

Dissaving of the Past via Reverse Mortgages^{*}

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We build a simple two-period general equilibrium model with incomplete markets which incorporates reverse market mortgages without appealing to the complicated framework required by the infinite horizon models. Two types of agents are considered: elderly agents and investors. The former are owners of physical assets (for instance housing) who will want to sell them to investors. For that end the elderly agents, who are assumed to not have any bequest motive, issue claims against physical assets they own. One of the claims issued will be interpreted as reverse mortgage (loan for seniors) and the other one as a call option written on the value of housing equity. By assuming that both the elderly agents and the investors are price takers, and by applying the generalized game approach, we show that the equilibrium in this economy always exists, providing the usual conditions on utilities and initial endowments are satisfied. We end with a remark on efficiency of the equilibrium.

Construímos um modelo simples de dois períodos com mercados incompletos. Nele, incorporamos um mercado de hipotecas reversas que não requer o complicado arcabouço exigido pelos modelos de horizonte infinito. Consideramos dois tipos de agentes: os idosos, proprietários de ativos físicos (imóveis, neste caso), e os investidores, compradores dos ativos dos idosos. Assumimos, para isso, que os idosos emitem títulos baseados em seus ativos físicos e que não possuem interesse em deixar herança (bequest motive). Um dos títulos emitidos é definido como a hipoteca reversa (empréstimo para idosos), enquanto o outro é uma opção de compra cujo

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ativo subjacente é o imóvel. Assumindo que tanto os idosos quanto os investidores são tomadores de preços, e aplicando a metodologia de jogos generalizados, demonstramos que nossa economia sempre tem um equilíbrio, desde que as condições usuais sobre as utilidades e as dotações iniciais sejam satisfeitas. Concluímos o artigo com um estudo acerca da eficiência do equilíbrio encontrado.

1. INTRODUCTION

One of the great problems that many countries' social security systems faces is how to sustain the elderly people who are ageing later than in previous generations. In many countries, in order to reduce costs for social security, they had to increase either the retirement age or the time of contribution for social security. It is also well known that the older the population is, the more there is spent on medical care (see [De Nardi, French, & Jones, 2010](#)). Hence, it is the government that bears the costs of medical care, especially when the retirement pension or the saving rate is low like in Brazil or in many countries of Latin America.

On the other hand, in these countries, particularly in Brazil, there are many ways elderly people can borrow money. Among them are payroll deductions. Although these types of borrowing solve elderly people's problems in the short term, they may compromise their future consumption pattern leading to a loss of well-being in the long term. Before this problem, many developed countries like the USA and the UK had created markets for reverse mortgages, which are loans for seniors (see [Cocco & Lopes, 2015](#) for a recent study). Due to the complexity of this new financial product, many countries were or are carrying out studies for its implementation (see for example [Caetano & Mata, 2009](#) for the Brazilian case). In Brazil, reverse mortgages are offered as forms of investments which are sold as options to fund retirements (see for instance [Macedo, 2015](#)). Lastly, reverse mortgages differ from classical mortgages in several aspects. For the sake of completeness, the main characteristics are illustrated in the two following paragraphs

The most important financial decision the typical household makes is buying a house. Such a decision depends on household wealth, housing prices, rental rates, the kind of financing, age and many other factors. In youth, the household decides whether to buy or rent a house and, if it buys, what sort of mortgage to choose. Whatever type of mortgage chosen, households would have a type of savings. To understand that we raise the following question: How does a mortgage move money around? First, a mortgage is a loan which transfers income from the future to the present. Second, after origination, a mortgage transfers money from the present to the future. When a borrower makes a mortgage payment, a portion of the payment goes to reduce the balance of the loan, thus increasing the net worth of the household. In this sense, a mortgage is a *savings scheme*, and for many households it is the main vehicle for life-cycle wealth accumulation.

However, the picture changes dramatically if we consider elderly homeowners. All elderly people face uncertainty regarding their lifespan, health, medical expenses and housing prices. Of course, they also have to make choices concerning consumption and financial saving. The housing decision is not as simple as it seems because it is both a consumption good and an investment good. Thus, the correct treatment of housing equity may not be very obvious in the retirement saving context. Whatever the situation, elderly homeowners need to make such decisions. Given the uncertainty faced by elderly persons, how should they finance such decisions about consumption and mainly, health care? Since the only asset that elderly homeowners in our model have is their homes, they should find a way to make them liquid. One option is to sell the house. The other is to fall into a reverse mortgage.

A reverse mortgage (or lifetime mortgage) is a loan available to seniors, and is used to release the home equity on the property in the form of one lump sum or multiple payments. The homeowner's

obligation to repay the loan is deferred until the owner leaves (e.g., into aged care), the home is sold or the owner dies. When an elderly homeowner receives cash flows due to the reverse mortgage, it is as if they were spending some portion of their housing equity (wealth product from their past), thus increasing the net worth of the household. In this sense, a reverse mortgage is a *dissavings scheme*, and for many elderly persons it is the main vehicle for life-cycle consumption-health financing.

In spite of the great volume of both theoretical and empirical studies about reverse mortgages (see e.g. [Cocco & Lopes, 2015](#)), few are the papers which deals with it in a general equilibrium framework. These have mainly been carried out in a life-cycle setting under a partial equilibrium analysis. Our main contribution is to show that reverse mortgages are compatible with well-functioning markets. That is, there exists a competitive general equilibrium in an economy in which the elderly people engage in reverse mortgages in order to fund their consumption. This may include, in addition to food consumption, medical expenditures and leisure. To reach that objective we build a simple two-period general equilibrium model with incomplete markets which incorporate reverse market mortgages without appealing to the complicated framework required by the infinite horizon models.

To reach such a goal, several things are necessary and some simplifications must be imposed: first, we need to borrow the financial structure to accommodate our financial instruments from [Allen & Gale \(1991\)](#). Second, we must assume that elderly homeowners have no bequest motives¹ so that they can issue derivatives on the remainder (if there are any) after the reverse mortgage ends.² For an analysis of the consequences of the absence or presence of the bequest motive in the elderly dissaving context, see [Ando, Guiso, & Terlizzese \(1994\)](#). Third, although our model has been inspired by the model of [Allen & Gale \(1991\)](#)—in relation to the classification³ of agents who interact in the economy—ours presents notable differences. Our model considers multiple goods and two kinds of claims, contrary to Allen and Gale who consider a variety of claims. Lastly, our concept of equilibrium maintains the characteristics of the Arrow–Debreu equilibrium in the context of general equilibrium with incomplete markets. That is, agents maximize their utility functions subject to their budget constraints, and all markets are clear. The concept of equilibrium used by Allen and Gale is a two-stage equilibrium which is more appropriate for the case of oligopolistic competition. We do not adopt this concept because we are assuming that both agents, seniors and lenders, are price takers. In our model physical assets are indexed by elderly agents. Even so, this does not break the anonymity of the markets because the investors are only interested in the durability of physical assets and not in the identity of the proprietors. Finally, we show the existence of equilibrium and briefly discuss its constrained efficiency.

The methodology used to reach our existence objective is the generalized game approach used by [Arrow & Debreu \(1954\)](#). More precisely, we define the generalized game played by a finite number of players. These players are elderly agents, lenders, and fictitious agents called auctioneers. In the first period, there is one auctioneer choosing first-period prices of goods and the prices of the two claims traded (reverse mortgage, call options) in order to maximize the total first-period excess demand. In the second period we have one auctioneer for each state who maximizes the excess demand of goods traded in the second period. Then, we demonstrate that equilibrium for the generalized game corresponds to equilibrium for our original economy.

1.1. Related Literature

One of the major financial innovations of recent decades, in markets of collateralized loans, has been to allow borrowers to use the collateral that backs their loans. For instance: mortgage markets and leasing. Since the pioneer work of [Dubey \(1995\)](#) up to their more recent version, [Geanakoplos & Zame](#)

¹This allows the model to not extend to more periods.

²Time when elderly agents move out or pass away.

³Elderly agents in our model correspond to entrepreneurs and lenders to the investors in their model.



(2014), many theoretical works have been written; be they of finite horizon or of infinite horizon. With respect to the latter, and particularly to those GEI models with infinite horizons where agents do not live forever (e.g. overlapping generations), agents are not allowed to trade in the financial markets in the last period of their lives. An exception is the paper of Seghir & Torres-Martínez (2008), which uses collateral (like in Geanakoplos & Zame, 2014) to enforce the promises.

Another major financial innovation in recent decades has been the development of reverse mortgage markets where elderly borrowers receive a loan against housing equity and are allowed to stay in their homes enjoying all the benefits thereof. Due to the recent development of this market⁴ a growing amount of literature has been interested in this topic. Studies on the potential demand for reverse mortgages goes back to Rasmussen, Megbolugbe, & Morgan (1995). See also Stucki (2005) who estimated the potential market at 13.2 million older households. For an ample and deep study about the recent expansion of the reverse mortgage market, see Shan (2011) and Nakajima (2012), respectively. Among the most recent papers about reverse mortgages, we highlight Cocco & Lopes (2015) who focus on the design of the reverse mortgage. The work of Nakajima & Telyukova (2011) also stood out. In it, a rich structural model of housing and saving/borrowing decisions in retirement is used. This literature, about reverse mortgages, has been developed through three major trends. Namely, life-cycle and precautionary savings, housing and portfolio choices and discrete choices. Using mathematical programming with an equilibrium constraints approach, Michelangeli (2010) solves consumption, housing, and mobility decisions within a dynamic structural life-cycle model.

The paper is organized as follows. In the next section we describe the model. In section 3 we define the concept of equilibrium and state our result on the existence of equilibria. In section 4 we demonstrate our results and we provide a short section discussing the constrained efficiency properties of equilibria. Finally, we end by offering some concluding remarks.

2. THE MODEL

We consider a two-period economy, where agents know the present but face an uncertain future. That is, it is assumed that in period 0 (the present) there is just one state of nature while in period 2 (the future) there are S states of nature. There are L commodities in each period and in each state of nature. Thus, the consumption set is $\mathbb{R}_+^{L(S+1)}$. Any element of this set will be denoted by a pair (a_o, \tilde{a}) where $a_o \in \mathbb{R}_+^L$ and $\tilde{a} \in \mathbb{R}_+^{SL}$, with $a_s \in \mathbb{R}_+^L$. The price system of commodities is denoted by (p_o, \tilde{p}) and is assumed to belong to $\mathbb{R}_+^{L(S+1)}$.

2.1. Agents

The economy is populated by a continuum of elderly agents and a finite number of investors (lenders). The elderly population is divided into a finite number V of age groups. To make things simpler, we consider a representative agent $h \in V$ in each age group so that we have a finite number of elderly agents. These agents own physical assets (e.g. their houses) as part of their initial endowments $\omega_o^h \in \mathbb{R}_+^L$ and are interested only in the first-period consumption. Thus, each elderly homeowner $h \in V$ is characterized by his/her utility function $U^h: \mathbb{R}_+^L \rightarrow \mathbb{R}$, his/her initial endowments, $\omega_o^h \in \mathbb{R}_+^L$, and by the way he/she deteriorates his/her physical assets (e.g. his/her house). Let $Y_s^h \in \mathbb{R}_+^{L \times L}$ be the linear transformation⁵ which represents the deterioration of physical assets in state s . There are $i \in I$ investors who value consumption in both periods. Thus, the utility function and initial endowments of each investor $i \in I$ are $u^i: \mathbb{R}_+^{L(S+1)} \rightarrow \mathbb{R}$ and $(\omega_o^i, \tilde{\omega}) \in \mathbb{R}_+^{L(S+1)}$.

⁴The reverse mortgage market was created in 1987 with the HUD (Department of Housing and Urban Development) program called Home Equity Conversion Mortgage (HECM).

⁵This transformation plays a similar role to the durability in Geanakoplos & Zame (2014). That is, physical assets are deteriorated by use.

2.2. Financial Structure

Let $C \in \mathbb{R}_+^L$ be the unit of measure of the physical assets of the economy. For instance, if the physical assets were houses, then C would represent the unit of the constructed area (e.g. m^2 or square feet). Each unit bundle $C \in \mathbb{R}_+^L$ is deteriorated according to the linear transformation Y_s^h which depends on state $s \in S$ and the elderly person⁶ $h \in V$. Thus, each unit bundle C becomes another bundle $Y_s^h C \in \mathbb{R}_+^L$.

Each elderly person $h \in V$ issues debts by using two kinds of financial instruments:⁷ reverse mortgages and derivatives. For each unit of reverse mortgage (borrowing against each unit bundle, $C \in \mathbb{R}_+^L$, owned) issued by elderly person h , he will have to return r_s if the state s occurs. Hence, in the second period the reverse mortgage lender keeps the minimum between the value of deteriorated physical assets and the outstanding debt. That is,

$$m_s^h = \min \{r_s, p_s Y_s^h C\}.$$

Since we are assuming no bequest motives, each elderly person also issues debts by selling derivatives on the remainder (if there are any) after the reverse mortgage ends. This means that for each unit of reverse mortgage issued, one unit of derivative is issued, whose payoff is

$$d_s^h = \max \{p_s Y_s^h C - r_s, 0\} = [p_s Y_s^h C - r_s]^+.$$

Therefore, each elderly person should issue the same number of reverse mortgages and derivatives. If θ^h and ϑ^h are amounts of reverse mortgages and derivatives issued by the elderly person h , then the cost of issuing them is $p_o C \varphi^h$.

It is useful to note that the derivative whose payoff is defined above is the same as a call option written on the unit of measure, $C \in \mathbb{R}_+^L$, of bundles of physical assets.

Let π_h and q_h be the prices of reverse mortgages and derivatives respectively.

An economy with reverse mortgage is defined by

$$\mathcal{E}_{rm} = \left[\left(U^h, \omega_o^h, Y^h \right)_{h \in V}; \left(u^i, \omega^i \right)_{i \in I}; F \right],$$

where each h represents a representative elderly agent whose characteristics were given above and each i represents an investor whose characteristics were also described above. Finally, F represents the financial structure which consists of only two claims—reverse mortgages and derivatives whose payoffs were described above.

2.3. Individuals' problems

2.3.1. Elderly Agents

Given the prices of commodities, reverse mortgages and derivatives, (p_o, π_h, q_h) , each elderly agent $h \in V$ chooses $(x_o, \varphi, \vartheta) \in \mathbb{R}_+^L \times \mathbb{R}_+ \times \mathbb{R}_+$ in order to maximize his/her utility

$$U^h(x_o + C\varphi)$$

subject to the following budget constraints:

$$p_o x_o \leq p_o (w_o^h - C\varphi) + \pi_h \varphi + q_h \vartheta. \quad (1)$$

⁶More precisely, each age group deteriorates their houses in a different way.

⁷In Allen & Gale (1991) would be claims.



In addition,

$$m_s^h \vartheta + d_s \varphi = p_s Y_s^h C \varphi, \quad \forall s \in S, \quad (2)$$

must be satisfied.

Equation (1) says that the consumption of elderly agents is financed by the value of wealth after having gone through the reverse mortgage process and after having issued the call option. Condition (2) says the financial structure (set of claims) must be binding. This last condition implies that for the financial structure (set claims issued) to be feasible (compatible with the physical asset), we must have $\varphi = \vartheta$.

2.3.2. Investors

Given the prices of claims $(\pi, q) \in \mathbb{R}^V \times \mathbb{R}^V$, and commodity prices $(p_o, \tilde{p}) \in \mathbb{R}_+^{L(S+1)}$ each investor $i \in I$ chooses a consumption-investment⁸ plan $(x_o, \tilde{x}, \theta, \phi) \in \mathbb{R}_+^{L(S+1)} \times \mathbb{R}_+^V \times \mathbb{R}_+^V$ in order to maximize his/her utility function $u^i(x_o, \tilde{x})$ subject to the following budget constraints:

$$p_o x_o + \sum_{h \in V} \pi_h \theta_h + \sum_{h \in V} q_h \phi_h \leq p_o w_o^i \quad (3)$$

$$p_s x_s \leq p_s w_s^i + \sum_{h \in V} m_s^h \theta_h + \sum_{h \in V} d_s^h \phi_h, \quad s \in S \quad (4)$$

Budget constraints (3) and (4) say that: in the first period each investor finances both consumption and investments via his/her initial endowments; and in the second period his/her consumption is financed by the value of initial endowments and the returns of his/her investments made in the first period.

3. EQUILIBRIUM

3.1. Definition

The equilibrium for an economy, \mathcal{E}_{rm} , consists of commodity price system (p_o, \tilde{p}) , reverse mortgage prices $\pi \in \mathbb{R}_+^V$, call option prices $q \in \mathbb{R}_+^V$; allocations

$$(x_o^h, \varphi^h, \vartheta^h)_{h \in V} \in (\mathbb{R}_+^L)^V \times (\mathbb{R}_+ \times \mathbb{R}_+)^V$$

and

$$(x_o^i, \tilde{x}^i, \theta^i, \phi^i)_{i \in I} \in (\mathbb{R}_+^{L(S+1)})^I \times (\mathbb{R}_+^V \times \mathbb{R}_+^V)^I$$

such that the following conditions are satisfied:

1. Each $h \in V$, $(x_o^h, \varphi^h, \vartheta^h)$ maximizes $U^h(x_o + C\varphi^h)$ subject to the budget constraint (1) and the compatibility condition (2).
2. Each $i \in I$, $(x_o^i, \tilde{x}^i, \theta^i, \phi^i)$ maximizes $U^i(x_o, \tilde{x})$ subject to budget constraints (3) and (4).
3. Commodity markets clear:

(i) In $t = 0$,

$$\sum_{i \in I} x_o^i + \sum_{h \in V} x_o^h + \sum_{h \in V} C\varphi^h = \sum_{i \in I} \omega_o^i + \sum_{h \in V} \omega_o^h.$$

In $t = 1$,

$$\sum_{i \in I} x_s^i = \sum_{i \in I} \omega_s^i + \sum_{h \in V} Y_s C\varphi^h, \quad s \in S.$$

⁸In reverse mortgage and call option.

(ii) Claim markets clear:

$$\begin{aligned} \sum_{i \in I} \theta_h^i &= \varphi^h, & h \in V; \\ \sum_{i \in I} \phi_h^i &= \vartheta^h, & h \in V. \end{aligned}$$

Condition (i) says all commodity markets are cleared. Condition (ii) says that everything that is demanded by investors must be equal to everything that is sold in both reverse mortgage and derivatives markets.

3.2. Existence

In this section we will give sufficient conditions which guarantee the existence of equilibrium for an economy with reverse mortgage. More precisely, we have the following theorem.

Theorem 1. *For an economy in which*

- (i) *for all agents $h \in V$ and $i \in I$, ω^h and ω^i belong to \mathbb{R}_{++}^L and $\mathbb{R}_{++}^{L(S+1)}$ respectively;*
- (ii) *the utility functions $U^h : \mathbb{R}_{++}^L \rightarrow R$ and $U^i : \mathbb{R}_{++}^{L(S+1)} \rightarrow R$ are continuous, strictly increasing and strictly quasi-concave;*
- (iii) *the unit of measure of the physical assets of the economy $C \in \mathbb{R}_+^L$ is different from zero and does not deteriorate completely.⁹*

there is always an equilibrium.

Remark: For the second part of item (iii) to be true it is sufficient, for instance, to assume that Y_s^h is non-singular.

4. RESULTS

Before proving our main result, we first establish the following lemma which allows us to bound the allocations satisfying the feasible conditions (market clear conditions) of the equilibrium definition. More precisely, we state and prove the following lemma.

Lemma 1. *Under hypotheses (i) and (ii) in Theorem 1, allocations $(x_o^h, \varphi^h, \vartheta^h)_{h \in V}$ and $(x_o^i, \tilde{x}^i, \theta^i, \phi^i)_{i \in I}$ in the \mathcal{E}_{rm} that satisfy the feasibility conditions of the equilibrium definition are bounded.*

Proof. From (i) of item 3 in the definition of equilibrium, in period $t = 0$, one has:

$$\sum_{i \in I} x_o^i + \sum_{h \in V} x_o^h + \sum_{h \in V} C \varphi^h = \sum_{i \in I} \omega_o^i + \sum_{h \in V} \omega_o^h.$$

Therefore, for each $l \in L$ the following holds:

$$\sum_{i \in I} x_{ol}^i + \sum_{h \in V} x_{ol}^h + \sum_{h \in V} C_l \varphi^h = \sum_{i \in I} \omega_{ol}^i + \sum_{h \in V} \omega_{ol}^h.$$

The right hand side of the previous inequality is lower than

$$W_{ol} := \max \left\{ I \max_{i \in I} \|\omega_o^i\|_{\max}, V \max_{h \in V} \|\omega_o^h\|_{\max} \right\}.$$

⁹That is, $Y_s^h C$ is not zero, for all $s \in S$.



Since x_{ol}^i and x_{ol}^h are both positive, it follows that

$$x_{ol}^i \leq W_{ol} \quad \text{and} \quad x_{ol}^h \leq W_{ol}.$$

Since $C \in \mathbb{R}_+^L$ is not the zero vector, there exists $l' \in L$ such that $C_{l'} > 0$. Let $K = \min_{l' \in L} C_{l'}$. Thus

$$\varphi^h \leq \frac{W_{ol}}{K}.$$

From (i) of item 3 in the definition of equilibrium, in period $t = 1$, one has:

$$\sum_{i \in I} x_s^i = \sum_{i \in I} \omega_s^i + \sum_{h \in V} Y_s C \varphi^h, \quad s \in S.$$

Therefore, for each $l \in L$ the following holds:

$$\sum_{i \in I} x_{ls}^i = \sum_{i \in I} \omega_{ls}^i + \sum_{h \in V} (Y_s C)_l \varphi^h, \quad s \in S.$$

From item (iii) of Theorem 1 it follows that the coordinate $(Y_s C)_l$ is lower than $\|Y_s^h C\|_{\max}$. Therefore,

$$\sum_{i \in I} \omega_{ls}^i + \sum_{h \in V} (Y_s C)_l \varphi^h \leq W_{ls} := I \max_{i \in I} \|\omega_s^i\|_{\max} + V \max_{h \in V} \|Y_s^h C\|_{\max} \frac{W_{ol}}{K}.$$

From the positivity of x_{ls}^i it follows that

$$x_{ls}^i \leq W_{ls}.$$

Lastly, from (ii) of item 3 in the definition of equilibrium, in period $t = 0$, one has:

$$\begin{aligned} \sum_{i \in I} \theta_h^i &= \varphi^h, & h \in V, \\ \sum_{i \in I} \phi_h^i &= \vartheta^h, & h \in V. \end{aligned}$$

As $\varphi^h = \vartheta^h$, $\forall h \in V$, one has that both θ_h^i and ϕ_h^i are bounded from above by

$$\frac{W_{ol}}{K}.$$

Hence, for each $h \in V$, $(x_o^h, \varphi^h, \vartheta^h)$ belongs to the box

$$\square^h := [0, W_o]^L \times \left[0, \frac{W_o}{K}\right]^2.$$

Similarly, for each $i \in I$, $(x_o^i, \tilde{x}^i, \theta^i, \phi^i)$ belongs to the box

$$\square^i := [0, W_o]^L \times [0, W_1]^{LS} \times \left[0, \frac{W_o}{K}\right]^{2V},$$

where $W_o = \max_{l \in L} W_{ol}$ and $W_1 = \max_{s, l} W_{sl}$. So, Lemma 1 follows. \square

In order to reach our goal (proof of Theorem 1), first we will define a generalized game, as in [Debreu \(1952\)](#). Then we will show that such a game has a Nash equilibrium; and lastly we will demonstrate that the equilibrium for the generalized game corresponds to the equilibrium for our economy.

4.1. The Generalized Game

As said above we will prove Theorem 1 by establishing the existence of equilibrium in a generalized game with a finite set of utility maximizing agents (elderly agents and investors) and auctioneers in each period, maximizing the value of the excess demand in the markets. Thus, we define the generalized game \mathcal{G} in the following way:

1. Each elderly agent $h \in H$ maximizes U^h in the constrained strategy set $B^h(p, \pi_h, q_h) \cap \square^h$ which consists of all choices $(x_o, \varphi, \vartheta) \in \mathbb{R}_+^L \times \mathbb{R}_+^2$ satisfying (1) and (2) and in addition are bounded from above by constants which were obtained from Lemma 1.

Similarly, each investor $i \in I$ maximizes U^i in the constrained strategy set $B^i(p, \pi, q) \cap \square^i$ which consists of all choices $(x, \theta, \phi) \in \mathbb{R}_+^{L(S+1)} \times \mathbb{R}_+^V \times \mathbb{R}_+^V$ satisfying (3) and (4) and in addition are bounded from above by constants which were obtained from Lemma 1.

2. The auctioneer of the first period chooses $(p_o, \pi, q) \in \Delta^{L+V+V-1}$ in order to maximize

$$p_o \left[\sum_{i \in I} x_o^i + \sum_{h \in V} x_o^h + \sum_{h \in V} C\varphi^h - \sum_{i \in I} \omega_o^i - \sum_{h \in V} \omega_o^h \right] + \sum_{h \in V} \pi_h \left(\sum_{i \in I} \theta_h^i - \varphi^h \right) + \sum_{h \in V} q_h \left(\sum_{i \in I} \phi_h^i - \vartheta^h \right).$$

3. The auctioneer of state s of the second period chooses $p_s \in \Delta^{L-1}$ in order to maximize

$$p_s \left(\sum_{i \in I} x_s^i - \sum_{i \in I} \omega_s^i - \sum_{h \in V} Y_s C\varphi^h \right).$$

The following lemma guarantees the existence of equilibrium of the generalized game \mathcal{G} .

Lemma 2. *Under the hypothesis of Theorem 1 there exists a pure strategy equilibrium for the generalized game \mathcal{G} .*

Proof. Lemma 1 follows from the equilibrium existence theorem in the generalized game of Debreu (1952). In fact, the objective functions of the agents are continuous and quasi-concave in their strategies. Furthermore, the objective functions of the auctioneers are continuous and linear in their own strategies, and therefore quasi-concave. The correspondence of admissible strategies, for the agents and for the auctioneers, has compact domain and compact, convex, and nonempty values. Such correspondences are upper semi-continuous, because they have compact values and a closed graph. The lower semi-continuity of interior correspondences follows from hypothesis (i) in Theorem 1 (see Hildenbrand, 1974, p.26, fact 4). Because the closure of a lower semi-continuity correspondence is also lower semi-continuous, the continuity of these set functions is guaranteed. We can apply Kakutani's fixed point theorem to the correspondence of optimal strategies in order to find the equilibrium. \square

Finally, the following lemma claims that the equilibrium of \mathcal{G} corresponds to the equilibrium of our economy.

Lemma 3. *If there exists an equilibrium for the generalized game \mathcal{G} , then there exists an equilibrium for the economy \mathcal{E}_{rm} .*

Proof. 1. **Feasibility:**

Assuming that $[p, \pi, p; (x_o^h, \varphi^h, \vartheta^h)_{h \in V}; (x_o^i, \tilde{x}^i, \theta^i, \phi^i)_{i \in I}]$ is an equilibrium for the generalized game,



we have that

$$p_o x_o^h \leq p_o (w_o^h - C\varphi^h) + \pi_h \varphi^h + q_h \vartheta^h \quad (5)$$

$$m_s^h \vartheta^h + d_s \varphi^h = p_s Y_s^h C\varphi^h, \quad \forall s \in S \quad (6)$$

$$p_o x_o^i + \sum_{h \in V} \pi_h \theta_h^i + \sum_{h \in V} q_h \phi_h^i \leq p_o w_o^i \quad (7)$$

$$p_s x_s^i \leq p_s w_s^i + \sum_{h \in V} m_s^h \theta_h^i + \sum_{h \in V} d_s^h \phi_h^i, \quad s \in S. \quad (8)$$

Adding in h gives

$$p_o \sum_{h \in V} [x_o^h + C_h \varphi^h - w_o^h] \leq \pi_h \sum_{h \in V} \varphi^h + q_h \sum_{h \in V} \vartheta^h \quad (9)$$

$$m_s^h \vartheta^h + d_s \varphi^h = p_s Y_s^h C\varphi^h, \quad \forall s \in S. \quad (10)$$

Summing in i we have

$$p_o \sum_{i \in I} (x_o^i - w_o^i) + \pi_h \sum_{i \in I} \sum_{h \in V} \theta_h^i + q_h \sum_{i \in I} \sum_{h \in V} \phi_h^i \leq 0 \quad (11)$$

$$p_s \sum_{i \in I} (x_s^i - w_s^i) \leq \sum_{i \in I} \sum_{h \in V} m_s^h \theta_h^i + \sum_{i \in I} \sum_{h \in V} d_s^h \phi_h^i, \quad s \in S. \quad (12)$$

Since $m_s = p_s Y_s^h C - d_s$, (12) becomes

$$p_s \sum_{i \in I} (x_s^i - w_s^i - \sum_{h \in V} Y_s C \theta_h^i) \leq \sum_{i \in I} \sum_{h \in V} d_s^h (\phi_h^i - \theta_h^i), \quad s \in S. \quad (13)$$

Summing (9) and (11) and grouping terms one has

$$p_o \left(\sum_{i \in I} x_o^i + \sum_{h \in V} x_o^h + \sum_{h \in V} C\varphi^h - \sum_{i \in I} \omega_o^i - \sum_{h \in V} \omega_o^h \right) + \sum_{h \in V} \pi_h \left(\sum_{i \in I} \theta_h^i - \varphi^h \right) + \sum_{h \in V} q_h \left(\sum_{i \in I} \phi_h^i - \vartheta^h \right) \leq 0. \quad (14)$$

Since (p_o, π, q) solves the first-period auctioneer's problem, we have

$$\sum_{i \in I} x_o^i + \sum_{h \in V} x_o^h + \sum_{h \in V} C\varphi^h - \sum_{i \in I} \omega_o^i - \sum_{h \in V} \omega_o^h \leq 0 \quad (15)$$

$$\sum_{i \in I} \theta_h^i - \varphi^h \leq 0, \quad \forall h \in V \quad (16)$$

$$\sum_{i \in I} \phi_h^i - \vartheta^h \leq 0, \quad \forall h \in V. \quad (17)$$

Notice that (13) and (14) hold with equality since (5), (7) and (8) also hold with equality. This last follows from the monotonicity of the utility functions.

Now we will show that (15), (16) and (17) hold with equality. In fact, suppose that there exists an $l \in L$ such that

$$\sum_{i \in I} x_{ol}^i + \sum_{h \in V} x_{ol}^h + \sum_{h \in V} C_l \varphi^h - \sum_{i \in I} \omega_{ol}^i - \sum_{h \in V} \omega_{ol}^h < 0, \quad (18)$$

which implies that $p_{ol} = 0$ and this in turn implies that the consumption x_o^h and x_o^i of both agents are the maximum available. This implies that both agents could increase their consumption, contradicting (15). Therefore (15) must hold with equality.

Now suppose that there is $h \in H$ such that (16) is a strict inequality. Then the price of this asset must be zero. That is, $\pi_h = 0$. This motivates the investors to purchase the maximum amount available, which contradicts the bounds already obtained for θ^i . Thus, (16) holds with equality. Similarly we can show that (17) is an equality as well. This implies that

$$\sum_{i \in I} \theta_h^i = \varphi^h, \quad \forall h \in V \quad (19)$$

$$\sum_{i \in I} \phi_h^i = \vartheta^h, \quad \forall h \in V. \quad (20)$$

Now we only need to prove conditions which clear markets in the second period. Before that, however, we notice that (2) implies that $\theta^h = \vartheta^h$ for all $h \in V$. From this it follows that the right hand side in (13) is zero. Therefore, one has

$$p_s \sum_{i \in I} \left(x_s^i - w_s^i - \sum_{h \in V} Y_s C \theta_h^i \right) \leq 0, \quad s \in S. \quad (21)$$

From the fact that p_s solves the second-period auctioneer's problem, we have

$$\sum_{i \in I} \left(x_s^i - w_s^i - \sum_{h \in V} Y_s C \theta_h^i \right) \leq 0, \quad s \in S. \quad (22)$$

Using the same argument to clear the goods markets in the first period, we prove that (22) holds with equality, which ends the feasibility.

2. Optimality:

We want to prove that $(x_o^h, \varphi^h, \vartheta^h)$ maximizes $U^h(x_o + C\varphi^h)$ on $B^h(p_o, \pi, p)$ and $(x_o^i, \tilde{x}^i, \theta^i, \phi^i)$ maximizes $U^i(x_o, \tilde{x})$ on $B^i(p, \pi, p)$. Suppose that what we have just said is untrue. Then, from the quasi-concavity of the utility functions of the agents and the interior of the solution in the generalized game on the part of agents, one finds a contradiction in the optimality of the players in \mathcal{G} . This is because $(x_o^h, \varphi^h, \vartheta^h)$ and $(x_o^i, \tilde{x}^i, \theta^i, \phi^i)$ satisfies the feasibility of Lemma 1. Therefore Lemma 3 follows. □

4.2. Remark on Efficiency

In this short section, we prove that the equilibrium allocations cannot be dominated in the Pareto sense by feasible allocations that satisfy the agents' budget restrictions, at equilibrium prices, at all states of the nature, except at a state where there may be some transference due to subsidies or taxes.

More precisely, we prove the following result

Theorem 2. *An equilibrium allocation $[(\bar{x}_o^h, \bar{\varphi}^h, \bar{\vartheta}^h)_{h \in V}; (\bar{x}_o^i, \bar{\tilde{x}}^i, \bar{\theta}^i, \bar{\phi}^i)_{i \in I}]$ for the economy \mathcal{E}_{rm} dominates, in the Pareto sense, any feasible allocation $[(x_o^h, \varphi^h, \vartheta^h)_{h \in V}; (x_o^i, \tilde{x}^i, \theta^i, \phi^i)_{i \in I}]$ that satisfies the budget restrictions of the agents at the original equilibrium prices.*



Proof. Suppose by contradiction that there is a feasible¹⁰ allocation $[(x_o^h, \varphi^h, \vartheta^h)_{h \in V}; (x_o^i, \tilde{x}^i, \theta^i, \phi^i)_{i \in I}]$ that belongs in the agent's budget set and

$$U^h(x_o^h + C\varphi^h) > U^h(\bar{x}_o^h + C\bar{\varphi}^h), \quad U^i(x_o, \tilde{x}) > U^i(\bar{x}_o, \tilde{\bar{x}}^h).$$

Then, it follows from standard arguments, such as the individual optimality of the equilibrium allocation and aggregation, that the allocation $[(x_o^h, \varphi^h, \vartheta^h)_{h \in V}; (x_o^i, \tilde{x}^i, \theta^i, \phi^i)_{i \in I}]$ is not feasible. Thus, Theorem 2 follows. \square

The efficiency of equilibrium allocations given by Theorem 2 is the weakly constrained efficiency, see Magill & Shafer (1991) for an ample discussion of this concept. In our model, besides the incompleteness of markets, there is another source of inefficiency which is introduced by the reverse mortgages. Trading of reverse mortgages entails a constrained transfer from lenders to borrowers (elderly people). However, this transfer could not be optimal since the housing property could increase above the value of the debt of the reverse mortgage. If this were the case, the lender would sell the housing to cover the debt and the surplus, which is the difference between the value of the house and debt, would be lost. The inefficiency introduced by the reverse mortgages may be corrected when a market for derivatives written on the value of the housing is created. Thus, the weakly constrained efficiency transfer from investors to homeowners is given by the equilibrium derivative prices. That is, the first-period income of the borrowers would increase due to the sale of the derivative by implying an increase in the investors' second-period income when the call option is in the money.

5. CONCLUDING REMARKS

In this paper we have constructed a simple two-period general equilibrium model which accommodates elderly agents who make use of the reverse mortgages to finance their consumption. We have also demonstrated the existence of equilibrium of this economy and have briefly provided some remarks about the constrained efficiency of the equilibrium allocation. Since we are not considering bequest motives, we postulate that the elderly agents issue derivatives whose payoff is the difference between the depreciated value of the housing and the value of the reverse mortgage. The methodology used to reach our goal has been that of the generalized game approach. In future research we will be analyzing the case in which senior citizens have descendants.

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