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A Note on Auctions with Compulsory Partnership*

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Contents: 1. Introduction; 2. The common value model; 3. The compulsory participation model; 4. Discussion; 5. Conclusions.

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We study a symmetric, profit share, common value auction with a twist: One (fixed) Bidder, if not winning the auction, has to enter a partnership with the winner, sharing both expenses and revenue at rate (say) $0 < \lambda < 1$. We show that it doesn't have an equilibrium in pure-strategies.

Nós estudamos um leilão de valor comum, simétrico com uma mudança: Um determinado licitante, se não vencer o leilão, deve entrar numa parceria com o vencedor, dividindo tanto receitas quanto despesas a uma taxa pré-determinada. Demonstramos que não há equilíbrio em estratégias puras

1. INTRODUCTION

Suppose we plan a mineral rights auction and we have a preferred Bidder. However we want to have some competition. If our preferred Bidder is the highest bidder okay. However if he is not the highest bidder we require that he shares with the winner the earnings and expenses at some fixed rate $\lambda \in (0,1)$. Thus a compulsory partnership. Is this a sensible approach? We would ask that a minimum requirement is, under usual assumptions, that equilibrium bidding strategies exist. The model we study is motivated by the 2013 Brazil's Libra oil field pre-salt auction. We refer to Araujo, Costellini, Damé, & Monteiro (2016) for more details. There are three main ingredients: (i) A fixed cash bonus; (ii) A profit share/revenue share auction, and (iii) compulsory partnership. Considering two firms, we establish—in the usual manner—the equilibrium bidding functions differential equations. However we show that, in general, there is no such equilibrium. The possible existence of equilibrium bidding functions that are not "nice" is not studied here.

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2. THE COMMON VALUE MODEL

A contract to explore a resource with random return $V \geqq 0$, is to be awarded through an auction. We suppose[1] that there are two firms, $i = 1,2$. Each bidder i receives a random estimate S_i. We suppose (S_1,S_2,V) has a distribution with density $f(s,v) = f(s_1,s_2,v)$, $0 \leqq s_1,s_2,v \leqq \overline{v}$.

Assumption 5. *(i) The density $f(s,v)$, can be written in the form*

$$f(s,v) = h(v)g(s_1|v)g(s_2|v); \tag{DA}$$

(ii) $g(u|v)$ has monotone likelihood ratio, $\int g(u|v)\,\mathrm{d}u = 1$;
(iii) $h(u) > 0$ and $\int g(u|v)h(v)\,\mathrm{d}v > 0$, $0 < u < \overline{v}$.

From $h(v) = \int f(s,v)\,\mathrm{d}s_1\,\mathrm{d}s_2$ we see that $h(\cdot)$ is the density of V. The condition (DA) says that S_1,S_2 is conditionally independent given V. If $f_{(S_i,V)\,|\,S_j}$ denote the conditional density of (S_i,V) given S_j $(i \neq j)$ we have

$$f_{(S_i,V)\,|\,S_j}\Big((s_i,v)\,\Big|\,s_j\Big) = \frac{g(s_1|v)g(s_2|v)h(v)}{\int g(s_j|v)h(v)\,\mathrm{d}v}. \tag{DC}$$

3. THE COMPULSORY PARTICIPATION MODEL

The auction is a profit share auction. The winner incurs a cost $f \geqq 0$ and pays the cash bonus $B > 0$. We suppose $\mathbb{E}[V] > f + B$. Bidder's 2 compulsory share is $0 < \lambda < 1$. If Bidder 1 bids share $b \in [0,1]$ and Bidder 2 bids $c \in [0,1]$ we have the following payoffs:

$$\begin{aligned}
\pi_1(b,c) &= \begin{cases} 0 & \text{if } b < c, \\ (1-\lambda)\Big(v - f - b(v-f)^+ - B\Big) & \text{if } b \geq c. \end{cases} \\
\pi_2(b,c) &= \begin{cases} \lambda\Big(v - f - b(v-f)^+ - B\Big) & \text{if } c \leq b, \\ v - f - c(v-f)^+ - B, & \text{if } c > b. \end{cases}
\end{aligned} \tag{1}$$

Thus if $c > b$ Bidder 1 gets nothing and Bidder 2 pays the bonus B and from the revenue $v - f$ pays royalties $c(v-f)^+$. Thus $\pi_2(b,c) = v - f - c(v-f)^+ - B$. This is the usual payoff formula. If Bidder 1 wins, he pays the share $(1-\lambda)$ of the bonus and from the revenue $v-f$ he gets $(1-\lambda)(v-f-b(v-f)^+)$. Bidder 2 pays the share λ of the bonus B and gets $\lambda(v - f - b(v-f)^+)$. Auction participants will be willing to enter bids if the expected payoff is non-negative. Given that $f + B > 0$ if signal s_i is low enough[2] Bidder i will not participate.

Definition 5. An equilibrium is a 4–uple $(s_1^*,s_2^*,b_1(\cdot),b_2(\cdot))$ such that for $i = 12$:

(i) Bidder i participate if and only if $s_i \geqq s_i^*$;
(ii) If i participates he bids $b_i(s_i)$;
(iii) $b_i\colon [s_i^*,\overline{v}] \to [0,1]$ is strictly increasing and differentiable, $b_i(s_i^*) = 0$.

Let $b := b_1$ and $c = b_2$. Let $f^* = f + B$.

[1] Our theorem on the non-existence of pure strategy equilibrium is reasonably general. There is little gain to consider an arbitrary number of firms.

[2] That is if $\mathbb{E}[V\,|\,S_i = s_i] < f + B$.

3.1. Equations for (s_1^*, s_2^*).

We suppose that if both bidders do not participate nothing is paid and nothing is received. If Bidder 1 participate and bids 0 his expected payoff is $\mathbb{E}\left[(V-B)1_{S_2<s_2^*}\middle|S_1=s_1\right]$. Thus we get the equation

$$\mathbb{E}\left[(V-f^*)1_{S_2<s_2^*}\middle|S_1=s_1^*\right]=0. \tag{2}$$

If Bidder 2 does not participate the expected payoff is

$$\mathbb{E}\left[\lambda(V-f^*-b(S_1)(V-f)^+)1_{S_1\geq s_1^*}\middle|S_2=s_2\right].$$

If Bidder 2 participate and bids 0 his expected payoff is

$$\mathbb{E}\left[(V-f^*)1_{S_1<s_1^*}\middle|S_2=s_2^*\right]+\mathbb{E}\left[\lambda(V-f^*-b(S_1)(V-f)^+)1_{S_1\geq s_1^*}\middle|S_2=s_2\right].$$

At $S_2=s_2^*$ he is indifferent between participating or not. Thus

$$\mathbb{E}\left[(V-f^*)1_{S_1<s_1^*}\middle|S_2=s_2^*\right]=0.$$

Summing up we have the system,

$$\mathbb{E}\left[(V-f^*)1_{S_2<s_2^*}\middle|S_1=s_1^*\right]=0, \tag{3}$$

$$\mathbb{E}\left[(V-f^*)1_{S_1<s_1^*}\middle|S_2=s_2^*\right]=0. \tag{4}$$

We introduce a new assumption:

Assumption 6. *The system* (3) *and* (4) *has a unique solution* $s_1^*=s_2^*=s^*$.

The following example satisfy both assumptions 5 and 6:

Example 1. Let $f(s_1,s_2,v)=\frac{1}{v^2}1_{[0,v]}(s_1)1_{[0,v]}(s_2)$ and $\overline{v}=1$. Let $s_1^*=x$ and $s_2^*=y$ solve (3) and (4). Without loss of generality $0<x\leqq y$. Suppose $x<y$. Then $g(y|v)=0$ if $v<y$.

$$\int_y^1\int_{s_1=0}^x (v-f^*)\frac{1}{v^2}1_{[0,v]}(s_1)\,\mathrm{d}s_1\,\mathrm{d}v=\mathbb{E}\left[(V-f^*)1_{S_1<x}\middle|S_2=y\right]=0.$$

Since $1_{[0,v]}(s_1)=1$ we have

$$x\int_y^1 (V-f^*)\frac{1}{v^2}\,\mathrm{d}v=0 \implies \int_y^1 (V-f^*)\frac{1}{v^2}\,\mathrm{d}v=0.$$

In particular $y<f^*$. Now

$$\int\int_{s_2=0}^y (V-f^*)\frac{1}{v^2}1_{[0,v]}(x)1_{[0,v]}(s_2)\,\mathrm{d}s_2\,\mathrm{d}v=0$$

implies

$$\int_{v=x}^y\int_{s_2=0}^y (V-f^*)\frac{1}{v^2}1_{[0,v]}(s_2)\,\mathrm{d}s_2\,\mathrm{d}v=\int_{v=x}^y\int_{s_2=0}^y (V-f^*)\frac{1}{v^2}1_{[0,v]}(s_2)\,\mathrm{d}s_2\,\mathrm{d}v + \int_{v=y}^1 y(V-f^*)\frac{1}{v^2}\,\mathrm{d}v=0.$$

An impossibility since $v-f^*<0$ if $v<y$ implies

$$\int_{v=x}^y\int_{s_2=0}^y (V-f^*)\frac{1}{v^2}1_{[0,v]}(s_2)\,\mathrm{d}s_2\,\mathrm{d}v\neq 0.$$

Finally if s^* is such that $\int_{s^*}^1\frac{v-f^*}{v^2}1\,\mathrm{d}v=0$ then $s_1^*=s_2^*=s^*$ is the unique solution.

Remark 1. For the symmetric distributions case we are studying, the assumption above is quite natural.

Remark 2. We remark for later use that $b(\overline{v}) = c(\overline{v})$. This is true since no one will bid more than just enough to win the auction.

3.2. Equations for $b(\cdot),c(\cdot)$.

To shorten the equations below we suppose from now on $f = 0$. Thus $f^* = B$. For example now $(v-f)^+ = v$. The problem of Bidder 1 is to choose, for every realization $S_1 = x$, a bid b, $0 \le b \le 1$, maximizing

$$\begin{aligned}\mathbb{E}\left[(V-f^*-bV)1_{b>c(s_2)}\,\Big|\,S_1=x\right] &= \int_{u<c^{-1}(b)}\int (v-f^*-bv)h(v)g(x|v)g(u|v)\,\mathrm{d}v\,\mathrm{d}u\\ &= \int (v-f^*-bv)h(v)g(x|v)G\left(c^{-1}(b)\,\Big|\,v\right)\mathrm{d}v.\end{aligned} \tag{R}$$

Analogously, the problem of Bidder 2 is to choose, for each realization $S_2 = y$, a bid $0 \le c \le 1$ maximizing

$$\begin{aligned}&\mathbb{E}\left[(V-f^*-c\cdot V)1_{c>b(s_1)}+\lambda(V-f^*-c\cdot V)1_{c<b(s_1)}\,\Big|\,S_2=y\right]\\ &\quad= \int\left((v-f^*-cv)1_{c>b(u)}+\lambda\left(v-f^*-b(u)v\right)1_{c<b(u)}\right)h(v)g(u|v)g(y|v)\,\mathrm{d}v\,\mathrm{d}u\\ &\quad= \int (v-f^*-cv)h(v)G\left(b^{-1}(c)\,\Big|\,v\right)g(y|v)\,\mathrm{d}v\\ &\qquad+\int_{u>b^{-1}(c)}\lambda\left(v-f^*-b(u)v\right)h(v)g(u|v)g(y|v)\,\mathrm{d}v\,\mathrm{d}u.\end{aligned} \tag{C}$$

3.3. First–order conditions

Differentiating in b the expression (R) and making it equal to zero:

$$(c^{-1})'(b)\int (v-f^*-bv)f\left(x,c^{-1}(b),v\right)\mathrm{d}v-\int vg(x|v)G\left(c^{-1}(b)\,\Big|\,v\right)h(v)\,\mathrm{d}v=0.$$

Here we used that $f(x,c^{-1}(b),v) = g(x|v)g(c^{-1}(b)\,|\,v)h(v)$. In equilibrium, $b = b(x)$:

$$(c^{-1})'(b(x)) = \frac{\int vh(v)g(x|v)G\left(c^{-1}(b(x))\,\Big|\,v\right)\mathrm{d}v}{\int\left(v-f^*-b(x)v\right)f\left(x,c^{-1}(b(x)),v\right)\mathrm{d}v}.$$

Differentiating in c the problem of (C):

$$\begin{aligned}&-\int vh(v)G\left(b^{-1}(c)\,\Big|\,v\right)g(y|v)\,\mathrm{d}v\\ &\quad+(b^{-1})'(c)\int (v-f^*-cv)h(v)G\left(b^{-1}(c)\,\Big|\,v\right)g(y|v)\,\mathrm{d}v\\ &\quad\quad-(b^{-1})'(c)\int \lambda(v-f^*-cv)h(v)G\left(b^{-1}(c)\,\Big|\,v\right)g(y|v)\,\mathrm{d}v=0.\end{aligned}$$

Simplifying,

$$(b^{-1})'(c)(1-\lambda)\int (v-f^*-cv)f\left(y,g\left(b^{-1}(c)\right),v\right)\mathrm{d}v=\int vh(v)G\left(b^{-1}(c)\,\Big|\,v\right)g(y|v)\,\mathrm{d}v.$$

In equilibrium $c = c(y)$ and therefore we get the system

$$(c^{-1})'(b(x)) = \frac{\int vh(v)g(x|v)G\Big(c^{-1}(b(x))\,\Big|\,v\Big)\,\mathrm{d}v}{\int \Big(v - f^* - b(x)v\Big) f\Big(x, c^{-1}(b(x)), v\Big)\,\mathrm{d}v},$$
$$(b^{-1})'(u) = \frac{\int vh(v)G\Big(b^{-1}(u)\,\Big|\,v\Big)g(y|v)\,\mathrm{d}v}{(1-\lambda)\int (v - f^* - c(y)v) f\Big(y, b^{-1}(u), v\Big)\,\mathrm{d}v}.$$

In equilibrium $\mathrm{range}\,(b(\cdot)) = \mathrm{range}\,(c(\cdot))$. Thus we may rewrite the system, changing variables to $u = b(x) = c(y)$ and get

$$\begin{aligned} (c^{-1})'(u) &= \frac{\int vh(v)g(b^{-1}(u)\,|\,v)G(c^{-1}(u)\,|\,v)\,\mathrm{d}v}{\int (v - f^* - uv) f\Big(b^{-1}(u), c^{-1}(u), v\Big)\,\mathrm{d}v}, \\ (b^{-1})'(u) &= \frac{\int vh(v)G(b^{-1}(u)\,|\,v)g(c^{-1}(u)\,|\,v)\,\mathrm{d}v}{(1-\lambda)\int (v - f^* - uv) f\Big(c^{-1}(u), b^{-1}(u), v\Big)\,\mathrm{d}v}. \end{aligned} \tag{5}$$

We have $b^{-1}(0) = s^*$ and $c^{-1}(0) = s^*$. We now reformulate system (5) in such a way that the second equation does not depend on first equation. So, defining $k(w) = c^{-1}(b(w))$, and using that $(b^{-1})'(b(w)) = \frac{1}{b'(w)}$, we obtain

$$b'(w) = (1-\lambda)\frac{\int \Big(v - f^* - b(w)v\Big) f\Big(w, k(w), v\Big)\,\mathrm{d}v}{\int vh(v)G(w|v)g(k(w)\,|\,v)\,\mathrm{d}v}. \tag{6}$$

Then,

$$\begin{aligned} k'(w) &= (c^{-1})'(b(w))b'(w) \\ &= \frac{\int vh(v)g(w|v)G(k(w)\,|\,v)\,\mathrm{d}v\,\mathrm{d}u}{\int \Big(v - f^* - b(w)v\Big) f\Big(w, k(w), v\Big)\,\mathrm{d}v} \times (1-\lambda)\frac{\int \Big(v - f^* - b(w)v\Big) f\Big(w, k(w), v\Big)\,\mathrm{d}v}{\int vh(v)G(u|v)G(k(w)\,|\,v)\,\mathrm{d}v\,\mathrm{d}u} \\ &= (1-\lambda)\frac{\int vh(v)g(w|v)G(k(w)\,|\,v)\,\mathrm{d}v\,\mathrm{d}u}{\int vh(v)G(w|v)G(k(w)\,|\,v)\,\mathrm{d}v\,\mathrm{d}u}. \end{aligned} \tag{7}$$

Therefore, we have the following system, where the last equation is independent:

$$b'(w) = (1-\lambda)\frac{\int \Big(v - f^* - b(w)v\Big) f\Big(w, k(w), v\Big)\,\mathrm{d}v}{\int vh(v)G(w|v)G(k(w)\,|\,v)\,\mathrm{d}v\,\mathrm{d}u}, \tag{8}$$

$$k'(w) = (1-\lambda)\frac{\int vh(v)g(w|v)G(k(w)\,|\,v)\,\mathrm{d}v\,\mathrm{d}u}{\int vh(v)G(w|v)G(k(w)\,|\,v)\,\mathrm{d}v\,\mathrm{d}u}. \tag{9}$$

Theorem 1. *There is no equilibrium.*

Proof. We have $k(s^*) = c^{-1}(b(s^*)) = c^{-1}(0) = s^*$. Therefore,

$$k'(s^*) = (1-\lambda)\frac{\int vh(v)g(s^*\,|\,v)G(s^*\,|\,v)\,\mathrm{d}v}{\int vh(v)G(s^*\,|\,v)g(s^*\,|\,v)\,\mathrm{d}v} = 1 - \lambda < 1. \tag{10}$$

Whenever $k(w) = w$, we have

$$k'(w) = (1-\lambda)\frac{\int vh(v)g(w|v)G(w|v)\,\mathrm{d}v}{\int vh(v)G(w|v)g(w|v)\,\mathrm{d}v} = 1-\lambda. \tag{11}$$

Necessarily $k(\overline{v}) = c^{-1}(b(\overline{v})) = c^{-1}(c(\overline{v})) = \overline{v}$. Let $x = \inf X$ where $X = \{z \in (s^*, \overline{v}] : k(z) = z\}$. Note that $\overline{v} \in X$. Since $k'(s^*) < 1$ and $k(s^*) = s^*$ we have $k(w) < w$ is $s^* < w$ if w is sufficiently near s^*. Thus $x > s^*$. However, $k(x) = x$ implies $k'(x) < 1$ and there is some $z < x$ sufficiently near x such that $k(z) > z$. But using the intermediate value theorem, this contradicts the definition of x. □

4. DISCUSSION

Existence The existence of monotonic bidding strategies is studied in great generality in Reny & Zamir (2014). A key condition is that only the winner pays and gets the good. In the compulsory partnership the compulsory partner bid may lose but he still shares with the winner the common value object and pays proportionally. Non-existence results as in Landsberger & Tsirelson (2000) rely on participation costs. In our paper participation is costless. So what drives non-existence? Once we are in the region with positive expected profits (i.e. $s_i > s^*$) the non compulsory Bidder gets a lower profit (at the rate $1-\lambda$) than the compulsory Bidder. This leads to $k(w) < w$ and thus, $b(w) < c(w)$. However if $b(w) = c(w)$ as it should be at least if $w = \overline{v}$ then $k'(w) = 1-\lambda < 1$. This conflict apparently drives the non-existence. Could there be a monotonic non-differentiable equilibrium? Discontinuous equilibria? Or one that is not increasing? Those are fair questions that we do not address.

Asymmetry Asymmetry might help existence. However if the asymmetry is such that participation cutoff $s_1^* > s_2^*$ the same proof as above works, since $k(s_1^*) = c^{-1}(b(s_1^*)) = s_2^* < s_1^*$. Thus $k(s_1^*) < s_1^*$ and the non-existence proof works fine.

5. CONCLUSIONS

The compulsory partnership model is an intriguing possibility. How would bidders play their strategies? We mention that, regretfully, Brazil's pre-salt auction wasn't successful. Bidders could form consortia (even including the compulsory partner Petrobras) and only one consortium was formed and bid the reserve price. Could it do better forbidding the consortia?

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