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ARTICLES

IDENTIFYING FIXED POINTS AND SOLUTION OF NONLINEAR FUNCTIONAL EQUATION IN NON-ARCHIMEDEAN INTUITIONISTIC MENGER PROBABILISTIC METRIC SPACES

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ABSTRACT. In this paper, we extend the concept of compatible maps of type J, (J-1) and (J-2) and prove fixed point theorems for compatible mappings in Non-Archimedian Intuitionistic Menger Probabilistic Metric Space. An example and an application to functional equation is provided to support the theoretical results.

Keywords: common fixed point, non-Archimedean intuitionistic Menger probabilistic metric spaces, compatible maps.

1 INTRODUCTION

The notion of probabilistic metric spaces (PMS) was introduced by Menger (1942) as a generalization of a metric space together with the concept of compatible maps of type (J-1) and type (J-2), which are equivalent to compatible maps under certain conditions, and illustrated some common fixed point theorems for such maps in this Space. It is also of fundamental importance in the probabilistic functional analysis. Cho et al. (1997) introduced the concepts of compatible maps in Non-Archimedian Menger Probabilistic Metric Space N-AIMPS proved some fixed point theorems for these maps. Many authors like Sehgal & Bharucha-Reid (1972), Hadzic (1980), Chauhan & Sharma (2021), Sharma & Garg (2020), Jafari & Shams (2015), Krishnakumar & Sanatammappa (2016), Khan (2011), Krishnakumar & Sanatammappa (2018), Roldán López de Hierro et al. (2021), Gupta et al. (2022) have proved fixed point theorems in various

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metric spaces. We further expanded and generalized the results of (Devi et al., 2018) by using different conditions for three compatible maps in Non-Archimedian Intuitionistic Menger Probabilistic Metric Space and obtained common fixed point theorems for nine maps along with an example and application to enhance the results in this space.

2 PRELIMINARIES

Definition 1. *(Devi et al., 2018) A triple (X,F,G) is said to be Non-Archimedian Intuitionistic Probabilistic Metric Space (shortly NAIPM-space) if X is a non empty set and F is a probabilistic distance and G is a probabilistic non-distance on X satisfying the following conditions: for all $x,y,z \in X$ and $t,s \geq 0$,*

(a) $F_{xy}(t) + G_{xy}(t) \leq 1$,

(b) $F_{xy}(t) = 1$ *if and only if* $x = y$,

(c) $F_{xy}(t) = F_{yx}(t)$,

(d) $F_{xy}(0) = 0$,

(e) $F_{xy}(t) = 1, F_{yz}(s) = 1 \implies F_{xz}(max\{t,s\}) = 1$,

(f) $G_{xy}(t) = 0$ *if and only if* $x = y$,

(g) $G_{xy}(t) = G_{yx}(t)$,

(h) $G_{xy}(0) = 1$,

(i) $G_{xy}(t) = 0, G_{yz}(s) = 0 \implies G_{xz}(min\{t,s\}) = 1$,

A 5-tuple $(X,F,G,*,\Diamond)$ is said to be Non-Archimedian Intuitionistic Menger Probabilistic Metric Space if (X,F,G) is a NAIPM-space and in addition the following inequalities hold for all $x,y,z \in X$ and $t,s > 0$,

(j) $F_{xz}(max\{t,s\}) \geq F_{xy}(t) * F_{yz}(s)$,

(k) $G_{xz}(min\{t,s\}) \geq G_{xy}(t) \Diamond G_{yz}(s)$,

*where * is a continuous t-norm and $\Diamond$ is a continuous t-conorm.*

Lemma 1. *(Devi et al., 2018) If a function $\Phi : [0,\infty) \to [0,\infty)$ satisfies the condition (Φ), then we have*
(a) For all $t \geq 0$, $lim_{n\to\infty}\Phi^n(t) = 0$, where $\Phi^n(t)$ is the n-th iteration of $\Phi(t)$,
(b) If $\{t_n\}$ is a non-decreasing sequence of real numbers and $t_{n+1} \leq \Phi(t_n), n = 1,2,3,...$ then $lim_{n\to\infty}t_n = 0$. In particular, if $t \leq \Phi(t)$ for all $t \geq 0$ then $t = 0$.

3 MAIN RESULT

In this section first we modify the definition of Non-Archimedian Intuitionistic Probabilistic Metric Space for three variables as follows:

Definition 2. *A triple (X,F,G) is said to be modified Non-Archimedian Intuitionistic Probabilistic Metric Space (shortly N-AIPMS) if X is a non empty set, F is a probabilistic distance and G is a probabilistic non-distance on X satisfying the following conditions for all $x,y,z \in X$ and $t,s \geq 0$,*

(a) $F_{xyz}(t)+G_{xyz}(t) \leq 1$,

(b) $F_{xyz}(t)=1$ *if and only if* $x=y=z$,

(c) $F_{xyz}(t)=F_{zyx}(t)$,

(d) $F_{xyz}(0)=0$,

(e) $F_{xy}(t)=1, F_{yz}(s)=1 \implies F_{xz}(max\{t,s\})=1$,

(f) $G_{xyz}(t)=0$ *if and only if* x,y,z *are pairwise equal,*

(g) $G_{xyz}(t)=G_{yxz}(t)=G_{zyx}(t)$,

(h) $G_{xyz}(0)=1$,

(i) $G_{xy}(t)=0, G_{yz}(s)=0 \implies G_{xz}(min\{t,s\})=1$,
A 5-tuple $(X,F,G,*,\diamondsuit)$ is said to be modified Non-Archimedian Intuitionistic Menger Probabilistic Metric Space (shortly N-AIMPMS) if (X,F,G) is a N-AIPMS and in addition the following inequalities hold for all $x,y,z \in X$ and $t,s>0$,

(j) $F_{xz}(max\{t,s\}) \geq F_{xy}(t)*F_{yz}(s)$,

(k) $G_{xz}(min\{t,s\}) \geq G_{xy}(t)\diamondsuit G_{yz}(s)$,

*where * is a continuous t-norm and $\diamondsuit$ is a continuous t-conorm.*

Lemma 2. *Let $\{y_n\}$ be a sequence in X, such that $lim_{n\to\infty}F_{y_n,y_{n+1},y_{n+2}}(t)=1$ and $lim_{n\to\infty}G_{y_n,y_{n+1},y_{n+2}}(t)=0$ for all $t>0$. If $\{y_n\}$ is not a Cauchy sequence in X, then there exists $\varepsilon_0>0, t_0>0$ and three sequence $\{m_i\},\{n_i\},\{p_i\}$ of positive integers such that*
(a) $m_i>n_i+1$ and $n_i\to\infty$ as $i\to\infty$, $n_i>p_i+1$ and $p_i\to\infty$ as $i\to\infty$;
(b) $F_{y_{m_i},y_{n_i},y_{p_i}}(t_0)<1-\varepsilon_0$ and $F_{y_{m_{i-2}},y_{n_{i-1}},y_{p_i}}(t_0)\geq 1-\varepsilon_0, i=1,2,3...$
(c) $G_{y_{m_i},y_{n_i},y_{p_i}}(t_0)>\varepsilon_0$ and $G_{y_{m_{i-2}},y_{n_{i-1}},y_{p_i}}(t_0)\leq\varepsilon_0, i=1,2,3...$

Definition 3. *Self maps A,B and C on a N-AIMPMS $(X,F,G,*,\diamondsuit)$ are said to be compatible if $g(F_{AB_{x_n},BC_{x_n},CA_{x_n}}(t))\to 0$ and $h(G_{AB_{x_n},BC_{x_n},CA_{x_n}}(t))\to 1$ for all $t>0$, whenever $\{x_n\}$ is a sequence in X such that $Ax_n,Bx_n,Cx_n\to q$ for some q in X, as $n\to\infty$.*

Definition 4. *Self maps A,B and C on a N-AIMPMS* $(X,F,G,*,\Diamond)$ *are said to be compatible of type (J) if* $g(F_{AB_{x_n},BC_{x_n},CC_{x_n}}(t)) \to 0$ *and* $g(F_{CB_{x_n},BA_{x_n},AA_{x_n}}(t)) \to 0$ *and* $h(G_{AB_{x_n},BC_{x_n},CC_{x_n}}(t)) \to 1$ *and* $h(G_{CB_{x_n},BA_{x_n},AA_{x_n}}(t)) \to 1$ *for all* $t > 0$, *whenever* $\{x_n\}$ *is a sequence in X such that* $Ax_n, Bx_n, Cx_n \to q$ *for some q in X, as* $n \to \infty$.

Definition 5. *Self maps A,B and C on a N-AIMPMS* $(X,F,G,*,\Diamond)$ *are said to be compatible of type (J-1) if* $g(F_{AB_{x_n},BC_{x_n},CC_{x_n}}(t)) \to 0$ *and* $h(G_{AB_{x_n},BC_{x_n},CC_{x_n}}(t)) \to 1$ *for all* $t > 0$, *whenever* $\{x_n\}$ *is a sequence in X such that* $Ax_n, Bx_n, Cx_n \to q$ *for some q in X, as* $n \to \infty$.

Definition 6. *Self maps A,B and C on a N-AIMPMS* $(X,F,G,*,\Diamond)$ *are said to be compatible of type (J-2) if* $g(F_{CB_{x_n},BA_{x_n},AA_{x_n}}(t)) \to 0$ *and* $h(G_{CB_{x_n},BA_{x_n},AA_{x_n}}(t)) \to 1$ *for all* $t > 0$, *whenever* $\{x_n\}$ *is a sequence in X such that* $Ax_n, Bx_n, Cx_n \to q$ *for some q in X, as* $n \to \infty$.

Proposition 1. *Let A,B and C are Self maps on a N-AIMPMS* $(X,F,G,*,\Diamond)$,
(a) If C is continuous, then the pair (A,C) *or* (B,C) *are compatible of type (J-1).*

Proof. *Let* $\{x_n\}$ *be a sequence in X such that* $Ax_n, Bx_n, Cx_n \to q$ *for some q in X as* $n \to \infty$ *and let the pair* (A,C) *or* (B,C) *be compatible of type (J-1) and (J-2). Since C is continuous, we have A,B,C are pair wise compatible and C is continuous,*

$$CAx_n \to Cq, CBx_n \to Cq, CCx_n \to Cq$$

and so,

$$\begin{aligned} g(F_{AB_{x_n},BC_{x_n},CA_{x_n}}(t)) &\le g(F_{AB_{x_n},BC_{x_n},CC_{x_n}}(t)) + g(F_{CC_{x_n},CB_{x_n},BA_{x_n}}(t)) \\ + g(F_{BA_{x_n},AC_{x_n},CA_{x_n}}(t)) &\to 0 + g(F_{Cq,Cq,Cq}(t) + g(F_{Bq,Cq,Cq}(t) \ as \ (ACx_n = CAx_n), \\ &\to 0+0+0 \ as \ g(F_{xyz}(t) = 0, \ if \ any \ two \ of \ x,y,z \ equal. \end{aligned}$$

$$\begin{aligned} h(G_{AB_{x_n},BC_{x_n},CA_{x_n}}(t)) &\ge h(G_{AB_{x_n},BC_{x_n},CC_{x_n}}(t)) + h(G_{CC_{x_n},CB_{x_n},BA_{x_n}}(t)) + \\ h(G_{BA_{x_n},AC_{x_n},CA_{x_n}}(t)) &\to 1 + h(G_{Cq,Cq,Bq}(t)) + h(G_{Bq,Aq,Aq}(t)) \\ &\to 1+0+0 = 1, \ as (ACx_n = CAx_n) \ and \ g(F_{xyz}(t) = 0. \end{aligned}$$

Then A,B,C are compatible of type J-I.
If A,B,C are pairwise compatible of type J-I and A,B,C are continuous.
To prove: A,B,C are pairwise compatible.

$$\begin{aligned} g(F_{AB_{x_n},BC_{x_n},CA_{x_n}}(t)) &\le g(F_{AB_{x_n},BC_{x_n},CC_{x_n}}(t)) + g(F_{AB_{x_n}BC_{x_n},CC_{x_n}}(t)) \\ &\quad + g(F_{AB_{x_n},CC_{x_n},CA_{x_n}}(t)) \\ &\ge 1 + h(F_{Cq,Cq,Cq}(t) + g(F_{Cq,Bq,Cq}, \ as \ A,C \ are \ continuous \\ &\ge 1+0+0+0 = 1 \ as \ g(F_{xyz}(t) = 0, \ if \ any \ two \ of \ x,y,z \ equal. \end{aligned}$$

$$\begin{aligned} h(G_{AB_{x_n},BC_{x_n},CA_{x_n}}(t)) &\ge h(G_{AB_{x_n},BC_{x_n},CC_{x_n}}(t)) + h(G_{CC_{x_n},CB_{x_n},BA_{x_n}}(t)) + \\ h(G_{BA_{x_n},AC_{x_n},CA_{x_n}}(t)) &\to 1 + h(G_{Cq,Cq,Bq}(t)) + h(G_{Bq,Aq,Aq}(t)) \\ &\to 1+0+0 = 1, \ as (ACx_n = CAx_n) \ and \ g(F_{xyz}(t) = 0. \end{aligned}$$

Hence, the mappings A,B,C are compatible of type (J-1).

Note: *Similarly, proof can be done for the other possible cases for mappings A,B,C.*

Theorem 1. *Let A,B,C,D,P,Q,R,U and V be Self maps on a complete N-AIMPMS* $(X,F,G,*,\Diamond)$ *satisfying:*

(a) $P(X) \subseteq UV(X), Q(X) \subseteq CD(X), R(X) \subseteq AB(X)$;

(b) $g(F_{Px,Qy,Rz}(t)) \leq \phi(g(F_{ABx,CDy,UVz}(t))$ *and*
$h(G_{Px,Qy,Rz}(t)) \geq \varphi(h(G_{ABx,CDy,UVz}(t))$;

(c) $g(F_{Px,Qy,Rz}(t)) \leq \phi[max\{g(F_{AB_x,CD_x,UV_x}(t)) + g(F_{P_x,AB_x,CD_y}(t)) + g(F_{Q_y,CD_y,UV_z}(t)) + g(F_{R_z,UV_z,AB_x}(t)), g(F_{P_x,AB_x,CD_y}(t)) + g(F_{Q_y,CD_y,UV_z}(t)) + g(F_{R_z,AB_x,CD_y}(t)), g(F_{P_x,CD_y,UV_z}(t)) + g(F_{Q_y,CD_y,UV_z}(t)) + g(F_{R_z,UV_z,AB_x}(t)), g(F_{P_x,AB_x,CD_y}(t)) + g(F_{Q_y,UV_z,AB_x}(t)) + g(F_{R_z,UV_z,AB_x}(t))\}]$;

(d) $h(G_{Px,Qy,Rz}(t)) \geq \varphi[min\{h(G_{AB_x,CD_x,UV_x}(t)) + h(G_{P_x,AB_x,CD_y}(t)) + h(G_{Q_y,CD_y,UV_z}(t)) + h(G_{R_z,UV_z,AB_x}(t)), h(G_{P_x,AB_x,CD_y}(t)) + h(G_{Q_y,CD_y,UV_z}(t)) + h(G_{R_z,AB_x,CD_y}(t)), h(G_{P_x,CD_y,UV_z}(t)) + h(G_{Q_y,CD_y,UV_z}(t)) + h(G_{R_z,UV_z,AB_x}(t)), h(G_{P_x,AB_x,CD_y}(t)) + h(G_{Q_y,UV_z,AB_x}(t)) + h(G_{R_z,UV_z,AB_x}(t))\}]$
for all $x,y \in X$ *and* $t > 0$ *, where a function* $\phi, \varphi : [0,\infty) \to [0,\infty)$ *satisfies the condition* (ϕ) *and* (φ);

(e) $AB = BA, CD = DC, UV = VU, PB = BP, QD = DQ, RV = VR$;

(f) *Either P or AB is continuous;*

(g) *The pairs* $(P,AB),(Q,CD),(R,UV)$ *are mutually compatible of type (J). Then, A,B,C,D,P,Q,R,U,V have a unique common fixed point.*

Proof. Let x_0 be an arbitrary point in X. By (a) there exists $x_1, x_2, x_3 \in X$ such that
$Px_0 = CDx_1 = UVx_2 = y_0$,
$Qx_1 = UVx_1 = ABx_2 = y_1$,
$Rx_2 = ABx_2 = CDx_3 = y_2$.
Inductively, we can construct sequences $\{x_n\}, \{y_n\}, \{z_n\} \in X$ such that,
$Px_{2n} = CDx_{2n+1} = UVx_{2n+2} = y_{2n}$,
$Qx_{2n+1} = UVx_{2n+2} = ABx_{2n+3} = y_{2n+1}$,
$Rx_{2n+2} = ABx_{2n+3} = CDx_{2n+4} = y_{2n+2}$, for $n = 0,1,2,....$

Step-1
We shall show that sequence $\{y_n\}$ is a Cauchy sequence.
Since, $Px_{2n} = CDx_{2n+1} = UVx_{2n+2}$, using (b), (c), (d), we have
$g(F_{y_{2n},y_{2n+1},y_{2n+2}}(t)) = g(F_{Px_{2n},Qy_{2n+1},Rz_{2n+2}}(t)) \leq \phi(g(F_{y_{2n},y_{2n+1},y_{2n+2}}(t))$ and

$h(G_{y_{2n},y_{2n+1},y_{2n+2}}(t)) = h(G_{Px_{2n},Qy_{2n+1},Rz_{2n+2}}(t)) \geq \varphi(h(G_{y_{2n},y_{2n+1},y_{2n+2}}(t))$

Since, $Qx_{2n+1} = UVx_{2n+2} = ABx_{2n+3}$, we also have
$g(F_{y_{2n},y_{2n-1},y_{2n-2}}(t)) = g(F_{Px_{2n},Qy_{2n-1},Rz_{2n-2}}(t)) \leq \phi(g(F_{y_{2n-2},y_{2n},y_{2n+3}}(t))$ and
$h(G_{y_{2n},y_{2n-1},y_{2n-2}}(t)) = h(G_{Px_{2n},Qy_{2n-1},Rz_{2n-2}}(t)) \geq \varphi(h(G_{y_{2n-2},y_{2n},y_{2n+3}}(t))$
Thus,
$g(F_{y_n,y_{n+1},y_{n+2}}(t)) \leq \phi(g(F_{y_{n-2},y_{n-1},y_{2n+3}}(t))$ and
$h(G_{y_n,y_{n+1},y_{n+2}}(t)) \geq \varphi(h(G_{y_{n-2},y_{n-1},y_n}(t))$ for $n = 0,1,2,....$
Hence,
$g(F_{y_n,y_{n+1},y_{n+2}}(t)) \leq \phi^n(g(F_{y_0,y_1,y_2}(t))$ and
$h(G_{y_n,y_{n+1},y_{n+2}}(t)) \geq \varphi^n(h(G_{y_0,y_1,y_2}(t))$ for $n = 0,1,2,....$

Therefore, from Lemma 2.2;

$$\begin{gathered} g(F_{y_n,y_{n+1},y_{n+2}}(t)) \to 0 \\ h(G_{y_n,y_{n+1},y_{n+2}}(t)) \to 1,\ as\ n \to \infty. \end{gathered} \tag{3.1.1}$$

Suppose $\{y_n\}$ is not a Cauchy sequence. Since g is strictly decreasing from lemma 2.3, so there exists $\varepsilon_0 > 0, t_0 > 0$ and three sequences $\{m_k\},\{n_k\},\{p_k\}$ of positive integers such that

(i) $m_k > n_{k+1} > p_{k+2}$ *and* $n_k \to \infty$ *as* $k \to \infty$,

(ii) $g(F_{y_{m_k},y_{n_k},y_{p_k}}(t_0)) > g(1-\varepsilon_0)$ and $g(F_{y_{m_{k-2}},y_{n_{k-1}},y_{p_k}}(t_0)) \leq g(1-\varepsilon_0)$, $h(G_{y_{m_k},y_{n_k},y_{p_k}}(t_0)) > h(\varepsilon_0)$ *and* $h(G_{y_{m_{k-2}},y_{n_{k-1}},y_{p_k}}(t_0)) \leq h(\varepsilon_0)$ *for* $k = 1,2,3,..$

Therefore,

$$\begin{aligned} g(1-\varepsilon_0) < g(F_{y_{m_k},y_{n_k},y_{p_k}}(t_0)) &\leq g(F_{y_{m_k},y_{m_{k-1}},y_{m_{k-2}}}(t_0)) + g(F_{y_{m_{k-2}},y_{n_{k-1}},y_{p_k}}(t_0)) \\ &\leq g(F_{y_{m_k},y_{m_{k-1}},y_{m_{k-2}}}(t_0)) + g(1-\varepsilon_0) \end{aligned}$$

and,

$$\begin{aligned} h(\varepsilon_0) > h(G_{y_{m_k},y_{n_k},y_{p_k}}(t_0)) &\geq h(G_{y_{m_k},y_{m_{k-1}},y_{m_{k-2}}}(t_0)) + h(G_{y_{m_{k-2}},y_{n_{k-1}},y_{p_k}}(t_0)) \\ &\geq h(G_{y_{m_k},y_{m_{k-1}},y_{m_{k-2}}}(t_0)) + h(\varepsilon_0). \end{aligned}$$

Letting $k \to \infty$ we have,

$$\begin{gathered} lim_{n\to\infty} g(F_{y_{m_k},y_{n_k},y_{p_k}}(t_0)) = g(1-\varepsilon_0) \\ and\ lim_{n\to\infty} h(G_{y_{m_k},y_{n_k},y_{p_k}}(t_0)) = h(\varepsilon_0). \end{gathered} \tag{3.1.2}$$

On the other hand, we have

$$\begin{aligned} g(1-\varepsilon_0) < g(F_{y_{m_k},y_{n_k},y_{p_k}}(t_0)) &\leq g(F_{y_{m_k},y_{n_{k+1}},y_{p_{k+2}}}(t_0)) \\ &+g(F_{y_{m_k},y_{p_{k+1}},y_{p_{k+2}}}(t_0)) + g(F_{y_{p_k},y_{p_{k+1}},y_{p_{k+2}}}(t_0)) \\ and\ h(\varepsilon_0) > h(G_{y_{m_k},y_{n_k},y_{p_k}}(t_0)) &\geq h(G_{y_{m_k},y_{n_{k+1}},y_{p_{k+2}}}(t_0)) \\ &+h(G_{y_{m_k},y_{p_{k+1}},y_{p_{k+2}}}(t_0)) + h(G_{y_{p_k},y_{p_{k+1}},y_{p_{k+2}}}(t_0)). \end{aligned} \tag{3.1.3}$$

Without loss of generality, assume that all the sequences $\{m_k\},\{n_k\},\{p_k\}$ are even, using (c), (d) we have,

$$\begin{aligned} g(F_{y_{m_k},y_{n_{k+1}},y_{p_{k+2}}}(t_0)) &= g(F_{Px_{m_k},Qx_{m_{k+1}},Rx_{m_{k+2}}}(t_0)) \\ \leq \phi[max\{g(F_{y_{m_{k-2}},y_{n_{k-1}},y_{p_k}}(t_0)) &+ g(F_{y_{m_k},y_{m_{k-2}},y_{n_{k-1}}}(t_0)) \\ +g(F_{y_{n_{k+1}},y_{n_{k-1}},y_{p_k}}(t_0)) &+ g(F_{y_{p_{k+2}},y_{p_k},y_{m_{k-2}}}(t_0)), \\ g(F_{y_{m_k},y_{m_{k-2}},y_{n_{k-1}}}(t_0)) &+ g(F_{y_{n_{k+1}},y_{n_{k-1}},y_{p_k}}(t_0)) \\ +g(F_{y_{p_{k+2}},y_{m_{k-2}},y_{n_{k-1}}}(t_0)), g(F_{y_{m_k},y_{n_{k-1}},y_{p_k}}(t_0)) \\ +g(F_{y_{n_{k+1}},y_{n_{k-1}},y_{p_k}}(t_0)) &+ g(F_{y_{p_{k+2}},y_{p_k},y_{m_{k-2}}}(t_0)), \\ g(F_{y_{m_k},y_{m_{k-2}},y_{n_{k-1}}}(t_0)) &+ g(F_{y_{n_{k+1}},y_{p_k},y_{m_{k-2}}}(t_0)) \\ &+g(F_{y_{p_{k+2}},y_{p_k},y_{m_{k-2}}}(t_0))\}] \end{aligned}$$

$$\begin{aligned} \leq \phi[max\{g(1-\varepsilon_0) &+ g(F_{y_{m_k},y_{m_{k-2}},y_{n_{k-1}}}(t_0)) \\ +g(F_{y_{n_{k+1}},y_{n_{k-1}},y_{p_k}}(t_0)) &+ g(F_{y_{p_{k+2}},y_{p_k},y_{m_{k-2}}}(t_0)), \\ g(F_{y_{m_k},y_{m_{k-2}},y_{n_{k-1}}}(t_0)) &+ g(F_{y_{n_{k+1}},y_{n_{k-1}},y_{p_k}}(t_0)) \\ +g(F_{y_{p_{k+2}},y_{m_{k-2}},y_{n_{k-1}}}(t_0)), g(F_{y_{m_k},y_{n_{k-1}},y_{p_k}}(t_0)) \\ +g(F_{y_{n_{k+1}},y_{n_{k-1}},y_{p_k}}(t_0)) &+ g(F_{y_{p_{k+2}},y_{p_k},y_{m_{k-2}}}(t_0)), \\ g(F_{y_{m_k},y_{m_{k-2}},y_{n_{k-1}}}(t_0)) &+ g(F_{y_{n_{k+1}},y_{p_k},y_{m_{k-2}}}(t_0)) \\ &+g(F_{y_{p_{k+2}},y_{p_k},y_{m_{k-2}}}(t_0))\}] \end{aligned}$$

and

$$\begin{aligned} h(G_{y_{m_k},y_{n_{k+1}},y_{p_{k+2}}}(t_0)) &= h(G_{Px_{m_k},Qx_{m_{k+1}},Rx_{m_{k+2}}}(t_0)) \\ \geq \varphi[min\{h(G_{y_{m_{k-2}},y_{n_{k-1}},y_{p_k}}(t_0)) &+ h(G_{y_{m_k},y_{m_{k-2}},y_{n_{k-1}}}(t_0)) \\ +h(G_{y_{n_{k+1}},y_{n_{k-1}},y_{p_k}}(t_0)) &+ h(G_{y_{p_{k+2}},y_{p_k},y_{m_{k-2}}}(t_0)), \\ h(G_{y_{m_k},y_{m_{k-2}},y_{n_{k-1}}}(t_0)) &+ h(G_{y_{n_{k+1}},y_{n_{k-1}},y_{p_k}}(t_0)) \\ +h(G_{y_{p_{k+2}},y_{m_{k-2}},y_{n_{k-1}}}(t_0)), h(G_{y_{m_k},y_{n_{k-1}},y_{p_k}}(t_0)) \\ +h(G_{y_{n_{k+1}},y_{n_{k-1}},y_{p_k}}(t_0)) &+ h(G_{y_{p_{k+2}},y_{p_k},y_{m_{k-2}}}(t_0)), \\ h(G_{y_{m_k},y_{m_{k-2}},y_{n_{k-1}}}(t_0)) &+ h(G_{y_{n_{k+1}},y_{p_k},y_{m_{k-2}}}(t_0)) \\ &+h(G_{y_{p_{k+2}},y_{p_k},y_{m_{k-2}}}(t_0))\}] \end{aligned}$$

$$\begin{aligned}
&\geq \varphi[min\{h(\varepsilon_0)+h(G_{y_{m_k},y_{m_{k-2}},y_{n_{k-1}}}(t_0))\\
&+h(G_{y_{n_{k+1}},y_{n_{k-1}},y_{p_k}}(t_0))+h(G_{y_{p_{k+2}},y_{p_k},y_{m_{k-2}}}(t_0)),\\
&h(G_{y_{m_k},y_{m_{k-2}},y_{n_{k-1}}}(t_0))+h(G_{y_{n_{k+1}},y_{n_{k-1}},y_{p_k}}(t_0))\\
&+h(G_{y_{p_{k+2}},y_{m_{k-2}},y_{n_{k-1}}}(t_0)),h(G_{y_{m_k},y_{n_{k-1}},y_{p_k}}(t_0))\\
&+h(G_{y_{n_{k+1}},y_{n_{k-1}},y_{p_k}}(t_0))+h(G_{y_{p_{k+2}},y_{p_k},y_{m_{k-2}}}(t_0)),\\
&h(G_{y_{m_k},y_{m_{k-2}},y_{n_{k-1}}}(t_0))+h(G_{y_{n_{k+1}},y_{p_k},y_{m_{k-2}}}(t_0))\\
&+h(G_{y_{p_{k+2}},y_{p_k},y_{m_{k-2}}}(t_0))\}].
\end{aligned}$$

Substituting this in (3.1.3), letting $k \to \infty$ and using (3.1.1), (3.1.2), we have

$$\begin{aligned}
&g(1-\varepsilon_0) \leq \phi(g(1-\varepsilon_0)) < g(1-\varepsilon_0) \text{ and}\\
&h(\varepsilon_0) \geq \varphi(h(\varepsilon_0)) > h(\varepsilon_0).
\end{aligned}$$

This is a contradiction. Hence, $\{y_n\}$ is a Cauchy sequence. Since, $(X,F,G,*,\Diamond)$ is complete, it converges to a point q in X. Also its subsequences converges as follows: $Px_{2n} \to q, ABx_{2n} \to q, Qy_{2n+1} \to q, CDy_{2n+1} \to q, Rz_{2n+2} \to q, UVz_{2n+2} \to q$.

Case-1 AB is continuous and (P, AB),(Q, CD), (R, UV) are compatible of type (J-1).
Since AB is continuous $AB(AB)x_{2n} \to ABq$ and $(AB)Px_{2n} \to ABq$ and (P,AB) are compatible of type (J-1), $PPx_{2n} \to ABq$.

Step-2
By taking $x = Px_{2n}, y = x_{2n+1}, z = x_{2n+2}$, we have,

$$\begin{aligned}
g(F_{PPx_{2n},Qx_{2n+1},Rx_{2n+2}}(t)) &\leq \phi[max\{g(F_{ABPx_{2n},CDx_{2n+1},UVx_{2n+2}}(t))+\\
&g(F_{PPx_{2n},ABPX_{2n},CDx_{2n+1}}(t))+g(F_{Qx_{2n+1},CDx_{2n+1},UVx_{2n+2}}(t))+\\
&g(F_{Rx_{2n+2},UVx_{2n+2},ABPx_{2n}}(t)),g(F_{PPx_{2n},ABPx_{2n},CDx_{2n+1}}(t))+\\
&g(F_{Qx_{2n+1},CDx_{2n+1},UVx_{2n+2}}(t))+g(F_{Rx_{2n+2},ABPx_{2n},CDx_{2n+1}}(t)),\\
&g(F_{PPx_{2n},CDx_{2n+1},UVx_{2n+2}}(t))+g(F_{Qx_{2n+1},CDx_{2n+1},UVx_{2n+2}}(t))+\\
&g(F_{Rx_{2n+2},UVx_{2n+2},ABPx_{2n}}(t)),g(F_{PPx_{2n},ABPx_{2n},CDx_{2n+1}}(t))+\\
&g(F_{Qx_{2n+1},UVx_{2n+2},ABPx_{2n}}(t))+g(F_{Rx_{2n+2},UVx_{2n+2},ABPx_{2n}}(t))\}].
\end{aligned}$$

Similarly,

$$\begin{aligned}
h(G_{PPx_{2n},Qx_{2n+1},Rx_{2n+2}}(t)) &\geq \varphi[min\{h(G_{ABPx_{2n},CDx_{2n+1},UVx_{2n+2}}(t))+\\
&h(G_{PPx_{2n},ABPX_{2n},CDx_{2n+1}}(t))+h(G_{Qx_{2n+1},CDx_{2n+1},UVx_{2n+2}}(t))+\\
&h(G_{Rx_{2n+2},UVx_{2n+2},ABPx_{2n}}(t)),h(G_{PPx_{2n},ABPx_{2n},CDx_{2n+1}}(t))+\\
&h(G_{Qx_{2n+1},CDx_{2n+1},UVx_{2n+2}}(t))+h(G_{Rx_{2n+2},ABPx_{2n},CDx_{2n+1}}(t)),\\
&h(G_{PPx_{2n},CDx_{2n+1},UVx_{2n+2}}(t))+h(G_{Qx_{2n+1},CDx_{2n+1},UVx_{2n+2}}(t))+\\
&h(G_{Rx_{2n+2},UVx_{2n+2},ABPx_{2n}}(t)),h(G_{PPx_{2n},ABPx_{2n},CDx_{2n+1}}(t))+\\
&h(G_{Qx_{2n+1},UVx_{2n+2},ABPx_{2n}}(t))+h(G_{Rx_{2n+2},UVx_{2n+2},ABPx_{2n}}(t))\}].
\end{aligned}$$

This implies that, as $n \to \infty$

$$\begin{aligned} g(F_{ABq,q,q}(t)) \leq \phi[max\{g(F_{ABq,q,q}(t)) + g(F_{q,ABq,q}(t)) + g(F_{q,q,q}(t)) + \\ g(F_{q,q,ABq}(t)), g(F_{q,ABq,q}(t)) + g(F_{q,q,q}(t)) + g(F_{q,ABq,q}(t)), \\ g(F_{q,q,q}(t)) + g(F_{q,q,q}(t)) + g(F_{q,q,ABq}(t)), g(F_{q,ABq,q}(t)) + \\ g(F_{q,q,ABq}(t)) + g(F_{q,q,ABq}(t))\}] \\ = \phi\{g(F_{ABq,q,q}(t))\} \end{aligned}$$

and

$$\begin{aligned} h(G_{ABq,q,q}(t)) \geq \varphi[min\{h(G_{ABq,q,q}(t)) + h(G_{q,ABq,q}(t)) + h(G_{q,q,q}(t)) + \\ h(G_{q,q,ABq}(t)), h(G_{q,ABq,q}(t)) + h(G_{q,q,q}(t)) + h(G_{q,ABq,q}(t)), \\ h(G_{q,q,q}(t)) + h(G_{q,q,q}(t)) + h(G_{q,q,ABq}(t)), h(G_{q,ABq,q}(t)) + \\ h(G_{q,q,ABq}(t)) + h(G_{q,q,ABq}(t))\}] \\ = \varphi\{h(G_{ABq,q,q}(t))\}. \end{aligned}$$

Thus by lemma-2.2,
$g(F_{ABq,q,q}(t)) = 0$ and $h(G_{ABq,q,q}(t)) = 1$ for all $t > 0$ and it follows that $q = ABq$.

Step-3
By taking $x = q, y = x_{2n+1}, z = x_{2n+2}$, we have,

$$\begin{aligned} g(F_{Pq,Qx_{2n+1},Rx_{2n+2}}(t)) \leq \phi[max\{g(F_{ABq,CDx_{2n+1},UVx_{2n+2}}(t)) + \\ g(F_{Pq,ABq,CDx_{2n+1}}(t)) + g(F_{Qx_{2n+1},CDx_{2n+1},UVx_{2n+2}}(t)) + \\ g(F_{Rx_{2n+2},UVx_{2n+2},ABq}(t)), g(F_{Pq,ABq,CDx_{2n+1}}(t)) + \\ g(F_{Qx_{2n+1},CDx_{2n+1},UVx_{2n+2}}(t)) + g(F_{Rx_{2n+2},ABq,CDx_{2n+1}}(t)), \\ g(F_{Pq,CDx_{2n+1},UVx_{2n+2}}(t)) + g(F_{Qx_{2n+1},CDx_{2n+1},UVx_{2n+2}}(t)) + \\ g(F_{Rx_{2n+2},UVx_{2n+2},ABq}(t)), g(F_{Pq,ABq,CDx_{2n+1}}(t)) + \\ g(F_{Qx_{2n+1},UVx_{2n+2},ABq}(t)) + g(F_{Rx_{2n+2},UVx_{2n+2},ABq}(t))\}]. \end{aligned}$$

Similarly,

$$\begin{aligned} h(G_{Pq,Qx_{2n+1},Rx_{2n+2}}(t)) \geq \varphi[min\{h(G_{ABq,CDx_{2n+1},UVx_{2n+2}}(t)) + \\ h(G_{Pq,ABq,CDx_{2n+1}}(t)) + h(G_{Qx_{2n+1},CDx_{2n+1},UVx_{2n+2}}(t)) + \\ h(G_{Rx_{2n+2},UVx_{2n+2},ABq}(t)), h(G_{Pq,ABq,CDx_{2n+1}}(t)) + \\ h(G_{Qx_{2n+1},CDx_{2n+1},UVx_{2n+2}}(t)) + h(G_{Rx_{2n+2},ABq,CDx_{2n+1}}(t)), \\ h(G_{Pq,CDx_{2n+1},UVx_{2n+2}}(t)) + h(G_{Qx_{2n+1},CDx_{2n+1},UVx_{2n+2}}(t)) + \\ h(G_{Rx_{2n+2},UVx_{2n+2},ABq}(t)), h(G_{Pq,ABq,CDx_{2n+1}}(t)) + \\ h(G_{Qx_{2n+1},UVx_{2n+2},ABq}(t)) + h(G_{Rx_{2n+2},UVx_{2n+2},ABq}(t))\}]. \end{aligned}$$

This implies that, as $n \to \infty$

$$\begin{aligned} g(F_{q,q,q}(t)) \leq \phi[max\{&g(F_{q,q,q}(t)) + g(F_{q,q,q}(t)) + g(F_{q,q,q}(t)) + g(F_{q,q,q}(t)),\\ &g(F_{q,q,q}(t)) + g(F_{q,q,q}(t)) + g(F_{q,q,q}(t)), g(F_{q,q,q}(t)) + g(F_{q,q,q}(t)) +\\ &g(F_{q,q,q}(t)), g(F_{q,q,q}(t)) + g(F_{q,q,q}(t)) + g(F_{q,q,q}(t))\}]\\ &= \phi\{g(F_{q,q,q}(t))\} \end{aligned}$$

and

$$\begin{aligned} h(G_{q,q,q}(t)) \geq \varphi[min\{&h(G_{q,q,q}(t)) + h(G_{q,q,q}(t)) + h(G_{q,q,q}(t)) + h(G_{q,q,q}(t)),\\ &h(G_{q,q,q}(t)) + h(G_{q,q,q}(t)) + h(G_{q,q,q}(t)), h(G_{q,q,q}(t)) + h(G_{q,q,q}(t)) +\\ &h(G_{q,q,q}(t)), h(G_{q,q,q}(t)) + h(G_{q,q,q}(t)) + h(G_{q,q,q}(t))\}]\\ &= \varphi\{h(G_{q,q,q}(t))\}. \end{aligned}$$

this gives that $q = Pq$ and therefore, $q = ABq = Pq$.

Step-4

By taking $x = Bq, y = x_{2n+1}, z = x_{2n+2}$,in (c) and (d) using (e) we have,

$$\begin{aligned} g(F_{PBq,Qx_{2n+1},Rx_{2n+2}}(t)) \leq \phi[max\{&g(F_{ABBq,CDx_{2n+1},UVx_{2n+2}}(t)) +\\ &g(F_{PBq,ABBq,CDx_{2n+1}}(t)) + g(F_{Qx_{2n+1},CDx_{2n+1},UVx_{2n+2}}(t)) +\\ &g(F_{Rx_{2n+2},UVx_{2n+2},ABBq}(t)), g(F_{PBq,ABBq,CDx_{2n+1}}(t)) +\\ &g(F_{Qx_{2n+1},CDx_{2n+1},UVx_{2n+2}}(t)) + g(F_{Rx_{2n+2},ABBq,CDx_{2n+1}}(t)),\\ &g(F_{PBq,CDx_{2n+1},UVx_{2n+2}}(t)) + g(F_{Qx_{2n+1},CDx_{2n+1},UVx_{2n+2}}(t)) +\\ &g(F_{Rx_{2n+2},UVx_{2n+2},ABBq}(t)), g(F_{PBq,ABBq,CDx_{2n+1}}(t)) +\\ &g(F_{Qx_{2n+1},UVx_{2n+2},ABBq}(t)) + g(F_{Rx_{2n+2},UVx_{2n+2},ABBq}(t))\}]. \end{aligned}$$

Similarly,

$$\begin{aligned} h(G_{PBq,Qx_{2n+1},Rx_{2n+2}}(t)) \geq \varphi[min\{&h(G_{ABBq,CDx_{2n+1},UVx_{2n+2}}(t)) +\\ &h(G_{PBq,ABBq,CDx_{2n+1}}(t)) + h(G_{Qx_{2n+1},CDx_{2n+1},UVx_{2n+2}}(t)) +\\ &h(G_{Rx_{2n+2},UVx_{2n+2},ABBq}(t)), h(G_{PBq,ABBq,CDx_{2n+1}}(t)) +\\ &h(G_{Qx_{2n+1},CDx_{2n+1},UVx_{2n+2}}(t)) + h(G_{Rx_{2n+2},ABBq,CDx_{2n+1}}(t)),\\ &h(G_{PBq,CDx_{2n+1},UVx_{2n+2}}(t)) + h(G_{Qx_{2n+1},CDx_{2n+1},UVx_{2n+2}}(t)) +\\ &h(G_{Rx_{2n+2},UVx_{2n+2},ABBq}(t)), h(G_{PBq,ABBq,CDx_{2n+1}}(t)) +\\ &h(G_{Qx_{2n+1},UVx_{2n+2},ABBq}(t)) + h(G_{Rx_{2n+2},UVx_{2n+2},ABBq}(t)).\}] \end{aligned}$$

This implies that, as $n \to \infty$

$$\begin{aligned} g(F_{Bq,q,q}(t)) \leq \phi[max\{&g(F_{Bq,q,q}(t)) + g(F_{q,Bq,q}(t)) + g(F_{q,q,q}(t)) + g(F_{q,q,Bq}(t)),\\ &g(F_{q,Bq,q}(t)) + g(F_{q,q,q}(t)) + g(F_{q,Bq,q}(t)), g(F_{q,q,q}(t)) + g(F_{q,q,q}(t)) +\\ &g(F_{q,q,Bq}(t)), g(F_{q,Bq,q}(t)) + g(F_{q,q,Bq}(t)) + g(F_{q,q,Bq}(t))\}]\\ &= \phi\{g(F_{Bq,q,q}(t))\} \end{aligned}$$

and

$$\begin{aligned}h(G_{Bq,q,q}(t)) \geq \varphi[min\{h(G_{Bq,q,q}(t))+h(G_{q,Bq,q}(t))+h(G_{q,q,q}(t))+h(G_{q,q,Bq}(t)),\\ h(G_{q,Bq,q}(t))+h(G_{q,q,q}(t))+h(G_{q,Bq,q}(t)),h(G_{q,q,q}(t))+h(G_{q,q,q}(t))+\\ h(G_{q,q,Bq}(t)),h(G_{q,Bq,q}(t))+h(G_{q,q,Bq}(t))+h(G_{q,q,Bq}(t))\}]\\ =\varphi\{h(G_{Bq,q,q}(t))\}.\end{aligned}$$

gives, $q = Bq$. Since, $q = ABq$, we have $q = Aq$ and therefore, $q = Aq = Bq = Pq$.

Step-5

Since, $Q(X) \subseteq CD(X)$ such that $q = Qq = Cdq$.
By taking $x = x_{2n}, y = j, z = x_{2n+1}$in (c) and (d), we get

$$\begin{aligned}g(F_{Px_{2n},Qj,Rx_{2n+1}}(t)) \leq \phi[max\{g(F_{ABx_{2n},CDj,UVx_{2n+1}}(t))+\\ g(F_{Px_{2n},ABx_{2n},CDj}(t))+g(F_{Qj,CDj,UVx_{2n+1}}(t))+\\ g(F_{Rx_{2n+1},UVx_{2n+1},ABx_{2n}}(t)),g(F_{Px_{2n},ABx_{2n},CDj}(t))+\\ g(F_{Qj,CDj,UVx_{2n+1}}(t))+g(F_{Rx_{2n+1},ABx_{2n},CDj}(t)),\\ g(F_{Px_{2n},CDj,UVx_{2n+1}}(t))+g(F_{Qj,CDj,UVx_{2n+1}}(t))+\\ g(F_{Rx_{2n+1},UVx_{2n+1},ABx_{2n}}(t)),g(F_{Px_{2n},ABx_{2n},CDj}(t))+\\ g(F_{Qj,UVx_{2n+1},ABx_{2n}}(t))+g(F_{Rx_{2n+1},UVx_{2n+1},ABx_{2n}}(t))\}].\end{aligned}$$

Similarly,

$$\begin{aligned}h(G_{Px_{2n},Qj,Rx_{2n+1}}(t)) \geq \varphi[min\{h(G_{ABx_{2n},CDj,UVx_{2n+1}}(t))+\\ h(G_{Px_{2n},ABx_{2n},CDj}(t))+h(G_{Qj,CDj,UVx_{2n+1}}(t))+\\ h(G_{Rx_{2n+1},UVx_{2n+1},ABx_{2n}}(t)),h(G_{Px_{2n},ABx_{2n},CDj}(t))+\\ h(G_{Qj,CDj,UVx_{2n+1}}(t))+h(G_{Rx_{2n+1},ABx_{2n},CDj}(t)),\\ h(G_{Px_{2n},CDj,UVx_{2n+1}}(t))+h(G_{Qj,CDj,UVx_{2n+1}}(t))+\\ h(G_{Rx_{2n+1},UVx_{2n+1},ABx_{2n}}(t)),h(G_{Px_{2n},ABx_{2n},CDj}(t))+\\ h(G_{Qj,UVx_{2n+1},ABx_{2n}}(t))+h(G_{Rx_{2n+1},UVx_{2n+1},ABx_{2n}}(t))\}].\end{aligned}$$

This implies that, as $n \to \infty$

$$\begin{aligned}g(F_{q,Qj,q}(t)) \leq \phi[max\{g(F_{q,q,q}(t))+g(F_{q,q,q}(t))+g(F_{Qj,q,q}(t))+g(F_{q,q,q}(t)),\\ g(F_{q,q,q}(t))+g(F_{Qj,q,q}(t))+g(F_{q,q,q}(t)),g(F_{q,q,q}(t))+g(F_{Qj,q,q}(t))+\\ g(F_{q,q,q}(t)),g(F_{q,q,q}(t))+g(F_{Qj,q,q}(t))+g(F_{q,q,q}(t))\}]\\ =\phi\{g(F_{q,Qj,q}(t))\}.\end{aligned}$$

Similarly,

$$\begin{aligned} h(G_{q,Qj,q}(t)) \geq \varphi[min\{&h(G_{q,q,q}(t)) + h(G_{q,q,q}(t)) + h(G_{Qj,q,q}(t)) + h(G_{q,q,q}(t)), \\ &h(G_{q,q,q}(t)) + h(G_{Qj,q,q}(t)) + h(G_{q,q,q}(t)), h(G_{q,q,q}(t)) + h(G_{Qj,q,q}(t)) + \\ &h(G_{q,q,q}(t)), h(G_{q,q,q}(t)) + h(G_{Qj,q,q}(t)) + h(G_{q,q,q}(t))\}] \\ &= \varphi\{h(G_{q,Qj,q}(t))\}. \end{aligned}$$

This implies that $q = Qj$. Hence, $CDj = q = Qj$. Since, (Q,CD) is compatible of type (J-1), we have $Q(CD)j = CD(CD)j$. Thus, $CDq = Qq$

Step-6

By taking $x = x_{2n}, y = q, z = x_{2n+1}$ in (c) and (d), we have

$$\begin{aligned} g(F_{Px_{2n},Qq,Rx_{2n+1}}(t)) \leq \phi[max\{&g(F_{ABx_{2n},CDq,UVx_{2n+1}}(t)) + \\ &g(F_{Px_{2n},ABx_{2n},CDq}(t)) + g(F_{Qq,CDq,UVx_{2n+1}}(t)) + \\ &g(F_{Rx_{2n+1},UVx_{2n+1},ABx_{2n}}(t)), g(F_{Px_{2n},ABx_{2n},CDq}(t)) + \\ &g(F_{Qq,CDq,UVx_{2n+1}}(t)) + g(F_{Rx_{2n+1},ABx_{2n},CDq}(t)), \\ &g(F_{Px_{2n},CDq,UVx_{2n+1}}(t)) + g(F_{Qq,CDq,UVx_{2n+1}}(t)) + \\ &g(F_{Rx_{2n+1},UVx_{2n+1},ABx_{2n}}(t)), g(F_{Px_{2n},ABx_{2n},CDq}(t)) + \\ &g(F_{Qq,UVx_{2n+1},ABx_{2n}}(t)) + g(F_{Rx_{2n+1},UVx_{2n+1},ABx_{2n}}(t))\}]. \end{aligned}$$

Similarly,

$$\begin{aligned} h(G_{Px_{2n},Qq,Rx_{2n+1}}(t)) \geq \varphi[min\{&h(G_{ABx_{2n},CDq,UVx_{2n+1}}(t)) + \\ &h(G_{Px_{2n},ABx_{2n},CDq}(t)) + h(G_{Qq,CDq,UVx_{2n+1}}(t)) + \\ &h(G_{Rx_{2n+1},UVx_{2n+1},ABx_{2n}}(t)), h(G_{Px_{2n},ABx_{2n},CDq}(t)) + \\ &h(G_{Qq,CDq,UVx_{2n+1}}(t)) + h(G_{Rx_{2n+1},ABx_{2n},CDq}(t)), \\ &h(G_{Px_{2n},CDq,UVx_{2n+1}}(t)) + h(G_{Qq,CDq,UVx_{2n+1}}(t)) + \\ &h(G_{Rx_{2n+1},UVx_{2n+1},ABx_{2n}}(t)), h(G_{Px_{2n},ABx_{2n},CDq}(t)) + \\ &h(G_{Qq,UVx_{2n+1},ABx_{2n}}(t)) + h(G_{Rx_{2n+1},UVx_{2n+1},ABx_{2n}}(t))\}]. \end{aligned}$$

This implies that, as $n \to \infty$

$$\begin{aligned} g(F_{q,Qq,q}(t)) \leq \phi[max\{&g(F_{q,q,q}(t)) + g(F_{q,q,q}(t)) + g(F_{Qq,q,q}(t)) + g(F_{q,q,q}(t)), \\ &g(F_{q,q,q}(t)) + g(F_{Qq,q,q}(t)) + g(F_{q,q,q}(t)), g(F_{q,q,q}(t)) + g(F_{Qq,q,q}(t)) + \\ &g(F_{q,q,q}(t)), g(F_{q,q,q}(t)) + g(F_{Qq,q,q}(t)) + g(F_{q,q,q}(t))\}] \\ &= \phi\{g(F_{q,Qq,q}(t))\}. \end{aligned}$$

Similarly,

$$\begin{aligned} h(G_{q,Qq,q}(t)) \geq \varphi[min\{h(G_{q,q,q}(t)) + h(G_{q,q,q}(t)) + h(G_{Qq,q,q}(t)) + h(G_{q,q,q}(t)), \\ h(G_{q,q,q}(t)) + h(G_{Qq,q,q}(t)) + h(G_{q,q,q}(t)), h(G_{q,q,q}(t)) + h(G_{Qq,q,q}(t)) + \\ h(G_{q,q,q}(t)), h(G_{q,q,q}(t)) + h(G_{Qq,q,q}(t)) + h(G_{q,q,q}(t))\}] \\ = \varphi\{h(G_{q,Qq,q}(t))\}. \end{aligned}$$

This means that $q = Qq$. Hence, $CDj = q = Qq$. Since, $CDq = Qq$, we have $q = CDq$. Therefore, $q = Aq = Bq = Pq = Qq = CDq$.

Step-7

By taking $x = x_{2n}, y = Dq, z = x_{2n+1}$ in (c) and (d), we have

$$\begin{aligned} g(F_{Px_{2n},QDq,Rx_{2n+1}}(t)) \leq \phi[max\{g(F_{ABx_{2n},CDDq,UVx_{2n+1}}(t)) + \\ g(F_{Px_{2n},ABx_{2n},CDDq}(t)) + g(F_{QDq,CDDq,UVx_{2n+1}}(t)) + \\ g(F_{Rx_{2n+1},UVx_{2n+1},ABx_{2n}}(t)), g(F_{Px_{2n},ABx_{2n},CDDq}(t)) + \\ g(F_{QDq,CDDq,UVx_{2n+1}}(t)) + g(F_{Rx_{2n+1},ABx_{2n},CDDq}(t)), \\ g(F_{Px_{2n},CDDq,UVx_{2n+1}}(t)) + g(F_{QDq,CDDq,UVx_{2n+1}}(t)) + \\ g(F_{Rx_{2n+1},UVx_{2n+1},ABx_{2n}}(t)), g(F_{Px_{2n},ABx_{2n},CDDq}(t)) + \\ g(F_{QDq,UVx_{2n+1},ABx_{2n}}(t)) + g(F_{Rx_{2n+1},UVx_{2n+1},ABx_{2n}}(t))\}]. \end{aligned}$$

Similarly,

$$\begin{aligned} h(G_{Px_{2n},QDq,Rx_{2n+1}}(t)) \geq \varphi[min\{h(G_{ABx_{2n},CDDq,UVx_{2n+1}}(t)) + \\ h(G_{Px_{2n},ABx_{2n},CDDq}(t)) + h(G_{QDq,CDDq,UVx_{2n+1}}(t)) + \\ h(G_{Rx_{2n+1},UVx_{2n+1},ABx_{2n}}(t)), h(G_{Px_{2n},ABx_{2n},CDDq}(t)) + \\ h(G_{QDq,CDDq,UVx_{2n+1}}(t)) + h(G_{Rx_{2n+1},ABx_{2n},CDDq}(t)), \\ h(G_{Px_{2n},CDDq,UVx_{2n+1}}(t)) + h(G_{QDq,CDDq,UVx_{2n+1}}(t)) + \\ h(G_{Rx_{2n+1},UVx_{2n+1},ABx_{2n}}(t)), h(G_{Px_{2n},ABx_{2n},CDDq}(t)) + \\ h(G_{QDq,UVx_{2n+1},ABx_{2n}}(t)) + h(G_{Rx_{2n+1},UVx_{2n+1},ABx_{2n}}(t))\}]. \end{aligned}$$

This implies that, as $n \to \infty$

$$\begin{aligned} g(F_{q,Dq,q}(t)) \leq \phi[max\{g(F_{q,q,q}(t)) + g(F_{q,q,q}(t)) + g(F_{Dq,q,q}(t)) + g(F_{q,q,q}(t)), \\ g(F_{q,q,q}(t)) + g(F_{Dq,q,q}(t)) + g(F_{q,q,q}(t)), g(F_{q,q,q}(t)) + g(F_{Dq,q,q}(t)) + \\ g(F_{q,q,q}(t)), g(F_{q,q,q}(t)) + g(F_{Dq,q,q}(t)) + g(F_{q,q,q}(t))\}] \\ = \phi\{g(F_{q,Dq,q}(t))\}. \end{aligned}$$

Similarly,

$$\begin{aligned}h(G_{q,Dq,q}(t)) \geq \varphi[min\{&h(G_{q,q,q}(t))+h(G_{q,q,q}(t))+h(G_{Dq,q,q}(t))+h(G_{q,q,q}(t)),\\ &h(G_{q,q,q}(t))+h(G_{Dq,q,q}(t))+h(G_{q,q,q}(t)),h(G_{q,q,q}(t))+h(G_{Dq,q,q}(t))+\\ &h(G_{q,q,q}(t)),h(G_{q,q,q}(t))+h(G_{Dq,q,q}(t))+h(G_{q,q,q}(t))\}]\\ &= \varphi\{h(G_{q,Dq,q}(t))\}.\end{aligned}$$

This gives, $q = Dq$. Since, $q = CDq$, we have $q = Dq$. Therefore, $q = Aq = Bq = Cq = Dq = Pq = Qq$.

Step-8

Since, $P(X) \subseteq UV(X)$, there exists $w \in X$ such that $q = Pq = UVw$.
By taking $x = x_{2n}, y = x_{2n+1}, z = w$ in (c) and (d), we have

$$\begin{aligned}g(F_{Px_{2n},Qx_{2n+1},Rw}(t)) \leq \phi[max\{&g(F_{ABx_{2n},CDx_{2n+1},UVw}(t))+\\ &g(F_{Px_{2n},ABx_{2n},CDx_{2n+1}}(t))+g(F_{Qx_{2n+1},CDx_{2n+1},UVw}(t))+\\ &g(F_{Rw,UVw,ABx_{2n}}(t)),g(F_{Px_{2n},ABx_{2n},CDx_{2n+1}}(t))+\\ &g(F_{Qx_{2n+1},CDx_{2n+1},UVw}(t))+g(F_{Rw,ABx_{2n},CDx_{2n+1}}(t)),\\ &g(F_{Px_{2n},CDx_{2n+1},UVw}(t))+g(F_{Qx_{2n+1},CDx_{2n+1},UVw}(t))+\\ &g(F_{Rw,UVw,ABx_{2n}}(t)),g(F_{Px_{2n},ABx_{2n},CDx_{2n+1}}(t))+\\ &g(F_{Qx_{2n+1},UVw,ABx_{2n}}(t))+g(F_{Rw,UVw,ABx_{2n}}(t))\}].\end{aligned}$$

Similarly,

$$\begin{aligned}h(G_{Px_{2n},Qx_{2n+1},Rw}(t)) \geq \varphi[min\{&h(G_{ABx_{2n},CDx_{2n+1},UVw}(t))+\\ &h(G_{Px_{2n},ABx_{2n},CDx_{2n+1}}(t))+h(G_{Qx_{2n+1},CDx_{2n+1},UVw}(t))+\\ &h(G_{Rw,UVw,ABx_{2n}}(t)),h(G_{Px_{2n},ABx_{2n},CDx_{2n+1}}(t))+\\ &h(G_{Qx_{2n+1},CDx_{2n+1},UVw}(t))+h(G_{Rw,ABx_{2n},CDx_{2n+1}}(t)),\\ &h(G_{Px_{2n},CDx_{2n+1},UVw}(t))+h(G_{Qx_{2n+1},CDx_{2n+1},UVw}(t))+\\ &h(G_{Rw,UVw,ABx_{2n}}(t)),h(G_{Px_{2n},ABx_{2n},CDx_{2n+1}}(t))+\\ &h(G_{Qx_{2n+1},UVw,ABx_{2n}}(t))+h(G_{Rw,UVw,ABx_{2n}}(t))\}].\end{aligned}$$

This implies that, as $n \to \infty$

$$\begin{aligned}g(F_{q,q,Rw}(t)) \leq \phi[max\{&g(F_{q,q,q}(t))+g(F_{q,q,q}(t))+g(F_{q,q,q}(t))+g(F_{Rw,q,q}(t)),\\ &g(F_{q,q,q}(t))+g(F_{q,q,q}(t))+g(F_{q,q,q}(t)),g(F_{q,q,q}(t))+g(F_{q,q,q}(t))+\\ &g(F_{Rw,q,q}(t)),g(F_{q,q,q}(t))+g(F_{q,q,q}(t))+g(F_{Rw,q,q}(t))\}]\\ &= \phi\{g(F_{q,q,Rw}(t))\}\end{aligned}$$

Similarly,

$$\begin{aligned} h(G_{q,q,Rw}(t)) \geq \varphi[min\{h(G_{q,q,q}(t)) + h(G_{q,q,q}(t)) + h(G_{q,q,q}(t)) + h(G_{Rw,q,q}(t)), \\ h(G_{q,q,q}(t)) + h(G_{q,q,q}(t)) + h(G_{q,q,q}(t)), h(G_{q,q,q}(t)) + h(G_{q,q,q}(t)) + \\ h(G_{Rw,q,q}(t)), h(G_{q,q,q}(t)) + h(G_{q,q,q}(t)) + h(G_{Rw,q,q}(t))\}] \\ = \varphi\{h(G_{q,q,Rw}(t))\}. \end{aligned}$$

This gives $q = Rw$. Hence, $Uvw = q = Rw$. Since, (R, ST) is compatible of type (J-1), we have $R(UV)w = UV(UV)w$. Thus, $UVq = Rq$.

Step-9

By taking $x = x_{2n}, y = x_{2n+1}, z = q$ in (c), (d) and using (e), we have

$$\begin{aligned} g(F_{Px_{2n},Qx_{2n+1},Rq}(t)) \leq \phi[max\{g(F_{ABx_{2n},CDx_{2n+1},UVq}(t)) + \\ g(F_{Px_{2n},ABx_{2n},CDx_{2n+1}}(t)) + g(F_{Qx_{2n+1},CDx_{2n+1},UVq}(t)) + \\ g(F_{Rq,UVq,ABx_{2n}}(t)), g(F_{Px_{2n},ABx_{2n},CDx_{2n+1}}(t)) + \\ g(F_{Qx_{2n+1},CDx_{2n+1},UVq}(t)) + g(F_{Rq,ABx_{2n},CDx_{2n+1}}(t)), \\ g(F_{Px_{2n},CDx_{2n+1},UVq}(t)) + g(F_{Qx_{2n+1},CDx_{2n+1},UVq}(t)) + \\ g(F_{Rq,UVq,ABx_{2n}}(t)), g(F_{Px_{2n},ABx_{2n},CDx_{2n+1}}(t)) + \\ g(F_{Qx_{2n+1},UVq,ABx_{2n}}(t)) + g(F_{Rq,UVq,ABx_{2n}}(t))\}]. \end{aligned}$$

Similarly,

$$\begin{aligned} h(G_{Px_{2n},Qx_{2n+1},Rq}(t)) \geq \varphi[min\{h(G_{ABx_{2n},CDx_{2n+1},UVq}(t)) + \\ h(G_{Px_{2n},ABx_{2n},CDx_{2n+1}}(t)) + h(G_{Qx_{2n+1},CDx_{2n+1},UVq}(t)) + \\ h(G_{Rq,UVq,ABx_{2n}}(t)), h(G_{Px_{2n},ABx_{2n},CDx_{2n+1}}(t)) + \\ h(G_{Qx_{2n+1},CDx_{2n+1},UVq}(t)) + h(G_{Rq,ABx_{2n},CDx_{2n+1}}(t)), \\ h(G_{Px_{2n},CDx_{2n+1},UVq}(t)) + h(G_{Qx_{2n+1},CDx_{2n+1},UVq}(t)) + \\ h(G_{Rq,UVq,ABx_{2n}}(t)), h(G_{Px_{2n},ABx_{2n},CDx_{2n+1}}(t)) + \\ h(G_{Qx_{2n+1},UVq,ABx_{2n}}(t)) + h(G_{Rq,UVq,ABx_{2n}}(t))\}]. \end{aligned}$$

This implies that, as $n \to \infty$

$$\begin{aligned} g(F_{q,q,Rq}(t)) \leq \phi[max\{g(F_{q,q,q}(t)) + g(F_{q,q,q}(t)) + g(F_{q,q,q}(t)) + g(F_{Rq,q,q}(t)), \\ g(F_{q,q,q}(t)) + g(F_{q,q,q}(t)) + g(F_{q,q,q}(t)), g(F_{q,q,q}(t)) + g(F_{q,q,q}(t)) + \\ g(F_{Rq,q,q}(t)), g(F_{q,q,q}(t)) + g(F_{q,q,q}(t)) + g(F_{Rq,q,q}(t))\}] \\ = \phi\{g(F_{q,q,Rq}(t))\} \end{aligned}$$

Similarly,

$$\begin{aligned}h(G_{q,q,Rq}(t)) \geq \varphi[min\{&h(G_{q,q,q}(t))+h(G_{q,q,q}(t))+h(G_{q,q,q}(t))+h(G_{Rq,q,q}(t)),\\&h(G_{q,q,q}(t))+h(G_{q,q,q}(t))+h(G_{q,q,q}(t)),h(G_{q,q,q}(t))+h(G_{q,q,q}(t))+\\&h(G_{Rq,q,q}(t)),h(G_{q,q,q}(t))+h(G_{q,q,q}(t))+h(G_{Rq,q,q}(t))\}]\\&= \varphi\{h(G_{q,q,Rq}(t))\}.\end{aligned}$$

This gives $q = Rq$. We have, $q = UVq$. Therefore, $q = Aq = Bq = Cq = Dq = Pq = Qq = Rq = UVq$

Step-10
By taking $x = x_{2n}, y = x_{2n+1}, z = Vq$ in (c), (d) and using (e), we have

$$\begin{aligned}g(F_{Px_{2n},Qx_{2n+1},RVq}(t)) \leq \phi[max\{&g(F_{ABx_{2n},CDx_{2n+1},UVVq}(t))+\\&g(F_{Px_{2n},ABx_{2n},CDx_{2n+1}}(t))+g(F_{VQx_{2n+1},CDx_{2n+1},UVVq}(t))+\\&g(F_{RVq,UVVq,ABx_{2n}}(t)),g(F_{Px_{2n},ABx_{2n},CDx_{2n+1}}(t))+\\&g(F_{VQx_{2n+1},CDx_{2n+1},UVVq}(t))+g(F_{RVq,ABx_{2n},CDx_{2n+1}}(t)),\\&g(F_{Px_{2n},CDx_{2n+1},UVVq}(t))+g(F_{VQx_{2n+1},CDx_{2n+1},UVVq}(t))+\\&g(F_{RVq,UVVq,ABx_{2n}}(t)),g(F_{Px_{2n},ABx_{2n},CDx_{2n+1}}(t))+\\&g(F_{VQx_{2n+1},UVVq,ABx_{2n}}(t))+g(F_{RVq,UVVq,ABx_{2n}}(t))\}].\end{aligned}$$

Similarly,

$$\begin{aligned}h(G_{Px_{2n},Qx_{2n+1},RVq}(t)) \geq \varphi[min\{&h(G_{ABx_{2n},CDx_{2n+1},UVVq}(t))+\\&h(G_{Px_{2n},ABx_{2n},CDx_{2n+1}}(t))+h(G_{VQx_{2n+1},CDx_{2n+1},UVVq}(t))+\\&h(G_{RVq,UVVq,ABx_{2n}}(t)),h(G_{Px_{2n},ABx_{2n},CDx_{2n+1}}(t))+\\&h(G_{VQx_{2n+1},CDx_{2n+1},UVVq}(t))+h(G_{RVq,ABx_{2n},CDx_{2n+1}}(t)),\\&h(G_{Px_{2n},CDx_{2n+1},UVVq}(t))+h(G_{VQx_{2n+1},CDx_{2n+1},UVVq}(t))+\\&h(G_{RVq,UVVq,ABx_{2n}}(t)),h(G_{Px_{2n},ABx_{2n},CDx_{2n+1}}(t))+\\&h(G_{VQx_{2n+1},UVVq,ABx_{2n}}(t))+h(G_{RVq,UVVq,ABx_{2n}}(t))\}].\end{aligned}$$

This implies that, as $n \to \infty$

$$\begin{aligned}g(F_{q,q,Vq}(t)) \leq \phi[max\{&g(F_{q,q,q}(t))+g(F_{q,q,q}(t))+g(F_{q,q,q}(t))+g(F_{Vq,q,q}(t)),\\&g(F_{q,q,q}(t))+g(F_{q,q,q}(t))+g(F_{q,q,q}(t)),g(F_{q,q,q}(t))+g(F_{q,q,q}(t))+\\&g(F_{Vq,q,q}(t)),g(F_{q,q,q}(t))+g(F_{q,q,q}(t))+g(F_{Vq,q,q}(t))\}]\\&= \phi\{g(F_{q,q,Vq}(t))\}.\end{aligned}$$

Similarly,

$$h(G_{q,q,Vq}(t)) \geq \varphi[min\{h(G_{q,q,q}(t))+h(G_{q,q,q}(t))+h(G_{q,q,q}(t))+h(G_{Vq,q,q}(t)),\\ h(G_{q,q,q}(t))+h(G_{q,q,q}(t))+h(G_{q,q,q}(t)),h(G_{q,q,q}(t))+h(G_{q,q,q}(t))+\\ h(G_{Vq,q,q}(t)),h(G_{q,q,q}(t))+\\ h(G_{q,q,q}(t))+h(G_{Vq,q,q}(t))\}]\\ =\varphi\{h(G_{q,q,Vq}(t))\}.$$

This implies $q = Vq$. Since $q = UVq$, we get $q = Vq$. Therefore, $q = Aq = Bq = Cq = Dq = Pq = Qq = Rq = Uq = Vq$, that is common fixed point of A,B,C,D,P,Q,R,U,V. Similarly, it is clear that q is also the common fixed point of A,B,C,D,P,Q,R,U,V in the case AB is continuous and (P,AB), (Q,CD), (R,UV) are compatible of type (J-2).

Case-2 AB is continuous and (P, AB),(Q, CD), (R, UV) are compatible of type (J-1). Since P is continuous and $(P,AB),(Q,CD),(R,UV)$ are compatible of type (J-1).
As P is continuous, $PPx_{2n} \rightarrow Pq$ and $P(AB)x_{2n} \rightarrow Pq$. Since, (P,AB) is compatible of type (J-1), $AB(AB)x_{2n} \rightarrow Pq$.

Step-11
By taking $x = ABx_{2n}, y = x_{2n+1}, z = x_{2n+2}$ in (c) and (d), we have

$$g(F_{P(AB)x_{2n},Qx_{2n+1},Rx_{n+2}}(t)) \leq \phi[max\{g(F_{AB(AB)x_{2n},CDx_{2n+1},UVx_{n+2}}(t))+\\ g(F_{P(AB)x_{2n},AB(AB)x_{2n},CDx_{2n+1}}(t))+g(F_{VQx_{2n+1},CDx_{2n+1},UVx_{n+2}}(t))+\\ g(F_{Rx_{n+2},UVx_{n+2},AB(AB)x_{2n}}(t)),g(F_{P(AB)x_{2n},AB(AB)x_{2n},CDx_{2n+1}}(t))+\\ g(F_{VQx_{2n+1},CDx_{2n+1},UVx_{n+2}}(t))+g(F_{Rx_{n+2},AB(AB)x_{2n},CDx_{2n+1}}(t)),\\ g(F_{P(AB)x_{2n},CDx_{2n+1},UVx_{n+2}}(t))+g(F_{VQx_{2n+1},CDx_{2n+1},UVx_{n+2}}(t))+\\ g(F_{Rx_{n+2},UVx_{n+2},AB(AB)x_{2n}}(t)),g(F_{P(AB)x_{2n},AB(AB)x_{2n},CDx_{2n+1}}(t))+\\ g(F_{VQx_{2n+1},UVx_{n+2},AB(AB)x_{2n}}(t))+g(F_{Rx_{n+2},UVx_{n+2},AB(AB)x_{2n}}(t))\}].$$

Similarly,

$$h(G_{P(AB)x_{2n},Qx_{2n+1},Rx_{n+2}}(t)) \geq \varphi[min\{h(G_{AB(AB)x_{2n},CDx_{2n+1},UVx_{n+2}}(t))+\\ h(G_{P(AB)x_{2n},AB(AB)x_{2n},CDx_{2n+1}}(t))+h(G_{VQx_{2n+1},CDx_{2n+1},UVx_{n+2}}(t))+\\ h(G_{Rx_{n+2},UVx_{n+2},AB(AB)x_{2n}}(t)),h(G_{P(AB)x_{2n},AB(AB)x_{2n},CDx_{2n+1}}(t))+\\ h(G_{VQx_{2n+1},CDx_{2n+1},UVx_{n+2}}(t))+h(G_{Rx_{n+2},AB(AB)x_{2n},CDx_{2n+1}}(t)),\\ h(G_{P(AB)x_{2n},CDx_{2n+1},UVx_{n+2}}(t))+h(G_{VQx_{2n+1},CDx_{2n+1},UVx_{n+2}}(t))+\\ h(G_{Rx_{n+2},UVx_{n+2},AB(AB)x_{2n}}(t)),h(G_{P(AB)x_{2n},AB(AB)x_{2n},CDx_{2n+1}}(t))+\\ h(G_{VQx_{2n+1},UVx_{n+2},AB(AB)x_{2n}}(t))+h(G_{Rx_{n+2},UVx_{n+2},AB(AB)x_{2n}}(t))\}].$$

This implies that, as $n \to \infty$

$$\begin{aligned} g(F_{Pq,q,q}(t)) \leq \phi[max\{g(F_{q,q,q}(t)) + g(F_{Pq,q,q}(t)) + g(F_{q,q,q}(t)) + g(F_{q,q,q}(t)), \\ g(F_{Pq,q,q}(t)) + g(F_{q,q,q}(t)) + g(F_{q,q,q}(t)), g(F_{Pq,q,q}(t)) + g(F_{q,q,q}(t)) + \\ g(F_{q,q,q}(t)), g(F_{Pq,q,q}(t)) + g(F_{q,q,q}(t)) + g(F_{q,q,q}(t))\}] \\ = \phi\{g(F_{Pq,q,q}(t))\}. \end{aligned}$$

Similarly,

$$\begin{aligned} h(G_{Pq,q,q}(t)) \geq \varphi[min\{h(G_{q,q,q}(t)) + h(G_{Pq,q,q}(t)) + h(G_{q,q,q}(t)) + h(G_{q,q,q}(t)), \\ h(G_{Pq,q,q}(t)) + h(G_{q,q,q}(t)) + h(G_{q,q,q}(t)), h(G_{Pq,q,q}(t)) + h(G_{q,q,q}(t)) + \\ h(G_{q,q,q}(t)), h(G_{Pq,q,q}(t)) + h(G_{q,q,q}(t)) + h(G_{q,q,q}(t))\}] \\ = \varphi\{h(G_{Pq,q,q}(t))\}. \end{aligned}$$

This means that $q = Pq$. Now using the step (4-9), we have, $q = Qq = CDq = Cq = Dq = Rq = UVq = Uq = Vq$.

Step-12

Since $R(X) \subseteq AB(X)$ there exists $l \in X$ such that $q = Rq = ABl$.
By taking $x = l, y = x_{2n+1}, z = x_{2n+2}$ in (c) and (d), we have

$$\begin{aligned} g(F_{Pl,Qx_{2n+1},Rx_{2n+2}}(t)) \leq \phi[max\{g(F_{ABl,CDx_{2n+1},UVx_{2n+2}}(t)) + \\ g(F_{Pl,ABl,CDx_{2n+1}}(t)) + g(F_{Qx_{2n+1},CDx_{2n+1},UVx_{2n+2}}(t)) + \\ g(F_{Rx_{2n+2},UVx_{2n+2},ABl}(t)), g(F_{Pl,ABl,CDx_{2n+1}}(t)) + \\ g(F_{Qx_{2n+1},CDx_{2n+1},UVx_{2n+2}}(t)) + g(F_{Rx_{2n+2},ABl,CDx_{2n+1}}(t)), \\ g(F_{Pl,CDx_{2n+1},UVx_{2n+2}}(t)) + g(F_{Qx_{2n+1},CDx_{2n+1},UVx_{2n+2}}(t)) + \\ g(F_{Rx_{2n+2},UVx_{2n+2},ABl}(t)), g(F_{Pl,ABl,CDx_{2n+1}}(t)) + \\ g(F_{Qx_{2n+1},UVx_{2n+2},ABl}(t)) + g(F_{Rx_{2n+2},UVx_{2n+2},ABl}(t))\}]. \end{aligned}$$

Similarly,

$$\begin{aligned} h(G_{Pl,Qx_{2n+1},Rx_{2n+2}}(t)) \geq \varphi[min\{h(G_{ABl,CDx_{2n+1},UVx_{2n+2}}(t)) + \\ h(G_{Pl,ABl,CDx_{2n+1}}(t)) + h(G_{Qx_{2n+1},CDx_{2n+1},UVx_{2n+2}}(t)) + \\ h(G_{Rx_{2n+2},UVx_{2n+2},ABl}(t)), h(G_{Pl,ABl,CDx_{2n+1}}(t)) + \\ h(G_{Qx_{2n+1},CDx_{2n+1},UVx_{2n+2}}(t)) + h(G_{Rx_{2n+2},ABl,CDx_{2n+1}}(t)), \\ h(G_{Pl,CDx_{2n+1},UVx_{2n+2}}(t)) + h(G_{Qx_{2n+1},CDx_{2n+1},UVx_{2n+2}}(t)) + \\ h(G_{Rx_{2n+2},UVx_{2n+2},ABl}(t)), h(G_{Pl,ABl,CDx_{2n+1}}(t)) + \\ h(G_{Qx_{2n+1},UVx_{2n+2},ABl}(t)) + h(G_{Rx_{2n+2},UVx_{2n+2},ABl}(t))\}]. \end{aligned}$$

This implies that, as $n \to \infty$

$$\begin{aligned} g(F_{Pl,q,q}(t)) \leq \phi[max\{g(F_{Pl,q,q}(t)) + g(F_{q,Pl,q}(t)) + g(F_{q,q,q}(t)) + g(F_{q,q,Pl}(t)),\\ g(F_{q,Pl,q}(t)) + g(F_{q,q,q}(t)) + g(F_{q,Pl,q}(t)), g(F_{q,q,q}(t)) + g(F_{q,q,q}(t)) +\\ g(F_{q,q,Pl}(t)), g(F_{q,Pl,q}(t)) + g(F_{q,q,Pl}(t)) + g(F_{q,q,Pl}(t))\}]\\ = \phi\{g(F_{Pl,q,q}(t))\end{aligned}$$

and

$$\begin{aligned} h(G_{Pl,q,q}(t)) \geq \varphi[min\{h(G_{Pl,q,q}(t)) + h(G_{q,Pl,q}(t)) + h(G_{q,q,q}(t)) + h(G_{q,q,Pl}(t)),\\ h(G_{q,Pl,q}(t)) + h(G_{q,q,q}(t)) + h(G_{q,Pl,q}(t)), h(G_{q,q,q}(t)) + h(G_{q,q,q}(t)) +\\ h(G_{q,q,Pl}(t)), h(G_{q,Pl,q}(t)) + h(G_{q,q,Pl}(t)) + h(G_{q,q,Pl}(t))\}]\\ = \varphi\{h(G_{Pl,q,q}(t))\}.\end{aligned}$$

This gives $q = Rl$. Since, $q = Rq$, we have $q = Aq = ABl = Pl = ABl$. Since P is continuous and $(P,AB),(Q,CD),(R,UV)$ are compatible of type (J-1), we have, $Pq = ABq$. Also,$q = Bq$ follows from step-3. Thus, $q = Aq = Bq = Pq$. Hence, q is common fixed point of the nine maps in this case also. Similarly, it is clear that q is also the common fixed point of A,B,C,D,P,Q,R,U,V in the case P is continuous and (P,AB), (Q,CD), (R,UV) are compatible of type (J-2).

Step-13

For uniqueness, let $u, v(u, v \neq q)$ be another common fixed point of A,B,C,D,P,Q,R,U,V. Taking $x = q, y = u, z = v$ in (c) and (d), we have

$$\begin{aligned} g(F_{Pq,Qu,Rv}(t)) \leq \phi[max\{g(F_{ABq,CDu,UVv}(t)) + g(F_{Pq,ABq,CDu}(t)) +\\ g(F_{Qu,CDu,UVv}(t)) + g(F_{Rv,UVv,ABq}(t)), g(F_{Pq,ABq,CDu}(t)) +\\ g(F_{Qu,CDu,UVv}(t)) + g(F_{Rv,ABq,CDu}(t)), g(F_{Pq,CDu,UVv}(t)) +\\ g(F_{Qu,CDu,UVv}(t)) + g(F_{Rv,UVv,ABq}(t)), g(F_{Pq,ABq,CDu}(t)) +\\ g(F_{Qu,UVv,ABq}(t)) + g(F_{Rv,UVv,ABq}(t))\}].\end{aligned}$$

Similarly,

$$\begin{aligned} h(G_{Pq,Qu,Rv}(t)) \geq \varphi[min\{h(G_{ABq,CDu,UVv}(t)) + h(G_{Pq,ABq,CDu}(t)) +\\ h(G_{Qu,CDu,UVv}(t)) + h(G_{Rv,UVv,ABq}(t)), h(G_{Pq,ABq,CDu}(t)) +\\ h(G_{Qu,CDu,UVv}(t)) + h(G_{Rv,ABq,CDu}(t)), h(G_{Pq,CDu,UVv}(t)) +\\ h(G_{Qu,CDu,UVv}(t)) + h(G_{Rv,UVv,ABq}(t)), h(G_{Pq,ABq,CDu}(t)) +\\ h(G_{Qu,UVv,ABq}(t)) + h(G_{Rv,UVv,ABq}(t))\}].\end{aligned}$$

This implies that, as $n \to \infty$

$$\begin{aligned} g(F_{q,u,v}(t)) \leq \phi[max\{g(F_{q,u,v}(t)) + g(F_{q,q,u}(t)) + g(F_{u,u,v}(t)) + g(F_{v,v,q}(t)),\\ g(F_{q,q,u}(t)) + g(F_{u,u,v}(t)) + g(F_{v,q,u}(t)), g(F_{q,u,v}(t)) + g(F_{u,u,v}(t)) +\\ g(F_{v,v,q}(t)), g(F_{q,q,u}(t)) + g(F_{u,v,q}(t)) + g(F_{v,v,q}(t))\}]\\ = \phi\{g(F_{q,u,v}(t))\}.\end{aligned}$$

Similarly,

$$\begin{aligned} h(G_{q,u,v}(t)) \geq \varphi[min\{&h(G_{q,u,v}(t))+h(G_{q,q,u}(t))+h(G_{u,u,v}(t))+h(G_{v,v,q}(t)), \\ &h(G_{q,q,u}(t))+h(G_{u,u,v}(t))+h(G_{v,q,u}(t)), h(G_{q,u,v}(t))+h(G_{u,u,v}(t))+ \\ &h(G_{v,v,q}(t)), h(G_{q,q,u}(t))+h(G_{u,v,q}(t))+h(G_{v,v,q}(t))\} \\ &= \varphi\{h(G_{q,u,v}(t))\}. \end{aligned}$$

So, we have, $q = u = v$. This completes the proof of the theorem.
If we take $A = B = C = D = U = V = I_X$ (the identity map on X) in Theorem-3.1, we have the following results:

Corollary-3.2
Let P, Q, R are self maps on complete N-AIMPMS $(X,F,G,*,\Diamond)$. If $g(F_{Px,Qy,Rz}(t)) \leq \phi(g(F_{x,y,z}(t))$ and $h(G_{Px,Qy,Rz}(t)) \geq \varphi(h(G_{x,y,z}(t))$
Therefore,

$$\begin{aligned} g(F_{Px,Qy,Rz}(t)) \leq \phi[max\{&g(F_{x,y,z}(t))+g(F_{x,x,y}(t))+g(F_{y,y,z}(t))+g(F_{z,z,x}(t)), g(F_{x,x,y}(t)) \\ &+g(F_{y,y,z}(t))+g(F_{z,x,y}(t)), g(F_{x,y,z}(t))+g(F_{y,y,z}(t))+ \\ &g(F_{z,z,x}(t)), g(F_{x,x,y}(t))+g(F_{y,z,x}(t))+g(F_{z,z,x}(t))\}] \end{aligned}$$

and

$$\begin{aligned} h(G_{Px,Qy,Rz}(t)) \geq \varphi[min\{&h(G_{x,y,z}(t))+h(G_{x,x,y}(t))+h(G_{y,y,z}(t))+h(G_{z,z,x}(t)), h(G_{x,x,y}(t)) \\ &+h(G_{y,y,z}(t))+h(G_{z,x,y}(t)), h(G_{x,y,z}(t))+h(G_{y,y,z}(t)) \\ &+h(G_{z,z,x}(t)), h(G_{x,x,y}(t))+h(G_{y,z,x}(t))+h(G_{z,z,x}(t))\}], \end{aligned}$$

for all $x,y,z \in X$ and $t > 0$, where functions $\phi, \varphi : [0,\infty) \to [0,\infty)$ satisfies the condition (ϕ) and (φ). Then, P,Q,R have a unique common fixed point. □

Example-3.3
Suppose $X = [-1,1] \subset \mathbb{R}$. Define $F : X \times X \to N$ by

$$\begin{aligned} g(F_{Px,Qy,Rz}(t)) &\leq \phi(g(F_{AB_x,CD_y,UV_z}(t))) \\ &= \begin{cases} (\frac{t}{t+2})^{|x-y+z|} & t > 0, \\ 0 & t \leq 0 \end{cases} \end{aligned}$$

for $x,y,z \in X$. It is easy to verify that $(X,F,G,*,\Diamond)$ is a N-AIMPMS. Now assume $t,q,j,x,y,z,w,l \in X$. Then we have,

$$\begin{aligned}
&\phi(g(F_{AB_{x,y,w}(t),CD_{l,w,y}(q),UV_{y,l,z}}(j))) \\
&= \phi\{max(\frac{t}{t+2})^{|x-3y+w|}(\frac{q}{q+2})^{|l-w+y|}(\frac{j}{j+2})^{|y-l+z|}\} \\
&\leq \phi\{\frac{max(t,q,j)}{max(t,q,j)+2}\}^{|x-3y+w|+|l-w+y|+|y-l+z|} \\
&\leq \phi\{\frac{max(t,q,j)}{max(t,q,j)+2}\}^{|x-3y+w+l-w+y+y-l+z|} \\
&\leq \phi\{\frac{max(t,q,j)}{max(t,q,j)+2}\}^{|x-3y+w+l-w+y+y-l+z|} \\
&\leq \phi\{\frac{max(t,q,j)}{max(t,q,j)+2}\}^{|x-y+z|} \\
&= g(F_{Px,Qy,Rz}(max(t,q,j)).
\end{aligned}$$

4 APPLICATION TO FUNCTIONAL EQUATIONS

There are many types of nonlinear functional equations for which fixed point theorems have been used to demonstrate their existence.

Let J and K be Banach spaces, $L \subseteq J$ be a state space and $N \subseteq K$ be a decision space. Now, by using the fixed point results obtained in previous section, we have;

$$T(x) = \sup_{y,z \in N} \{g(x,y,z) + K(x,y,z,T(\tau(x,y,z)))\},$$

where $\tau : L \times N \to T, g : L \times N \to \mathbb{R}, K : L \times N \times \mathbb{R} \to \mathbb{R}$.
Let $H(L)$ denote the space of all bounded real valued function on L. Clearly, this space endowed with the metric given by

$$h(t,q,j) = \sup_{x \in L} \mid t(x) - q(x) + j(x) \mid$$

for all $t,q,j \in H(L)$, is a complete metric space.

Now, define

$$G_{t,q,j}(t) = \begin{cases} e^{-(\frac{h(t,q,j)}{t})} & t > 0 \\ 0 & t \leq 0, \end{cases}$$

where $t,q,j \in H(L)$, then $(H(L),F,G,*,\Diamond)$ is a complete N-AIMPMS with

$$\begin{aligned}
h(G_{Px,Qy,Rz} &\geq \psi\{h(G_{ABx,CDy,UVz}(t)\} \\
&= h(G_{Px,Qy,Rz}(min(t,q,j)),
\end{aligned}$$

where $t,q,j \in [-1,1] \subset \mathbb{R}$.

Functional equations is an essential tool for describing the nature of physical universe. It has multiple real-world applications from sports to engineering to astronomy and space travel, which can be solved by reducing them to equivalent fixed-point problems. Newton's Laws of motion and gravitational, astronomical science, Investment plans, global mappings, constructing tracks, Relation of income and market prediction are based on Functional equations. One example of a functional equation that may be used in the real world is the Euler-Lagrange equation. The shortest path between two points on a manifold is one way to approach this problem. The equation is a line when the two points are on a Cartesian plane. A great circle is obtained if the two points lie on a sphere. Airlines don't typically travel on straight paths due to this, so the journey is in fact longer. Geographically, the shortest distance/path is called a geodesic in both cases.

5 CONCLUSIONS

We obtained common fixed point theorems for nine maps by introducing three types of compatible maps in N-AIMPMS and expanded the results of Devi et al. (2018) under certain conditions using the concept of compatible maps of type (J-1) and (J-2). An example and application is provided to stake the applicability of our results.

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