Revista Brasileira de Ensino de Física, vol. 46, e20240113 (2024)
www.scielo.br/rbef
DOI: https://doi.org/10.1590/1806-9126-RBEF-2024-0113

Letter to the Editor

Closed-form solutions of generalized linear first-order differential equations by Picard's method

Soluções em forma fechada de equações diferenciais lineares de primeira ordem generalizadas pelo método de Picard

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Received on March 31, 2024. Accepted on May 10, 2024.

For homogeneous linear first-order differential equations, it is shown that Picard's method of successive approximations is effective to furnish a closed-form solution even if the coefficient is an arbitrary function.
Keywords: Picard's method, successive approximations, first-order differential equation.

Para equações diferenciais lineares de primeira ordem homogêneas, é demonstrado que o método de sucessivas aproximações de Picard é eficaz para fornecer uma solução em forma fechada mesmo quando o coeficiente é uma função arbitrária.
Palavras-chave: Método de Picard, sucessivas aproximações, equação diferencial de primeira ordem.

In a recent didactic paper, Diniz [1] presented twelve different ways to solve the well-known simple harmonic oscillator problem. Among these methods lies Picard's method of successive approximations, commonly utilized for resolving particular cases of first-order differential equations (see, e.g. [2–7]). Here, after providing a succinct overview of this method, we demonstrate its effectiveness in yielding a closed-form solution for a homogeneous linear first-order differential equation, even when the coefficient is an arbitrary function.

Differential equations akin to

$$\frac{dy\left(x\right)}{dx}=f\left[x,y\left(x\right)\right] \tag{1}$$

can be reformulated as integral equations:

$$y\left(x\right)=y_0+\int_{x_0}^{x}d\zeta\, f\left[\zeta,y\left(\zeta\right)\right]. \tag{2}$$

These integral equations can be solved iteratively. In Picard's method, we derive a sequence of functions $\{y_n\left(x\right)\}_{n=0,1,2,\ldots,N}$, each satisfying the condition $y_n\left(x\right)|_{x=x_0}=y_0$. It is supposed that there is an interval about y_0 on which this sequence approaches the solution $y\left(x\right)$ as $N\to\infty$ and that it is the only continuous solution which does so (see, e.g. [2–7]). Because each succeeding function improves the prior one, this method is termed the method of successive approximations. The initial approximation is $y_0\left(x\right)=y_0$. Subsequent approximations are obtained as

$$y_{n+1}\left(x\right)=y_0+\int_{x_0}^{x}d\zeta\; f\left[\zeta,y_n\left(\zeta\right)\right], \tag{3}$$

resulting in

$$y\left(x\right)=\lim_{n\to\infty}y_{n+1}\left(x\right)=y_0+\int_{x_0}^{x}d\zeta\; f\left[\zeta,\lim_{n\to\infty}y_n\left(\zeta\right)\right], \tag{4}$$

For the general homogeneous linear first-order differential equation

$$\frac{dy\left(x\right)}{dx}+Q\left(x\right)y\left(x\right)=0, \tag{5}$$

where $y(0)=y_0$ and the coefficient $Q\left(x\right)$ is an arbitrary function, Picard's method of successive approximations yields

$$y_{n+1}\left(x\right)=y_0-\int_{0}^{x}d\zeta\; Q\left(\zeta\right)y_n\left(\zeta\right), \tag{6}$$

with $y_n(0)=y_0$. The first approximation is $y_0\left(x\right)=y_0$, and the subsequent approximations follow suit:

$$y_1\left(x\right)=y_0\left[1-\int_{0}^{x}dx_1\,Q\left(x_1\right)\right], \tag{7}$$

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$$y_2(x) = y_0 \left[1 - \int_0^x dx_1\, Q(x_1) + \int_0^x dx_1 \int_0^{x_1} dx_2\, Q(x_1)\, Q(x_2)\right], \tag{8}$$

$$y_3(x) = y_0 \left[1 - \int_0^x dx_1\, Q(x_1) + \int_0^x dx_1 \int_0^{x_1} dx_2\, Q(x_1)\, Q(x_2) - \int_0^x dx_1 \int_0^{x_1} dx_2 \int_0^{x_2} dx_3\, Q(x_1)\, Q(x_2)\, Q(x_3)\right], \tag{9}$$

so that

$$y_n(x) = y_0 \left[1 - \int_0^{x_0} dx_1\, Q(x_1) + \int_0^{x_0} dx_1 \int_0^{x_1} dx_2\, Q(x_1)\, Q(x_2) - \int_0^{x_0} dx_1 \int_0^{x_1} dx_2 \int_0^{x_2} dx_3\, Q(x_1)\, Q(x_2)\, Q(x_3) + \cdots + (-1)^n \int_0^{x_0} dx_1 \int_0^{x_1} dx_2 \int_0^{x_2} dx_3 \ldots \int_0^{x_{n-1}} dx_n\, Q(x_1)\, Q(x_2)\, Q(x_3) \ldots Q(x_n)\right], \tag{10}$$

where we have defined $x_0 = x$. This can be compactly written as

$$y_n(x) = y_0 \left[1 + \sum_{k=1}^{n} (-1)^k I_k(x)\right], \tag{11}$$

where

$$I_k(x) = \prod_{j=1}^{k} \int_0^{x_{j-1}} dx_j\, Q(x_j). \tag{12}$$

The definite integral

$$I_2(x) = \int_0^x dx_1 \int_0^{x_1} dx_2\, Q(x_1)\, Q(x_2) \tag{13}$$

is depicted in Figure 1 as an integral over the triangle above the dashed line $x_2 = x_1$ for $0 < x_2 < x$. The first integral is over the area of a horizontal slice of width dx_2 ranging from 0 to x_1, whereas the second integral adds up all the contributions from these horizontal slices from 0 to x. Because x_1 and x_2 are dummy variables, $I_2(x)$ can also be written as

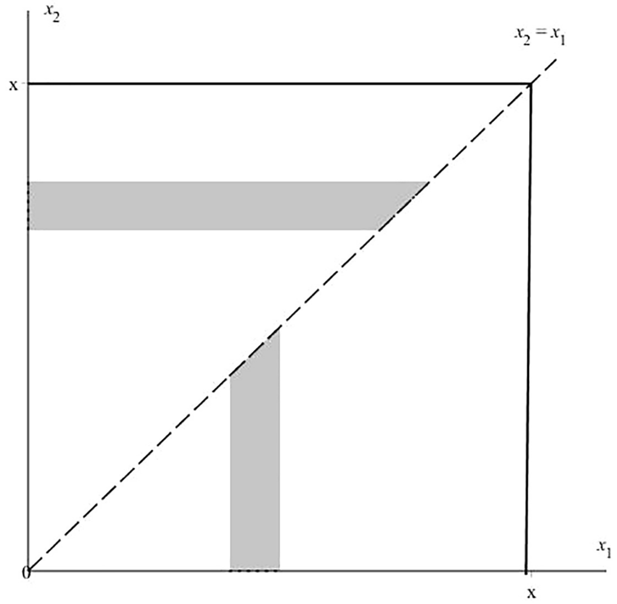

Figure 1: Graphical representation of the double integral $\int_0^x dx_1 \left(\int_0^{x_1} dx_2\, Q(x_1)\, Q(x_2)\right)$ is over the triangle above the dashed line $x_2 = x_1$ for $0 < x_2 < x$, and $\int_0^x dx_2 \left(\int_0^{x_2} dx_1\, Q(x_2)\, Q(x_1)\right)$ is represented over the triangle below the dashed line for $0 < x_1 < x$. Both are equivalent to half of the integral over a square with sides equal to x.

$$I_2(x) = \int_0^x dx_2 \int_0^{x_2} dx_1\, Q(x_2)\, Q(x_1). \tag{14}$$

Now, we can see an integral of the very same integrand over the triangle below the line $x_2 = x_1$. The first integral is over the area of a vertical slice of width dx_1 ranging from 0 to x_2. The second integral adds up all the vertical slices from 0 to x. Concisely, $I_2(x)$ represents half of the integral covering the square with $0 < x_1 < x$ and $0 < x_2 < x$, i.e.

$$I_2(x) = \frac{1}{2} \left(\int_0^x d\zeta\, Q(\zeta)\right)^2. \tag{15}$$

It is instructive to note that, even without resorting to geometry, this result can be analytically derived by identifying

$$u(x) = \int_0^x d\zeta\, Q(\zeta), \tag{16}$$

and subsequently applying integration by parts of $u(x)\, du(x)/dx$.

 DOI: https://doi.org/10.1590/1806-9126-RBEF-2024-0113

In general,

$$I_k(x) = \frac{1}{k!}\left(\int\limits_0^x d\zeta\, Q(\zeta)\right)^k, \tag{17}$$

because there are $k!$ identical terms of the type (13), corresponding to the $k!$ possible ways of interchanging the dummy variables $x_1, x_2, \ldots, x_k$. As a result,

$$y_n(x) = y_0 \sum_{k=0}^{n} \frac{1}{k!}\left(-\int\limits_0^x d\zeta\, Q(\zeta)\right)^k, \tag{18}$$

such that $\lim\limits_{n\to\infty} y_n(x)$ equals

$$y(x) = y_0 \exp\left(-\int\limits_0^x d\zeta\, Q(\zeta)\right), \tag{19}$$

which represents the general solution of (5).

It can be confirmed that incorporating a constant nonhomogeneous term into equation (5) poses no additional challenge for Picard's method. With diligent application of Leibniz's theorem for differentiation under the integral sign and integration by parts repeatedly, we can smoothly include any arbitrary nonhomogeneous term.

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