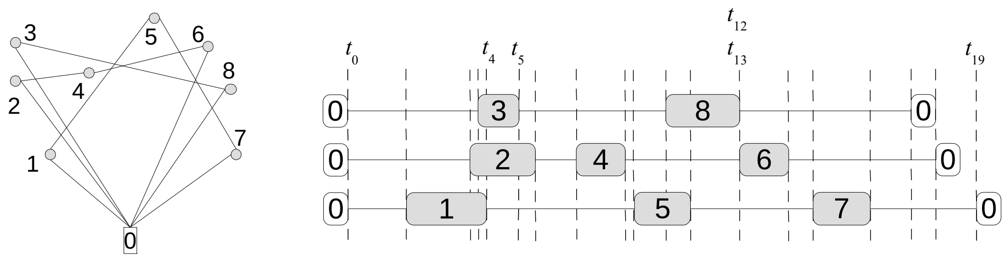


Figure 1 – A set of tours and its time slices. The left diagram presents a set of tours disposed in a geographic representation, while the right diagram presents a temporal representation of the same set. Traversal times are illustrated as horizontal lines and service times are illustrated as boxes identified with vertices numbers. All timestamps that serve as boundaries for time slices are identified by vertical dashed lines.

For an instance with $m+1$ vertices and h vehicles, there are $1+2m+h$ timestamps and $2m+h$ time slices in a solution, and we use a set $\mathscr{T} = \{\tau : \tau = 0, 1, \ldots, 2m+h\}$ to index both timestamps and time slices. The total length of a time slice $\tau > 0$ is given by $T_\tau = ts_\tau - ts_{\tau-1}, \forall \tau \in \mathscr{T} \setminus \{0\}$, where $ts_{\tau-1}$ and ts_τ define the initial and final boundaries.

We can now rewrite the displacement equation for a vehicle k to consider movement within a single time slice.

$$X_k(t) = X_k(t_{\tau-1}) + t\ v_{xk\tau}\ ,\ Y_k(t) = Y_k(t_{\tau-1}) + t\ v_{yk\tau}\ ,\ t \in [t_{\tau-1}, t_\tau] \tag{2}$$

Another vehicle $k' \in K$, traversing another edge $e_{i'j'} \in E$ at the same time slice, will have its instantaneous distance to vehicle k at time t calculated as presented in Equation (3).

$$d_{kk'}(t) = \sqrt{[X_k(t) - X_{k'}(t)]^2 + [Y_k(t) - Y_{k'}(t)]^2} \tag{3}$$

We integrate Equation (3) over the length T_τ of a slice to obtain the local (inside slice) dispersion between the pair of vehicles.

$$D_{kk'\tau} = \int_0^{T_\tau} \sqrt{[X_k(t) - X_{k'}(t)]^2 + [Y_k(t) - Y_{k'}(t)]^2} dt \tag{4}$$

Finally, the dispersion associated to a time slice is the maximum local dispersion between all pairs of vehicles, and the total dispersion is given by the sum of dispersion values associated to all time slices. MDRP objective function is given by the solution dispersion S, defined in Equation (5).

$$S = \sum_{\tau=1}^{2m+h} \max_{\substack{k=1,\dots,h-1 \\ k'=k+1,\dots,h}} (D_{kk'\tau}) \tag{5}$$

This objective function is minimized in order to keep vehicles traveling with reduced relative distance. A formal definition of MDRP can be found in Dhein et al. (2019), where a mathematical model is provided.

3 REPLACING VELOCITY VECTOR BY DISPLACEMENT VECTOR IN THE OBJECTIVE FUNCTION OF MDRP

Our proof is based on the transformation of ETSP instances into MDRP instances. But to demonstrate that an optimal solution for MDRP is also an optimal solution for ETSP, we must establish a correlation between the dispersion metric and the total length of a tour. The following proposition is essential for this.

Proposition 1. *The dispersion between two vehicles in a time slice can be calculated as the average instantaneous distance between vehicles multiplied by the duration of the time slice.*

Proof. Given $(x_{0k\tau}, y_{0k\tau})$ and $(x_{1k\tau}, y_{1k\tau})$, initial and final positions of vehicle k in a time slice τ, the components of the respective displacement vector $\vec{d}_{k\tau}$ are $d_{xk\tau} = x_{1k\tau} - x_{0k\tau}$ and $d_{yk\tau} = y_{1k\tau} - y_{0k\tau}$. For a second vehicle k', with initial and final positions for the same time slice given by $(x_{0k'\tau}, y_{0k'\tau})$ and $(x_{1k'\tau}, y_{1k'\tau})$, the components of the displacement vector $\vec{d}_{k'\tau}$ are given by $d_{xk'\tau} = x_{1k'\tau} - x_{0k'\tau}$ and $d_{yk'\tau} = y_{1k'\tau} - y_{0k'\tau}$.

Equation (6) presents the integral in Equation (4) expanded according to those definitions for vehicles k and k'. The length of time slice τ is represented by T_τ.

$$D_{kk'\tau} = \int_0^{T_\tau} \sqrt{[x_{0k\tau} - x_{0k'\tau} + (v_{xk\tau} - v_{xk'\tau}) \cdot t]^2 + [y_{0k\tau} - y_{0k'\tau} + (v_{yk\tau} - v_{yk'\tau}) \cdot t]^2} dt \tag{6}$$

Variable t can be replaced by $T_\tau \cdot u$, where u is non-dimensional, which implies in $dt = T_\tau du$, $t = 0 \Rightarrow u = 0$ and $t = T_\tau \Rightarrow u = 1$. Replacing the variable, we can define $D_{kk'\tau}$ according to Equation (7).

$$D_{kk'\tau} = T_\tau \cdot \int_0^1 \sqrt{[x_{0k\tau} - x_{0k'\tau} + (v_{xk\tau} - v_{xk'\tau}) \cdot T_\tau \cdot u]^2 + [y_{0k\tau} - y_{0k'\tau} + (v_{yk\tau} - v_{yk'\tau}) \cdot T_\tau \cdot u]^2} du \tag{7}$$

The multiplication of the velocity vector by time T_τ gives us displacement.

$$D_{kk'\tau} = T_\tau \cdot \int_0^1 \sqrt{[x_{0k\tau} - x_{0k'\tau} + (d_{xk\tau} - d_{xk'\tau}) \cdot u]^2 + [y_{0k\tau} - y_{0k'\tau} + (d_{yk\tau} - d_{yk'\tau}) \cdot u]^2} du \tag{8}$$

Equation (8) can be rewritten as Equation (9).

$$D_{kk'\tau} = T_\tau \cdot \int_0^1 \sqrt{[X_k(u) - X_{k'}(u)]^2 + [Y_k(u) - Y_{k'}(u)]^2} du \tag{9}$$

As Equation (9) is equivalent to Equation (4), the equality presented in Equation (10) holds.

$$\int_0^1 \sqrt{[X_k(u) - X_{k'}(u)]^2 + [Y_k(u) - Y_{k'}(u)]^2} du = \frac{\int_0^{T_\tau} \sqrt{[X_k(t) - X_{k'}(t)]^2 + [Y_k(t) - Y_{k'}(t)]^2} dt}{T_\tau} \tag{10}$$

Because the velocity magnitude is unitary, we can consider the dispersion of vehicles k and k' during a time slice τ, as expressed in Equation (9), as numerically equal to the average distance between those vehicles multiplied by the duration of the time slice τ. $\square$

We present in Figure 2 a set of images to illustrate this characteristic. Thin gray lines represent the discretization of instantaneous distances for vehicles traveling edges represented as black lines. Starting and finishing points for each vehicle are presented together with the result of Equation (8). As the velocity has a unitary magnitude, the T_τ is numerically equivalent to the traveled distance in the timeslice.

4 REDUCTION FROM EUCLIDEAN TSP TO MDRP

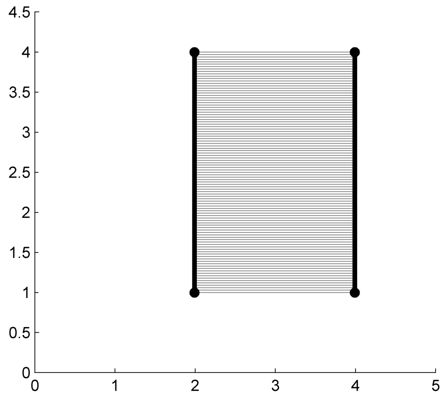
In this section, we demonstrate that an Euclidean Traveling Salesman Problem instance can be reduced to an MDRP instance in polynomial time.

In ETSP, all clients are visited by one salesman only. On the other hand, as defined in section 2, MDRP is only possible with a set of at least 2 vehicles, since there is no dispersion if $h = 1$. The main idea of the proof that follows is essentially this: (i) to duplicate the graph that defines an ETSP instance to generate an MDRP instance, and to consider a set of two vehicles in this instance, (ii) to find logically separated but spatially equivalent tours in the new graph, and (iii) to impose a little delay to one of the vehicles while traveling its tour, in order to generate a controlled distance between vehicles and, as a consequence, a dispersion easy to calculate. The resulting minimum dispersion pair of tours will be shown to be composed by tours with minimum travel costs, which represent the best solution for the original ETSP.

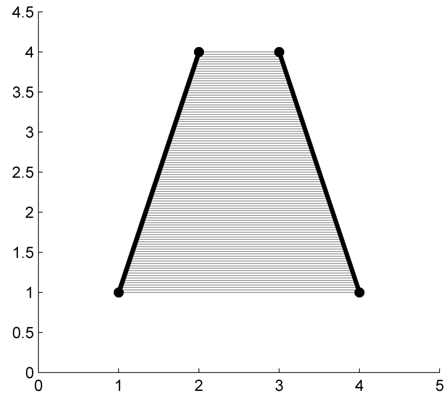
Proposition 2. *MDRP is NP-Hard.*

Proof. An ETSP instance can be defined as follows. Let $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ be the graph with the set of $\mathscr{V} = \{1, \cdots, n\}$ vertices (clients) to be visited by the salesman and the set $\mathscr{E} = \{e_{ij} : i, j \in \mathscr{V}, i \neq j\}$ of edges connecting those vertices in $\mathscr{V}$. Each edge $e_{ij} \in \mathscr{E}$ has an associated cost c_{ij} representing the distance between vertices i and j, and each vertex $i \in \mathscr{V}$ is located on a plane according to coordinates (x_i, y_i).

Such configuration is illustrated in Figure 3.



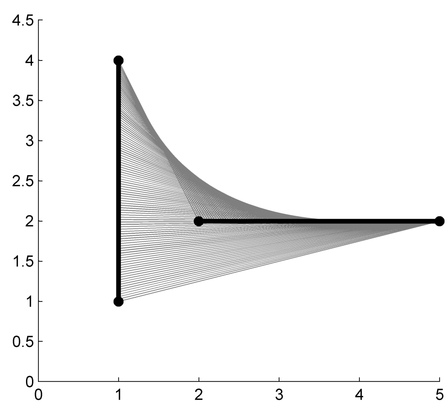
(a) (2,1) to (2,4) and (4,1) to (4,4);
Dispersion calculation result: $T_\tau \cdot 2$



(b) (1,1) to (2,4) and (4,1) to (3,4);
Dispersion calculation result: $T_\tau \cdot 2$

(c) (1,1) to (1,4) and (1,1) to (4,1);
Dispersion calculation result: $T_\tau \cdot 2.12$

(d) (1,1) to (4,1) and (1,4) to (1,1);
Dispersion calculation result: $T_\tau \cdot 2.43$



(e) (1,4) to (1,1) and (2,2) to (5,2);
Dispersion calculation result: $T_\tau \cdot 2.75$

Figure 2 – Five examples of dispersion calculation.

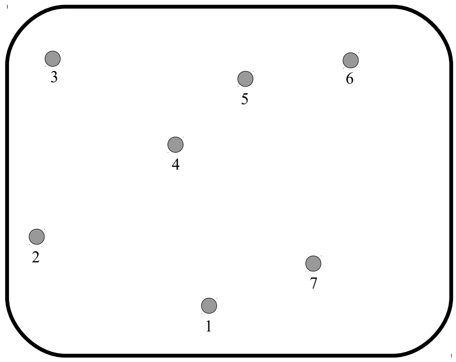


Figure 3 – Set $\mathscr{V}$ of seven vertices in an ETSP instance.

Part One: graph construction for the MDRP instance

To generate a graph for MDRP, we first create a new set of vertices $\mathscr{V}' = \{i+n : \forall i \in \mathscr{V}\}$. Each vertex $i+n \in \mathscr{V}'$ replicates vertex $i \in \mathscr{V}$ so they share the same coordinates, i.e., $x_{i+n} = x_i$ and $y_{i+n} = y_i$. Pairs of vertices $i+n \in \mathscr{V}'$ and $i \in \mathscr{V}$ will be referred to as *corresponding* vertices. Set $\mathscr{E}'$ is also created after set $\mathscr{E}$: an *corresponding* edge $e_{i+n,j+n} \in \mathscr{E}'$ is created if and only if there is an edge $e_{ij} \in \mathscr{E}$. Corresponding edges have the same associated cost, i.e., $c_{i+n,j+n} = c_{ij}$.

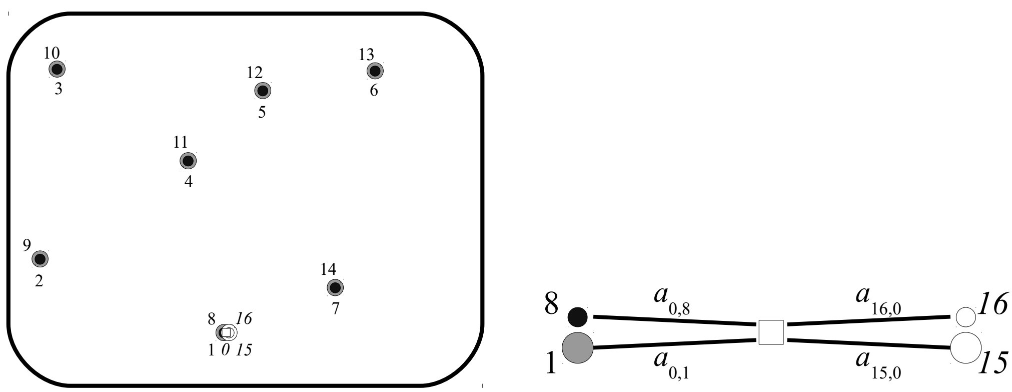
Vertex 1 is arbitrarily chosen as the initial position of the ETSP tour. In MDRP, all vehicles start and finish their tours in the same vertex, the depot. A configuration in which the first vehicle starts and finishes its tour in vertex 1 while the second vehicle starts and finishes its tour in vertex $n+1$ does not constitute a valid MDRP solution. To avoid this problem, we create a vertex 0 (the MDRP depot) with the same coordinates as vertex 1. We also create two more vertices $2n+1$ and $2n+2$, also coincident to vertex 1. Those vertices are important to allow both vehicles to return to the same position of vertex 1, closing what would be a valid ETSP tour, before returning to vertex 0. Without those vertices, the vehicles would only be able to travel ETSP equivalent tours if they visited vertices 1 and $n+1$ twice before returning to the depot, which is not allowed in MDRP.

New edges are created to connect those three new vertices: edges $e_{0,1}$, $e_{0,n+1}$, $e_{0,2n+1}$ and $e_{0,2n+2}$ connect the depot to vertices 1, $1+n$, $2n+1$ and $2n+2$; edges $\{e_{i,2n+1}\ \forall i \in \mathscr{V}\}$ connect all vertices in $\mathscr{V}$ to vertex $2n+1$; and edges $\{e_{i,2n+2}\ \forall i \in \mathscr{V}'\}$ connect all vertices in $\mathscr{V}'$ to vertex $2n+2$.

There are no edges connecting vertices in $\mathscr{V} \cup \{2n+1\}$ to vertices in $\mathscr{V}' \cup \{2n+2\}$, so the only way for two vehicles to visit all vertices is to assign each of those sets to a different vehicle. Also, a tour built to visit all vertices in $\mathscr{V} \cup \{2n+1\}$ will start with vertex 1 and finish with vertex $2n+1$ or start with vertex $2n+1$ and finish with vertex 1, since those are the only vertices connected to the depot. In the remainder of the text, we will refer to those vertices as the *extreme* vertices of the first tour, and $1+n$ and $2n+2$ are the extreme vertices of the second tour.

Notice that the first tour, the one starting with vertex 1 or vertex $2n+1$ after departure from the depot, will be composed essentially by vertices already in the ETSP instance, except for depot and vertex $2n+1$. We can understand this tour as the salesman's tour. On the other hand, the second tour, which starts with a visit to vertex $1+n$ or to vertex $2n+2$, will be constructed with the set of mirrored vertices, which are artificial if we consider ETSP but are essential for MDRP because dispersion only exists with at least two vehicles performing two travels. The reference to the tour servicing vertices in $\mathscr{V} \cup \{2n+1\}$ as the first tour is totally arbitrary, used only to clarify the explanation.

The resulting MDRP instance, created after an ETSP instance and illustrated in Figure 4, presents a graph $G = (V,E)$ with $V = \mathscr{V} \cup \mathscr{V}' \cup \{0, 2n+1, 2n+2\}$ and $E = \mathscr{E} \cup \mathscr{E}' \cup \{e_{0,1}, e_{0,1+n}, e_{0,2n+1}, e_{0,2n+2,0}\} \cup \{e_{i,2n+1}\ \forall i \in \mathscr{V}\} \cup \{e_{i,2n+2}\ \forall i \in \mathscr{V}'\}$. The time necessary to traverse an edge is numerically the same as the distance between vertices, and the vehicles set is defined as $K = \{1,2\}$.



(a) Vertices in the MDRP instance

(b) Detailed representation of edges incident to depot

Figure 4 – An MDRP instance created after the ETSP instance with 7 vertices.

Figure 4a illustrates the MDRP instance created after the ETSP instance presented in Figure 3. *Original* vertices in $\mathscr{V}$ are represented by gray circles, *duplicated* vertices in $\mathscr{V}'$ are represented by smaller black circles, vertices $2n+1$ and $2n+2$ are represented by white circles and vertex 0 is represented by a white square. Figure 4b details the configuration of vertex 0 and *extreme* vertices 1, $n+1$, $2n+1$ and $2n+2$, and edges connecting those vertices. Notice that those vertices are presented with some displacement in Figures 4a and 4b to evince their existence, but they share identical coordinates.

For an ETSP instance with n vertices, the resulting MDRP instance will have $2n+3$ vertices: those in $\mathscr{V}$ and $\mathscr{V}'$ and vertices 0, $2n+1$ and $2n+2$. As there are two vehicles and $2n+2$ vertices to visit, a solution for this instance will have $4n+7$ timestamps and $4n+6$ time slices.

Part Two: imposing a delay between vehicles

Let us initially consider $s_i = 0\ \forall i \in V$, i.e., the service time is zero for all vertices. Observe that with such configuration, if the pair of vehicles always remains grouped together, i.e., if the second tour mirrors the salesman's tour and both vehicles visit all corresponding vertices concomitantly, total dispersion will be zero for all possible sequences of visits, and getting optimal MDRP solutions becomes trivial. For the same reason, the solution is also trivial if service times are not 0, but the same for all vertices except depot.

In our MDRP instance created after an ETSP instance, those trivial solutions are avoided by imposing a δ delay to one vehicle at the beginning of the tours, which is done by setting service times for extreme vertices. For extreme vertices on the first tour, service times are zero, i.e., $s_1 = 0$ and $s_{2n+1} = 0$, while other extreme vertices have a service time set to δ, i.e., $s_{1+n} = \delta$ and $s_{2n+2} = \delta$.

With this configuration of service times, if all non-extreme vertices share the same service time and vehicles travel grouped together (mirrored tours), the one that starts in vertex 1 or vertex $2n+1$ will always be δ time units (or distance units, since we defined that MDRP edges, measured in time units, are numerically equivalent to ETSP edges, measured in distance units) ahead of the other vehicle until they reach their final extreme vertices.

Observe that it would be impossible to guarantee this delay without vertices $2n+1$ and $2n+2$. The depot must be connected to at least two vertices in each tour because each vehicle must depart from and return to the depot. As we have an undirected graph, connections between the depot and other vertices in the graph would allow vehicles to avoid delay and perform tours without dispersion. Our four extreme vertices give only two alternative starting points for each vehicle, but the choice among those alternatives is irrelevant because the vertices have the same positions and the same service time for each vehicle.

Part Three: service time for non-extreme vertices

This proof is based on the dispersion caused by a delay imposed on one vehicle by distinct service times in extreme vertices. As aforementioned, the distance between vehicles must be controlled to generate a dispersion that is easy to identify and calculate. But, for this to be true, the service time defined for non-extreme vertices in the MDRP instance must be at least as long as the delay. We use this part of the proof to present the reasons why this is necessary.

If all the remaining (non-extreme) vertices receive identical service times, the delay initially imposed will be sustained until vehicles reach their final extreme vertices, which is fundamental to our proof. Observe in Figure 5 a situation that might occur if service times are set to values lower than δ, in which instantaneous distances between vehicles, and consequently dispersion, become complex to measure. With too small service times, the vehicle ahead departs from a vertex before the delayed one arrives at the corresponding vertex, and we have our vehicles traveling non-corresponding edges at the same instant. In such situations, the dispersion becomes dependent on the angle between those edges, and consequently, its calculation becomes arduous.

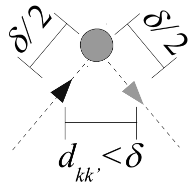


Figure 5 – Distance between vehicles k e k' when traversing non-corresponding edges. In this situation, we consider service time set to 0 on the vertex, but similar situations occur with non-zero service times lower than δ.

To avoid this difficulty, we set service times of all non-extreme vertices also to δ. With such a configuration, the arrival of the delayed vehicle in a vertex occurs exactly when the other vehicle departs from a corresponding vertex, and it becomes impossible to have vehicles traversing non-corresponding edges at the same time.

While traversing corresponding edges, the distance between vehicles remains constant and exactly δ. On the other hand, while the vehicle ahead is at a vertex during its service time and the delayed vehicle is arriving at the same position (to visit the corresponding vertex), the distance between vehicles decreases from δ to 0. The inverse situation occurs when the vehicle ahead departs from a vertex while the delayed one is starting its service time at the same position, and the distance between vehicles varies from 0 to δ. This predictable behavior allows us to always calculate the dispersion between vehicles based on the edges sizes and δ value.

Part Four: definition of δ value

Our final goal is to obtain an optimal solution for an ETSP instance from an optimal solution obtained for the derived MDRP instance. To achieve this goal, we must avoid scattering. Vehicles must travel their tours grouped together, i.e., the tour composed by vertices in $\mathscr{V}'$ must mirror the tour traveled by the salesman, composed by vertices in $\mathscr{V}$. This can be guaranteed by the definition of a sufficiently small but non-zero value for δ.

To justify the necessity of a small δ to keep vehicles grouped, we start our explanation by returning to the hypothetical situation with no delay between vehicles ($\delta = 0$). In this case, vehicles traveling together always present both instantaneous distance and general dispersion equal to zero, an ideal situation for MDRP but an undesirable situation for this proof.

But if vehicles diverge at some point, as illustrated in Figure 6, they will necessarily start traversing non-corresponding edges departing from a pair of corresponding vertices i e $i+n$, the first one heading towards a vertex $l \in \mathscr{V}$ and the second one heading towards a vertex $j+n \in \mathscr{V}'$, with $j \neq l$.

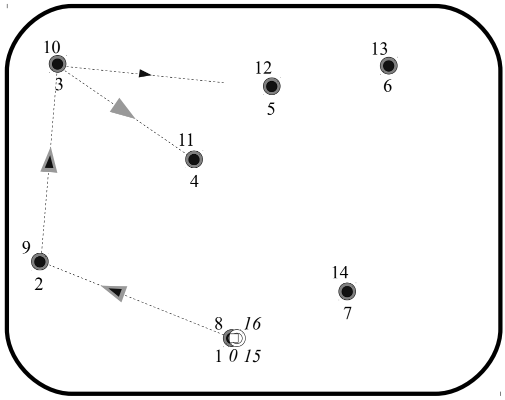


Figure 6 – In the illustrated tours segments, vehicles are grouped together while traveling from corresponding vertices 1 and 8 to corresponding vertices 2 and 9, and then to corresponding vertices 3 and 10. At this point, they separate, and the following edges are not corresponding. One vehicle is now heading towards vertex 4 while the other one is heading towards vertex 12. If we consider the generic description provided in the text, vertices 3, 10, 4 and 12 are, respectively, i, $i+n$, l and $j+n$.

Such a situation causes at least one time slice after the scattering of vehicles with non-zero dispersion, calculated as described in Section 2. This dispersion would certainly be avoided in an optimal solution for an instance as the hypothetical one used in Figure 6, with corresponding vertices and no delay. But, as aforementioned, $\delta \neq 0$ is essential to generate a non-zero objective function value.

Observe that the minimum non-zero dispersion found for the first time slice after a scattering of vehicles, considering all possible combinations of i, l and j, defines a lower bound LB for the total dispersion (objective function value) of any of the possible solutions (or tours configurations) in which the vehicles do not travel grouped together at any moment. If we can define a value for δ that is small enough to generate a total dispersion that does not overcome this lower bound with vehicles always traveling grouped together, we can ensure that the second tour will mirror the first one and the solution will not present any separation point.

In Section 3, we have demonstrated that the total dispersion of an MDRP solution can be viewed as the average distance between vehicles multiplied by the total travel time. In a configuration with delay set to $\delta = LB/(n \cdot e_{max})$, where e_{max} is the largest cost associated to an edge in E and $n \cdot e_{max}$ is an upper bound for the total cost of an ETSP tour, the dispersion in a solution in which vehicles travel grouped together will never be superior to LB. Also, if there is an edge $e_{i,j} \in E$ with an associated cost $c_{i,j}$ that is positive but inferior than $LB/(n \cdot e_{max})$, we adopt a δ with this value.

With a sufficiently small δ value, vehicles are kept grouped because any separation would increase the total dispersion if compared to the dispersion caused by the imposed delay. As with the solution for the hypothetical instance with $\delta = 0$ referred before, the optimal solution for the instance with *small* δ will also be obtained with vehicles always traversing corresponding edges concomitantly, although in this case with little delay. This delay gives us a non-zero dispersion value, and the solution is not trivial because the sequence of visits becomes relevant to minimize the total travel time and, consequently, total dispersion.

Finally, it is important to notice that edges with zero cost do not invalidate our method to define δ value as the minimum between $LB/(n \cdot e_{max})$ and the minimum non-zero edge cost. The traversal of an edge e_{ij} with $c_{ij} = 0$ results in the following timestamps: (i) arrival of the first vehicle to vertex i; (ii) arrival of the other vehicle to the corresponding vertex $i+n$; (iii) departure of the first vehicle from i, and, because $c_{ij} = 0$, its immediate arrival at vertex j; (iv) departure of the second vehicle from $i+n$ and its immediate arrival at $j+n$; (v) departure of the first vehicle from j; and (vi) departure of the second vehicle from $j+n$.

In such a situation, each vehicle visits two consecutive vertices without traversal time between them, staying at the same place for 2δ time units. The vehicles are separated in space only before the arrival at $i+n$ of the late vehicle (i.e., during the time slice between timestamps i and ii) and after the departure from j of the vehicle ahead (i.e., during the time slice between timestamps v and vi). There is no difference in terms of dispersion generation between this situation and those where the vehicles visit one vertex each. It could be interpreted as a visit to a pair of corresponding vertices with service of time 2δ.

Part Five: time slices in our MDRP configuration with vehicles traveling grouped together with a δ delay

To understand how the MDRP solution is evaluated to calculate its objective function, we must understand a pattern of time slices generated by vehicles while performing a pair of mirrored tours with an imposed delay. Figure 7 presents temporal characteristics of a pair of tours that represent a solution for an MDRP instance constructed according to the description made so far. The image illustrates the sequence of timestamps related to every moment a vehicle reaches or leaves a vertex.

All four initial timestamps occur at the same instant, caused by: concomitant departures from the depot (t_0), arrivals to initial extreme vertices (t_1 e t_2), and the departure of the first vehicle, that visited an extreme vertex with no service time (t_3). It is correct to state that dispersion only starts after the first vehicle departs from its initial extreme vertex, and timestamps t_0 to t_3 can be considered as one without any loss for the calculation of total dispersion.

A similar situation occurs with the last three timestamps caused by the vehicle ahead, identified as t_{29} to t_{31} in Figure 7: the arrival to the final extreme vertex, the departure from this vertex to depot and the arrival at depot. The extreme vertex involved has a service time of 0 and it is actually at the final coordinates, so all events are concomitant. The arrival at the final extreme vertex can be considered as the arrival at the depot, also without any loss.

Both extreme vertices in the delayed tour have service time set to δ, and the delayed vehicle arrives at the depot 2δ time units after the first one. But we must consider that extreme vertices and depot, where the other vehicle finished its tour, have the same coordinates. Even with a

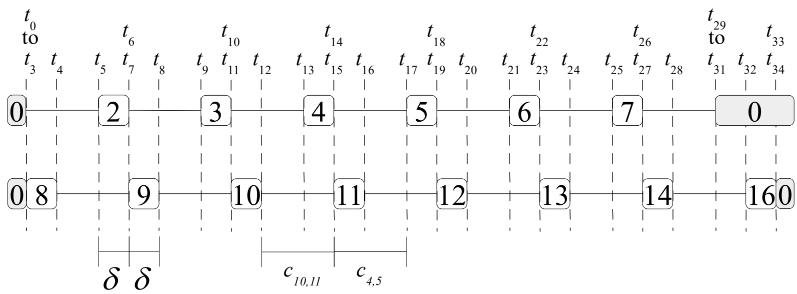


Figure 7 – Temporal characteristics of a solution for the same example instance already used in previous figures. Displacements between vertices are illustrated with the same size as a simplification. In this solution, one vehicle visits the following sequence of vertices: 0, 1, 2, 3, 4, 5, 6, 7, 15, 0. Vehicles travel mirrored tours separated only by a δ delay, so the sequence of vertices in the second tour is 0, 8, 9, 10, 11, 12, 13, 14, 16, 0. As there are 17 vertices and 2 vehicles, there are 35 timestamps and 34 time slices. Vertices 1 and 15 are not represented because their service times are 0, but timestamps related to arrival at and departure from those vertices are considered. Arrival at vertex 1 defines timestamp t_1 and immediate departure defines timestamp t_3, both simultaneous to timestamps t_0 (departure from the depot) and t_2 (arrival at vertex 8). The arrival at vertex 15 occurs at timestamp t_{29} and departure at timestamp t_{30}. Notice that there is no distance between vehicles after timestamp t_{32}.

time slice determined by two non-concomitant timestamps, there is actually no distance between vehicles to generate dispersion to be considered.

If we suppress those time slices that do not contribute to the objective function, it is possible to present Figure 8 with a simplified configuration of timestamps and time slices.

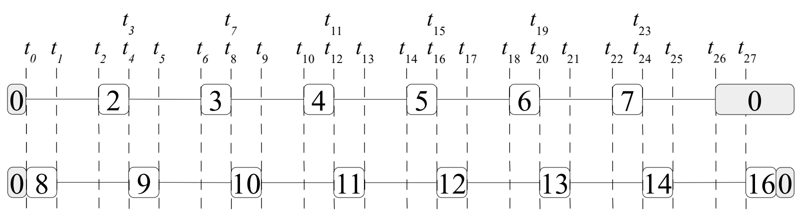
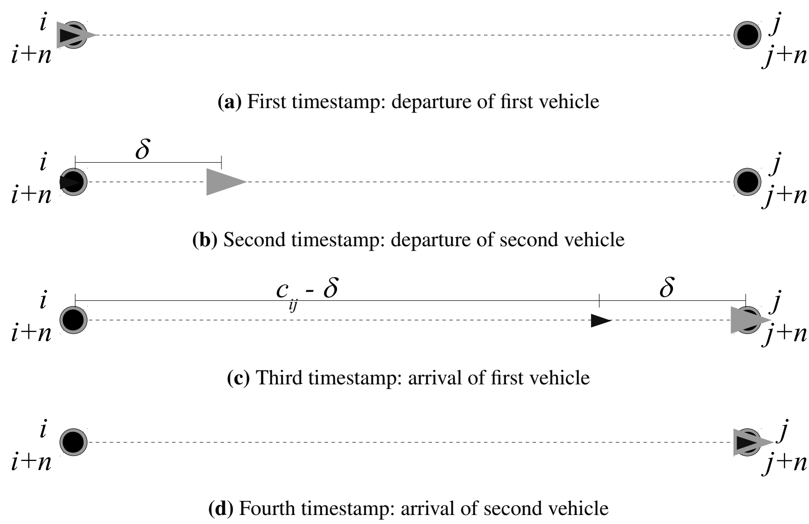


Figure 8 – Temporal characteristics of the solution without initial and final time slices that do not contribute to the objective function. Timestamp t_0 combines the initial set of arrivals and departures, depicted in Figure 7 as t_0 to t_3, and can be interpreted as the departure that creates a delay between vehicles. Timestamp t_{26} combines timestamps identified as t_{29} to t_{31} in Figure 7 and can be interpreted as the first arrival at depot. Finally, timestamp t_{27} represents the instant identified as t_{32} in Figure 7, and the following timestamps are ignored because vehicles share the same position after this moment.

Part Six: how to calculate the total dispersion for our MDRP solution

Figure 8 exposes the existence of a pattern of four successive events that is repeated for every pair of corresponding edges: two departures from corresponding vertices, with timestamps separated by δ, and two arrivals at another pair of corresponding vertices. Timestamps t_0 and t_1 mark departures and are followed by arrivals at t_2 and t_3. Timestamps t_4 to t_7 repeat those departure and arrival events and this pattern is repeated until both vehicles reach the same position at timestamp 27. Notice, also, that the event that finishes the traversal of a corresponding pair of edges (arrival of the late vehicle) is concomitant with the event that marks the start of another occurrence (departure of the vehicle ahead), so there are no other timeslices to be considered in the dispersion computation that are not part of this pattern of four timestamps and three time slices.

Once we understand how to calculate the dispersion for the traversal of one pair of corresponding edges, the calculation of the total dispersion for a solution becomes straightforward. Figure 9 depicts in details the pattern, presenting the configuration at each timestamp, i.e., the relative position of vehicles when each of the four events happens.



(a) First timestamp: departure of first vehicle

(b) Second timestamp: departure of second vehicle

(c) Third timestamp: arrival of first vehicle

(d) Fourth timestamp: arrival of second vehicle

Figure 9 – Timestamps for the events that occur during the traversal of a pair of corresponding edges. Four images depict a sequence where the first vehicle (gray triangle) travels from vertex i to vertex j (gray circles) and the second vehicle (black triangle) travels from vertex $i+n$ to vertex $j+n$ (black circles).

In a complete tour, n pairs of edges will be traversed and generate dispersion to be calculated. Those are the edges originally in the ETSP instance and their corresponding edges. Remember that, as discussed before, the traversal of *artificial* edges between the depot and extreme vertices do not present dispersion, and the respective timestamps have already been removed from Figure 8.

We have defined a numerical equivalence between the cost of edges in the original ETSP instance, given in distance units, and the cost of edges defined for the derived MDRP instance, given in time units. Also, as presented in Section 3, the dispersion can be calculated for a time slice as the product of its total time and the average distance between vehicles.

The first time slice in the pattern, finished by the departure of the second vehicle, is δ time units long and presents an average distance between vehicles of $\delta/2$, contributing to the total dispersion of the solution with $\delta \cdot (\delta/2)$. It is followed by a time slice closed by the arrival of the first vehicle at its destination vertex, $c_{ij} - \delta$ time units later. Vehicles maintain a constant δ distance between them in this part of the traversal, so the dispersion in this time slice is $\delta \cdot (c_{ij} - \delta)$. The last time slice, finished by the arrival of the second vehicle to vertex $j+n$ contributes to the objective function with a dispersion identical to the one generated in the first time slice.

As aforementioned, the timestamp that marks the end of this last time slice occurs at the same instant that another departure starts a new and identical set of events. We could consider those concomitant timestamps as the limits of another timeslice, but it does not contribute to the final evaluation of a solution because its duration is 0.

In short, for each pair of corresponding edges $e_{ij} \in \mathscr{E}$ and $e_{i+n,j+n} \in \mathscr{E}'$ traversed by two vehicles with a δ delay between departures, we have the total dispersion of $\delta \cdot (\delta/2) + \delta \cdot (c_{ij} - \delta) + \delta \cdot (\delta/2)$. This equation can be reduced to $\delta \cdot c_{ij}$, and it becomes explicit that, for a pair of corresponding edges, the only variable value is the cost of edges being traversed.

Part Seven: the ETSP solution

The objective function value for the solution of our MDRP instance created after an ETSP instance is the sum of dispersion values found in n repetitions of the pattern of three time slices. This total dispersion can be represented by Equation 11, where x_{ij} is a binary variable that indicates whether edge e_{ij} is a part of the solution or not.

$$C = \sum_{e_{ij} \, \in \, \mathscr{E} \cup \{e_{i,2n+1} \, : \, \forall i \in \mathscr{V}\}} \delta \; c_{ij} x_{ij} \tag{11}$$

As δ is a constant, it is possible to represent the same dispersion calculation as in Equation 12, where it becomes evident that the solution that minimizes the total dispersion is also the one that minimizes total routing cost. We have demonstrated, therefore, that the optimal solution for the MDRP instance is also optimal for the original ETSP instance.

$$C = \delta \sum_{e_{ij} \, \in \, \mathscr{E} \cup \{e_{i,2n+1} \, : \, \forall i \in \mathscr{V}\}} c_{ij} x_{ij} \tag{12}$$

The tour that represents the ETSP solution is obtained through a simple procedure: the second tour, in which edges in $\mathscr{E}'$ are traversed, is discarded, together with the edges connecting the depot to extreme vertices. The last edge before the final extreme vertex, $e_{j,2n+1}$, is replaced by edge $e_{j,1}$, so the circuit is closed.

To create the MDRP instance from an ETSP instance, it was necessary to create $n+3$ new vertices and a total of $n^2 + 2n + 4$ new edges: n^2 edges in $\mathscr{V}'$, $2n$ edges connecting vertex $2n+1$ to vertices in $\mathscr{V}$ and vertex $2n+2$ to vertices in $\mathscr{V}'$, and 4 edges to connect the depot to extreme vertices. The calculation of δ value requires $(n^3 - 3n^2 + 2n)/2$ steps to compare all possible combinations of three vertices.

As the reduction can be executed in polynomial time, and ESTP is a known *NP-Hard* problem, MDRP is proven to be also *NP-Hard*. □

5 CONCLUSIONS

MDRP presents an original objective function when compared to other routing problems. It is not based on travel costs but computed with a metric that quantifies instantaneous distances among vehicles.

On the one hand, developing heuristic solution methods implies a major difficulty. Constant synchronization between a set of tours is hard to obtain and easy to lose, even with small neighborhood movements.

On the other hand, the unusual objective function made it difficult to understand the relation of this new problem to other routing problems. As a consequence, determining its computational complexity is not straightforward. In this work, we demonstrate that ETSP instances can be solved as MDRP instances. As ETSP is known to be an NP-Hard problem, we prove that MDRP is also NP-Hard.

It is important to highlight that the proof development presented in this paper is not about solving a usual MDRP instance, but about solving an ETSP instance through an MDRP solver. The final goal was to obtain an optimal solution for an ETSP instance, and this goal was only possible because of careful construction of a derived MDRP instance, in which an *artificial* (from ETSP point of view) vehicle traveled an *artificial* tour, maintaining a controlled distance to the original salesman (the other vehicle) visiting its original clients (in another tour). A series of characteristics imposed on the derived MDRP instance allowed us to demonstrate that the minimum dispersion solution was also the solution with minimum travel cost for the salesman, essentially because of this controlled distance.

Usual MDRP instances do not present corresponding vertices and edges, identical service times, and controlled distance between only two vehicles, and its objective function calculation is usually computationally expensive. The characteristic exponential complexity of NP-Hard problems, the difficulty of constant synchronization, and the computationally expensive calculation of the objective function make the Minimum Dispersion Routing Problem challenging for researchers willing to find fast solution methods.

Finally, we understand that our dispersion metric can be used in the context of other problems. It can be used, for example, to assess the dissimilarity between paths in new formulations for the dissimilar path-finding problem, as an objective function to be maximized rather than minimized. All characteristics and peculiarities discussed in this work might be helpful for other researchers interested in a dispersion metric that provides continuous tracking of vehicles positions during the execution of tours and paths.

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